Filomat 37:2 (2023), 643–660
https://doi.org/10.2298/FIL2302643Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the domain of q-Euler matrix in c and c_0 with its point spectra

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Abstract. We introduce new Banach spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ defined as the domain of generalized q-Euler matrix $\mathsf{E}^{\alpha,\beta}(q)$ in the spaces c_0 and c, respectively. Some topological properties and inclusion relations related to the newly defined spaces are exhibited. We determine the bases and obtain Köthe duals of the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$. We characterize certain matrix mappings from the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ to the space $\mathsf{S} \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. We compute necessary and sufficient conditions for a matrix operator to be compact from the space $\mathsf{e}_0^{\alpha,\beta}(q)$ to the space $\mathsf{S} \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$ using Hausdorff measure of non-compactness. Finally, we give point spectrum of the matrix $\mathsf{E}^{\alpha,\beta}(q)$ in the space c.

1. Introduction and preliminaries

The q-analog of a mathematical expression means the generalization of that expression using the parameter q. The generalized expression returns the original expression when q approaches 1. The study of q-calculus dates back to the time of Euler. It is a wide and an interesting area of research in recent times. Several researchers are engaged in the field of q-calculus due to its vast applications in mathematics, physics and engineering sciences. In the field of mathematics, it is widely used by researchers in approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, etc.

Let $0 < q < 1$. Then the q-number is defined by

$$v[q] = \begin{cases} \frac{1-q^v}{1-q} & (v > 0), \\ 1 & (v = 0). \end{cases}$$

One may notice that, when $q \to 1$ then $v[q] = v$ if $v > 0$.

The q-analog of binomial coefficient or q-binomial coefficient is defined by

$$\begin{bmatrix} r \\ v \end{bmatrix}_q = \begin{cases} \frac{r[q]!}{(r-v)[q]!v[q]!} & (r \geq v), \\ 0 & (v > r), \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 46A45, 46B45, 47A10, 47B07, 47B37.
Keywords. Sequence space; q-Euler matrix; Köthe duals; Matrix mappings; Compact operators; Point spectrum.
Received: 18 December 2020; Accepted: 23 November 2021
Communicated by Pratulananda Das
The research of the first author (T. Yaying) is supported by SERB, DST, New Delhi, India under the grant number EEQ/2019/000082.
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where q-factorial $v[q]!$ of v is given by

$$v[q]! = v[q](v-1)[q]\dots 2[q]1[q].$$

Lemma 1.1. *The q-analog of binomial formula or Gauss's binomial formula is given by*

$$(\alpha+\beta)_q^r = \begin{cases} (\alpha+\beta)(q\alpha+\beta)(q^2\alpha+\beta)\dots(q^{r-1}\alpha+\beta) & (r \geq 1), \\ 1 & (r=0) \end{cases}$$
$$= \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v}, \text{ where } \binom{v}{2} = 0 \text{ for } v < 2.$$

1.1. Sequence spaces

A linear subspace of w, the set of all real-valued sequences, is called a sequence space. Few examples of classical sequence spaces are ℓ_k (k-absolutely summable sequences, $1 \leq k < \infty$), ℓ_∞ (bounded sequences), c_0 (null sequences), c (convergent sequences), etc. Further the spaces of all bounded, null and convergent series are denoted by bs, cs_0 and cs, respectively. A Banach sequence space having continuous coordinates is called a BK sequence space. We are well aware that the spaces c_0 and c are Banach spaces endowed with the supremum norm.

It is well known that the matrix mappings between BK-spaces are continuous. Because of this celebrated property, the theory of matrix mappings has an important place in the study of sequence spaces. Let S and T be two sequence spaces and $\Phi = (\phi_{rv})$ be an infinite matrix of real entries. Further let Φ_v denote the v^{th} row of the matrix Φ. The sequence $\Phi s = \{(\Phi s)_r\} = \left\{ \sum_{v=0}^{\infty} \phi_{rv} s_v \right\}$ is called Φ-transform of the sequence $s = (s_v) \in \mathsf{S}$, provided that the series $\sum_{v=0}^{\infty} \phi_{rv} s_v$ exists. Further, if $\Phi s \in \mathsf{T}$ for every sequence $s \in \mathsf{S}$, then the matrix Φ is said to define a matrix mapping from S to T. The notation $(\mathsf{S} \to \mathsf{T})$ shall represent the family of all matrices that map from S to T. Furthermore, the matrix $\Phi = (\phi_{rv})$ is called a triangle if $\phi_{rr} \neq 0$ and $\phi_{rv} = 0$ for $r < v$.

The matrix domain S_Φ of matrix Φ in the space S is defined by

$$\mathsf{S}_\Phi = \{s \in w : \Phi s \in \mathsf{S}\}. \tag{1}$$

The set S_Φ is a sequence space. This property plays a significant role in constructing new sequence spaces. Moreover if Φ is a triangle and S is a BK-space then the sequence space S_Φ is also a BK-space equipped with the norm $\|s\|_{\mathsf{S}_\Phi} = \|\Phi s\|_{\mathsf{S}}$. Several authors applied this celebrated theory in the past to construct new Banach (or BK) sequence spaces by using some special triangles. For relevant literature, we refer the papers [2, 3, 6, 15, 16, 23, 28, 34, 45–47] and textbooks [7, 33, 43].

1.2. Compact operators and Hausdorff measure of non-compactness (Hmnc)

Let S and T be two Banach spaces. The set of all bounded linear operators $\mathsf{C} : \mathsf{S} \to \mathsf{T}$ will be denoted by $B(\mathsf{S} \to \mathsf{T})$, which is again a Banach space equipped with the norm $\|\mathsf{C}\| = \sup_{s \in B_{\mathsf{S}}} \|\mathsf{C}s\|$, where the notation B_{S} denote open ball in S. Further, we denote $\|\varsigma\|_{\mathsf{S}}^{\dagger} = \sup_{s \in B_{\mathsf{S}}} \left| \sum_{v=0}^{\infty} \varsigma_v s_v \right|$. In this case, we observe that $\varsigma = (\varsigma_v) \in \mathsf{S}^\beta$, provided that the supremum exists.

Now we recall the definitions of compact operator and **Hmnc** of a bounded set.

Definition 1.2. *An operator $\mathsf{C} : \mathsf{S} \to \mathsf{T}$ is said to be compact if the domain of S is all of S and for every bounded sequence (s_r) in S, the sequence $(\mathsf{C}(s_r))$ has a convergent subsequence in T.*

Definition 1.3. *The **Hmnc** of a bounded set H in a metric space S is defined by*

$$\chi(H) = \inf \left\{ \varepsilon > 0 : H \subset \cup_{v=0}^{r} B(s_v, a_v), s_v \in \mathsf{S}, a_v < \varepsilon \ (v = 0, 1, 2, \dots, r), r \in \mathbb{N}_0 \right\}.$$

where $B(s_v, a_v)$ is the open ball centered at s_v and radius a_v for each $v = 0, 1, 2, \dots, r$.

The compact operator and **Hmnc** are closely related. An operator $\mathsf{C} : \mathsf{S} \to \mathsf{T}$ is compact if and only if $\|\mathsf{C}\|_\chi = 0$, where $\|\mathsf{C}\|_\chi$ denotes **Hmnc** of the operator C and is defined by $\|\mathsf{C}\|_\chi = \chi(\mathsf{C}(B_{\mathsf{S}}))$. Using **Hmnc**, several authors obtained necessary and sufficient conditions for matrix operators to be compact between BK-spaces. For relevant literature, we refer to [9, 10, 35–38].

1.3. Euler sequence spaces

The Euler matrix $\mathsf{E}^\alpha = (\mathsf{e}^\alpha_{rv})$ of order α is defined by

$$\mathsf{e}^\alpha_{rv} = \begin{cases} \binom{r}{v}(1-\alpha)^{r-v}\alpha^v & (0 \le v \le r), \\ 0 & (v > r), \end{cases}$$

for all $r, v \in \mathbb{N}_0$, where $0 < \alpha < 1$.

Let α and β be two non-zero real numbers such that $\alpha + \beta \neq 1$, then binomial matrix $\mathsf{E}^{\alpha,\beta} = (\mathsf{e}^{\alpha,\beta}_{rv})$ is defined by

$$\mathsf{e}^{\alpha,\beta}_{rv} = \begin{cases} \frac{1}{(\alpha+\beta)^r}\binom{r}{v}\alpha^r\beta^{r-v} & (0 \le v \le r), \\ 0 & (v > r). \end{cases}$$

One may observe that the binomial matrix $\mathsf{E}^{\alpha,\beta}$ generalizes the Euler matrix E^α.

Several research publications can be found in the literature concerning sequence spaces generated by using Euler matrix E^α. Altay and Başar [2] introduced the Euler sequence spaces $\mathsf{e}^\alpha_0 = (c_0)_{\mathsf{E}^\alpha}$ and $\mathsf{e}^\alpha_\infty = (\ell_\infty)_{\mathsf{E}^\alpha}$. One may refer Table 1 that contains publications dealing with Euler sequence spaces. Before proceeding to the table, we define the operators $\nabla = (\delta_{rv})$, $\nabla^i = (\delta^i_{rv})$, $B^{(i)} = (b^{(i)}_{rv})$, $B^{(i)}_n = (b^{(i),n}_{rv})$ and $\nabla^f = (\delta^f_{rv})$, that are used in the table:

$$\delta_{rv} = \begin{cases} (-1)^{r-v} & (r-1 \le v \le r), \\ 0 & (v > r). \end{cases} \qquad \delta^{(i)}_{rv} = \begin{cases} (-1)^{r-v}\binom{i}{r-v} & (\max\{0, r-i\} \le v \le r), \\ 0 & (0 \le s < \max\{0, r-i\} \text{ or } v > r). \end{cases}$$

$$b^{(i)}_{rv} = \begin{cases} \binom{i}{r-v}\alpha^{i-r+v}\beta^{r-v} & (\max\{0, r-i\} \le v \le r), \\ 0 & (0 \le v \le \max\{0, r-i\}) \text{ or } (v > r), \end{cases}$$

$$b^{(i),n}_{rv} = \begin{cases} \binom{i}{r-v}a^{i-r+v}\beta^{r-v}n_v & (\max\{0, r-i\} \le v \le r), \\ 0 & (0 \le v \le \max\{0, r-i\}) \text{ or } (v > r), \end{cases}$$

$$\delta^f_{rv} = \begin{cases} (-1)^{r-v}\frac{\Gamma(f+1)}{(r-v)!\Gamma(f-r+v+1)} & (0 \le v \le r), \\ 0 & (s > r), \end{cases}$$

where $i \in \mathbb{N}_0$, $f \in \mathbb{R}$ and $n = (n_v)$ is any fixed sequence of real numbers.

Let $0 < q < 1$, then the q-Cesàro matrix $C(q) = (c^q_{rv})$ [1, 11] is defined by

$$c^q_{rv} = \begin{cases} \frac{q^v}{(r+1)[q]} & (0 \le v \le r), \\ 0 & (v > r). \end{cases}$$

The construction of sequence spaces using q-analog $C(q)$ of Cesàro matrix has been studied recently by Demiriz and Şahin [18]. The authors studied the domains $X_0(q) = (c_0)_{C(q)}$ and $X_c(q) = (c)_{C(q)}$. More recently Yaying et al. [47] studied Banach spaces $X^q_k = (\ell_k)_{C(q)}$ and $X^q_\infty = (\ell_\infty)_{C(q)}$, and studied associated operator ideals. For studies in q-Hausdorff matrices, we refer [1, 5, 11, 14, 40]. We stricly refer to [20] for detailed studies in q-calculus.

Motivated by the above studies, we construct BK sequence spaces $\mathsf{e}^{\alpha,\beta}_0(q)$ and $\mathsf{e}^{\alpha,\beta}_c(q)$ derived by the q-analog $\mathsf{E}^{\alpha,\beta}(q)$ of the matrix $\mathsf{E}^{\alpha,\beta}$. We exhibit some topological properties, inclusion relations and determine

Table 1: Euler spaces

Euler Spaces	References
$\mathsf{e}_0^\alpha = (c_0)_{\mathsf{E}^\alpha}$, $\mathsf{e}_c^\alpha = (c)_{\mathsf{E}^\alpha}$	[2]
$\mathsf{e}_k^\alpha = (\ell_k)_{\mathsf{E}^\alpha}$, $\mathsf{e}_\infty^\alpha = (\ell_\infty)_{\mathsf{E}^\alpha}$	[3, 34]
$\mathsf{e}_0^\alpha(\nabla) = (c_0)_{\mathsf{E}^\alpha\nabla}$, $\mathsf{e}_c^\alpha(\nabla) = (c)_{\mathsf{E}^\alpha\nabla}$	[4]
$\mathsf{e}_0^\alpha(\nabla^i) = (c_0)_{\mathsf{E}^\alpha\nabla^i}$, $\mathsf{e}_c^\alpha(\nabla^i) = (c)_{\mathsf{E}^\alpha\nabla^i}$, $\mathsf{e}_\infty^\alpha(\nabla^i) = (\ell_\infty)_{\mathsf{E}^\alpha\nabla^i}$	[39]
$\mathsf{e}_k^\alpha(\nabla^f) = (\ell_k)_{\mathsf{E}^\alpha\nabla^f}$, $\mathsf{e}_0^\alpha(\nabla^f) = (c_0)_{\mathsf{E}^\alpha\nabla^f}$, $\mathsf{e}_c^\alpha(\nabla^f) = (c)_{\mathsf{E}^\alpha\nabla^f}$, $\mathsf{e}_\infty^\alpha(\nabla^f) = (\ell_\infty)_{\mathsf{E}^\alpha\nabla^f}$	[21]
$\mathsf{e}^\alpha(k) = (\ell(k))_{\mathsf{E}^\alpha}$	[24]
$\mathsf{e}_0^\alpha(\nabla, k) = (c_0(k))_{\mathsf{E}^\alpha\nabla}$, $\mathsf{e}_c^\alpha(\nabla, k) = (c(k))_{\mathsf{E}^\alpha\nabla}$ and $\mathsf{e}_\infty^\alpha(\nabla, k) = (\ell_\infty(k))_{\mathsf{E}^\alpha\nabla}$	[25]
$\mathsf{e}_0^\alpha(\nabla^i, k) = (c_0(k))_{\mathsf{E}^\alpha\nabla^i}$, $\mathsf{e}_c^\alpha(\nabla^i, k) = (c(k))_{\mathsf{E}^\alpha\nabla^i}$, $\mathsf{e}_\infty^\alpha(\nabla^i, k) = (\ell_\infty(k))_{\mathsf{E}^\alpha\nabla^i}$	[26]
$f_0(\mathsf{E}^\alpha) = (f_0)_{\mathsf{E}^\alpha}$, $f(\mathsf{E}^\alpha) = (f)_{\mathsf{E}^\alpha}$	[27]
$\mathsf{e}_0^\alpha(B^{(i)}) = (c_0)_{\mathsf{E}^\alpha B^{(i)}}$, $\mathsf{e}_c^\alpha(B^{(i)}) = (c)_{\mathsf{E}^\alpha B^{(i)}}$, $\mathsf{e}_\infty^\alpha(B^{(i)}) = (\ell_\infty)_{\mathsf{E}^\alpha B^{(i)}}$	[22]
$\mathsf{e}_0^\alpha(B_n^{(i)}) = (c_0)_{\mathsf{E}^\alpha B_n^{(i)}}$, $\mathsf{e}_c^\alpha(B_n^{(i)}) = (c)_{\mathsf{E}^\alpha B_n^{(i)}}$	[30]
$\mathsf{e}_k^{\alpha,\beta} = (\ell_k)_{\mathsf{E}^{\alpha,\beta}}$, $\mathsf{e}_0^{\alpha,\beta} = (c_0)_{\mathsf{E}^{\alpha,\beta}}$, $\mathsf{e}_c^{\alpha,\beta} = (c)_{\mathsf{E}^{\alpha,\beta}}$, $\mathsf{e}_\infty^{\alpha,\beta} = (\ell_\infty)_{\mathsf{E}^{\alpha,\beta}}$	[12, 13]
$\mathsf{e}_0^{\alpha,\beta}(B^{(i)}) = (c_0)_{\mathsf{E}^{\alpha,\beta}B^{(i)}}$, $\mathsf{e}_c^{\alpha,\beta}(B^{(i)}) = (c)_{\mathsf{E}^{\alpha,\beta}B^{(i)}}$, $\mathsf{e}_\infty^{\alpha,\beta}(B^{(i)}) = (\ell_\infty)_{\mathsf{E}^{\alpha,\beta}B^{(i)}}$	[32]
$\mathsf{e}_0^{\alpha,\beta}(\nabla^f) = (c_0)_{\mathsf{E}^{\alpha,\beta}\nabla^f}$, $\mathsf{e}_c^{\alpha,\beta}(\nabla^f) = (c)_{\mathsf{E}^{\alpha,\beta}\nabla^f}$, $\mathsf{e}_\infty^{\alpha,\beta}(\nabla^f) = (\ell_\infty)_{\mathsf{E}^{\alpha,\beta}\nabla^f}$	[31, 44]
$\mathsf{e}_k^{\alpha,\beta}(\nabla^f) = (\ell_k)_{\mathsf{E}^{\alpha,\beta}\nabla^f}$	[46]

bases for the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$. In the section 3, we compute Köthe duals (α-, β- and γ-duals) of the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$. In the section 4, we characterize some matrix mappings from the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ to the space $\mathsf{T} \in \{\ell_\infty, c, c_0, \ell_1, cs, cs_0, bs\}$. In the section 5, we obtain necessary and sufficient conditions for the matrix $\mathsf{E}^{\alpha,\beta}$ to be compact from the space $\mathsf{e}_0^{\alpha,\beta}(q)$ to the space $\mathsf{T} \in \{\ell_\infty, c, c_0, \ell_1, cs, cs_0, bs\}$. In the final section, we compute point spectrum of the matrix $\mathsf{E}^{1,1}(q)$ in the space c.

2. The sequence spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$

Let α, β be two non-negative real numbers with $\alpha + \beta \neq 1$, then the q-analog of the binomial matrix $\mathsf{E}^{\alpha,\beta}(q) = (\mathsf{e}_{rv}^{\alpha,\beta}(q))$ of order (α, β) is defined by

$$\mathsf{e}_{rv}^{\alpha,\beta}(q) = \begin{cases} \frac{1}{(\alpha+\beta)_q^r}\begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}}\alpha^v\beta^{r-v} & (0 \leq v \leq r), \\ 0 & (v > r), \end{cases}$$

where $0 < q < 1$. Clearly when $q \to 1$, the matrix $\mathsf{E}^{\alpha,\beta}(q)$ reduces to binomial matrix $\mathsf{E}^{\alpha,\beta}$. Thus $\mathsf{E}^{\alpha,\beta}(q)$ generalizes binomial matrix $\mathsf{E}^{\alpha,\beta}$ in the sense of q-theory. Hence we may call the matrix $\mathsf{E}^{\alpha,\beta}(q)$ as the q-analog of binomial matrix $\mathsf{E}^{\alpha,\beta}$. Furthermore, we also realize that the matrix $\mathsf{E}^{\alpha,\beta}(q)$ reduces to the triangle $\mathsf{E}^\alpha(q)$ with entries $\frac{1}{(\alpha+(1-\alpha))_q^r}\begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}}\alpha^v(1-\alpha)^{r-v}$, when $\beta = 1 - \alpha$.

More explicitely

$$\mathsf{E}^{\alpha,\beta}(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{\beta}{(\alpha+\beta)_q} & \frac{\alpha}{(\alpha+\beta)_q} & 0 & 0 & \cdots \\ \frac{\beta^2}{(\alpha+\beta)_q^2} & \frac{(1+q)\alpha\beta}{(\alpha+\beta)_q^2} & \frac{q\alpha^2}{(\alpha+\beta)_q^2} & 0 & \cdots \\ \frac{\beta^3}{(\alpha+\beta)_q^3} & \frac{(1+q+q^2)\alpha\beta^2}{(\alpha+\beta)_q^3} & \frac{q(1+q+q^2)\alpha^2\beta}{(\alpha+\beta)_q^3} & \frac{q^3\alpha^3}{(\alpha+\beta)_q^3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let us consider the sequence $\epsilon_r = \left(\frac{\beta^r}{(\alpha+\beta)_q^r}\right)$ in the first column of the matrix $\mathsf{E}^{\alpha,\beta}(q)$. It is known from [20, p. 29] that the infinite product $(1+\alpha)_q^\infty = (1+\alpha)(1+q\alpha)(1+q^2\alpha)\dots$ converges to a finite limit. In the light of this result and some minor calculation, we realize that unlike its classical version, the sequence ϵ_r converges to a finite limit greater than zero, as $r \to \infty$. Similarly, the other columns of the matrix converge to a finite limit. This immediately gives us the following result:

Lemma 2.1. *$\mathsf{E}^{\alpha,\beta}(q)$ is a conservative matrix. In otherwords, $\mathsf{E}^{\alpha,\beta}(q)$ maps c to c.*

Now we define the generalized q-Euler sequence spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ by

$$\mathsf{e}_0^{\alpha,\beta}(q) = \left\{ s = (s_v) \in w : \lim_{r\to\infty} \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} s_v = 0 \right\},$$

$$\mathsf{e}_c^{\alpha,\beta}(q) = \left\{ s = (s_v) \in w : \lim_{r\to\infty} \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} s_v \text{ exists} \right\}.$$

In the notation of (1), we redefine the above sequence spaces by

$$\mathsf{e}_0^{\alpha,\beta}(q) = (c_0)_{\mathsf{E}^{\alpha,\beta}(q)} \quad \text{and} \quad \mathsf{e}_c^{\alpha,\beta}(q) = (c)_{\mathsf{E}^{\alpha,\beta}(q)}. \tag{2}$$

We emphasize that the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ reduce to some of the well known Euler sequence spaces in literature:

1. When q approaches 1, the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ reduce to the binomial sequence spaces $\mathsf{e}_0^{\alpha,\beta}$ and $\mathsf{e}_c^{\alpha,\beta}$, respectively, as studied by Bişgin [12].
2. When $\beta = 1-\alpha$, the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ reduce to q-Euler spaces $\mathsf{e}_0^{\alpha}(q) = (\ell_k)_{\mathsf{E}^{\alpha}(q)}$ and $\mathsf{e}_c^{\alpha}(q) = (\ell_\infty)_{\mathsf{E}^{\alpha}(q)}$, respectively, which further reduce to well known Euler sequence spaces e_0^{α} and e_c^{α}, respectively, when $q \to 1$, as studied by Altay and Başar [2].

Now define the sequence $t = (t_v)$ in terms of the sequence $s = (s_v)$ by

$$t_r = (\mathsf{E}^{\alpha,\beta}(q)s)_r = \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} s_v, \tag{3}$$

for each $r \in \mathbb{N}_0$. The sequence t is called the $\mathsf{E}^{\alpha,\beta}(q)$-transform of the sequence s. Further, on using (3), we write

$$s_v = \sum_{j=0}^{v} (-1)^{v-j} \frac{\begin{bmatrix} v \\ j \end{bmatrix}_q q^{\binom{v-j}{2}} \beta^{v-j} (\alpha+\beta)_q^j}{\alpha^v q^{\binom{v}{2}}} t_j, \tag{4}$$

for each $v \in \mathbb{N}_0$.

Now we state our first result:

Theorem 2.2. *$\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ are BK-spaces endowed with the same norm defined by*

$$\|s\|_{\mathsf{e}_0^{\alpha,\beta}(q)} = \|s\|_{\mathsf{e}_c^{\alpha,\beta}(q)} = \sup_{r\in\mathbb{N}_0} \left| \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} s_v \right|.$$

Proof. The proof is a routine verification and hence omitted. □

Theorem 2.3. *$\mathsf{e}_0^{\alpha,\beta}(q) \cong c_0$ and $\mathsf{e}_c^{\alpha,\beta}(q) \cong c$.*

Proof. Since the proofs are similar for both the spaces, hence we provide the proof of the first case only. Define the mapping $\pi : \mathsf{e}_0^{\alpha,\beta}(q) \to c_0$ by $\pi s = \mathsf{E}^{\alpha,\beta}(q)s$ for all $s \in \mathsf{e}_0^{\alpha,\beta}(q)$. Clearly, π is linear and $1-1$. Let $t = (t_r)$ be a sequence in c_0 and $s = (s_v)$ be as defined in (4). Then, we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} s_v \\
&= \lim_{r\to\infty} \frac{1}{(\alpha+\beta)_q^r} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} \alpha^v \beta^{r-v} \left(\sum_{j=0}^{v} (-1)^{v-j} \frac{\begin{bmatrix} v \\ j \end{bmatrix}_q q^{\binom{v-j}{2}} \beta^{v-j} (\alpha+\beta)_q^j}{\alpha^v q^{\binom{v}{2}}} t_j \right) \\
&= \lim_{r\to\infty} t_r = 0, \text{ since } t \in c_0.
\end{aligned}$$

Thus we realize that s is a sequence in $\mathsf{e}_0^{\alpha,\beta}(q)$ and the mapping $\pi : \mathsf{e}_0^{\alpha,\beta}(q) \to c_0$ is onto and norm preserving. Hence $\mathsf{e}_0^{\alpha,\beta}(q) \cong c_0$. This completes the proof. □

Theorem 2.4. *$c_0 \not\subseteq \mathsf{e}_0^{\alpha,\beta}(q)$. However, the inclusion $c \subset \mathsf{e}_c^{\alpha,\beta}(q)$ holds.*

Proof. The result immediately follows from Lemma 2.1. □

To end this section, we construct bases for the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$. We recall that domain S_Φ of a triangle Φ in the space S has a basis if and only if S has a basis. This statement together with Theorem 2.3 gives us the following result:

Theorem 2.5. *For every fixed $v \in \mathbb{N}_0$, define the elements of the sequence $z^{(v)}(q) = (z_r^{(v)}(q))$ in the space $\mathsf{e}_0^{\alpha,\beta}(q)$ by*

$$z_r^{(v)}(q) = \begin{cases} (-1)^{r-v} \dfrac{\begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{r-v}{2}} \beta^{r-v} (\alpha+\beta)_q^v}{\alpha^r q^{\binom{r}{2}}} & (v \le r), \\ 0 & (v > r). \end{cases}$$

Then

(a) the set $\left\{z^{(0)}(q), z^{(1)}(q), z^{(2)}(q), \ldots\right\}$ forms basis for the space $\mathsf{e}_0^{\alpha,\beta}(q)$ and every $s \in \mathsf{e}_0^{\alpha,\beta}(q)$ has a unique representation $s = \sum_{v=0}^{\infty} t_v z^{(v)}(q)$.

(b) the set $\left\{e, z^{(0)}(q), z^{(1)}(q), z^{(2)}(q), \ldots\right\}$ forms a basis for the space $\mathsf{e}_c^{\alpha,\beta}(q)$ and every $s \in \mathsf{e}_c^{\alpha,\beta}(q)$ can be uniquely expressed in the form $s = \xi e + \sum_{v=0}^{\infty} (t_v - \xi) z^{(v)}(q)$, where $\xi = \lim_{v\to\infty} t_v = \lim_{v\to\infty} (\mathsf{E}^{\alpha,\beta}(q)s)_v$.

3. Köthe duals

In the current section, we compute Köthe duals (α-, β-, γ-duals) of the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$. Since the computation of duals is similar for both the spaces, we shall omit the proof for the space $\mathsf{e}_c^{\alpha,\beta}(q)$. Before proceeding, we recall the definitions of Köthe duals.

Definition 3.1. *The Köthe-Toeplitz duals or $\alpha-$, $\beta-$ and $\gamma-$duals of subset $\mathsf{S} \subset w$ are defined by*

$$\begin{aligned}
\mathsf{S}^\alpha &= \{\varsigma = (\varsigma_v) \in w : \varsigma s = (\varsigma_v s_v) \in \ell_1 \text{ for all } s \in \mathsf{S}\}, \\
\mathsf{S}^\beta &= \{\varsigma = (\varsigma_v) \in w : \varsigma s = (\varsigma_v s_v) \in cs \text{ for all } s \in \mathsf{S}\} \text{ and} \\
\mathsf{S}^\gamma &= \{\varsigma = (\varsigma_v) \in w : \varsigma s = (\varsigma_v s_v) \in bs \text{ for all } s \in \mathsf{S}\},
\end{aligned}$$

respectively.

In the rest of the paper, $\mathcal{N}$ will denote the family of all finite subsets of $\mathbb{N}_0$. First we note the following lemmas due to Stielglitz and Tietz [41] that are necessary for obtaining the duals:

Lemma 3.2. *$\Phi = (\phi_{rv}) \in (c_0 \to \ell_1)$ if and only if*

$$\sup_{R\in\mathcal{N}}\left(\sum_{v=0}^{\infty}\left|\sum_{r\in R}\phi_{rv}\right|\right) < \infty.$$

Lemma 3.3. *$\Phi = (\phi_{rv}) \in (c_0 \to c)$ if and only if*

$$\sup_{r\in\mathbb{N}_0}\sum_{v=0}^{r}\left|\phi_{rv}\right| < \infty, \tag{5}$$

$$\lim_{r\to\infty}\phi_{rv} \text{ exists for each } v\in\mathbb{N}_0. \tag{6}$$

Lemma 3.4. *$\Phi = (\phi_{rv}) \in (c_0 \to \ell_\infty)$ if and only if (5) holds.*

Theorem 3.5. *The set $\nu_1(q)$ defined by*

$$\nu_1(q) = \left\{\varsigma = (\varsigma_v) \in w : \sup_{R\in\mathcal{N}}\sum_{v=0}^{\infty}\left|\sum_{r\in R}(-1)^{r-v}\frac{\begin{bmatrix} r\\ v\end{bmatrix}_q q^{\binom{r-v}{2}}\beta^{r-v}(\alpha+\beta)_q^v}{\alpha^r q^{\binom{r}{2}}}\varsigma_r\right| < \infty\right\}.$$

is the α-dual of the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$.

Proof. Consider the following equality

$$\begin{aligned}\varsigma_r s_r &= \sum_{v=0}^{r}(-1)^{r-v}\frac{\begin{bmatrix} r\\ v\end{bmatrix}_q q^{\binom{r-v}{2}}\beta^{r-v}(\alpha+\beta)_q^v}{\alpha^r q^{\binom{r}{2}}}\varsigma_r t_s\\ &= (\mathsf{A}^{\alpha,\beta}(q)t)_r\end{aligned} \tag{7}$$

for all $r \in \mathbb{N}_0$, where the sequence $t = (t_v)$ is the $\mathsf{E}^{\alpha,\beta}$-transform of the sequence $s = (s_v)$ and the matrix $\mathsf{A}^{\alpha,\beta}(q) = (a_{rv}^q)$ is defined by

$$a_{rv}^q = \begin{cases}(-1)^{r-s}\frac{\begin{bmatrix} r\\ v\end{bmatrix}_q q^{\binom{r-v}{2}}\beta^{r-v}(\alpha+\beta)_q^v}{\alpha^r q^{\binom{r}{2}}}\varsigma_r & (0\le v\le r),\\ 0 & (v > r).\end{cases}$$

We realize on using Eq. 7 that $\varsigma s = (\varsigma_r s_r) \in \ell_1$ whenever $s \in \mathsf{e}_0^{\alpha,\beta}(q)$ if and only if $\mathsf{A}^{\alpha,\beta}(q)t \in \ell_1$ whenever $t \in c_0$. Thus we deduce that $\varsigma = (\varsigma_r)$ is a sequence in α-dual of $\mathsf{e}_0^{\alpha,\beta}(q)$ if and only the matrix $\mathsf{E}^{\alpha,\beta}(q)$ belongs to the class $(c_0 \to \ell_1)$. Thus we conclude from Lemma 3.2 that $\left[\mathsf{e}_0^{\alpha,\beta}(q)\right]^\alpha = \nu_1(q)$. This completes the proof. □

Theorem 3.6. *Define the sets $\nu_2(q)$, $\nu_3(q)$ and $\nu_4(q)$ by*

$$\nu_2(q) = \left\{\varsigma = (\varsigma_r) \in w : \sum_{r=v}^{\infty}(-1)^{r-v}\frac{\begin{bmatrix} r\\ v\end{bmatrix}_q q^{\binom{r-v}{2}}\beta^{r-v}(\alpha+\beta)_q^v}{\alpha^r q^{\binom{r}{2}}}\varsigma_r \text{ exists for each } v\in\mathbb{N}_0\right\},$$

$$\nu_3(q) = \left\{\varsigma = (\varsigma_r) \in w : \sup_{r\in\mathbb{N}_0}\sum_{v=0}^{r}\left|\sum_{m=v}^{r}(-1)^{m-v}\frac{\begin{bmatrix} m\\ v\end{bmatrix}_q q^{\binom{m-v}{2}}\beta^{m-v}(\alpha+\beta)_q^v}{\alpha^m q^{\binom{m}{2}}}\varsigma_m\right| < \infty\right\},$$

$$\nu_4(q) = \left\{\varsigma = (\varsigma_r) \in w : \lim_{r\to\infty}\sum_{v=0}^{r}\sum_{m=v}^{r}(-1)^{m-v}\frac{\begin{bmatrix} m\\ v\end{bmatrix}_q q^{\binom{m-v}{2}}\beta^{m-v}(\alpha+\beta)_q^v}{\alpha^m q^{\binom{m}{2}}}\varsigma_m \quad \text{exists}\right\},$$

Then $\left[\mathsf{e}_0^{\alpha,\beta}(q)\right]^\beta = \nu_2(q) \cap \nu_3(q)$ *and* $\left[\mathsf{e}_c^{\alpha,\beta}(q)\right]^\beta = \nu_2(q) \cap \nu_3(q) \cap \nu_4(q)$.

Proof. Consider the following equality

$$\begin{aligned}\sum_{v=0}^{r} \varsigma_v s_v &= \sum_{v=0}^{r} \left\{ \sum_{m=0}^{v} (-1)^{v-m} \frac{\left[{v \atop m}\right]_q q^{\binom{v-m}{2}} \beta^{v-m} (\alpha+\beta)_q^m}{\alpha^v q^{\binom{v}{2}}} t_m \right\} \varsigma_v \\ &= \sum_{v=0}^{r} \left\{ \sum_{m=v}^{r} (-1)^{m-v} \frac{\left[{m \atop v}\right]_q q^{\binom{m-v}{2}} \beta^{m-v} (\alpha+\beta)_q^v}{\alpha^m q^{\binom{m}{2}}} \varsigma_m \right\} t_v \\ &= (\mathsf{B}^{\alpha,\beta}(q)t)_r \end{aligned} \tag{8}$$

for each $r \in \mathbb{N}_0$, where the sequence $t = (t_v)$ is the $\mathsf{E}^{\alpha,\beta}(q)$-transform of the sequence $s = (s_v)$ and the matrix $\mathsf{B}^{\alpha,\beta}(q) = (b_{rv}^q)$ is defined by

$$b_{rv}^q = \begin{cases} \sum\limits_{m=v}^{r} (-1)^{m-v} \frac{\left[{m \atop v}\right]_q q^{\binom{m-v}{2}} \beta^{m-v} (\alpha+\beta)_q^v}{\alpha^m q^{\binom{m}{2}}} \varsigma_m & (0 \le v \le r), \\ 0 & (v > r), \end{cases}$$

for all $r, v \in \mathbb{N}_0$. Thus on using Eq. 8, we realize that $\varsigma s = (\varsigma_r s_r) \in cs$ whenever $s = (s_r) \in \mathsf{e}_0^{\alpha,\beta}(q)$ if and only if $\mathsf{B}^{\alpha,\beta} t \in c$ whenever $t = (t_v) \in c_0$. This yields that $\varsigma = (\varsigma_r)$ is a sequence in β-dual of $\mathsf{e}_0^{\alpha,\beta}(q)$ if and if only the matrix $\mathsf{B}^{\alpha,\beta}(q)$ belongs to the class $(c_0 \to c)$. This in turn implies on using Lemma 3.3 that

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{r} \left| b_{rv}^{\alpha,\beta} \right| < \infty \text{ and } \lim_{r\to\infty} b_{rv}^{\alpha,\beta} \text{ exists for each } v \in \mathbb{N}_0.$$

Thus $\mathsf{e}_0^{\alpha,\beta}(q) = \nu_2(q) \cap \nu_3(q)$. This completes the proof. □

Theorem 3.7. *The γ-dual of the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ is $\nu_3(q)$.*

Proof. The proof is similar to the previous theorem except that Lemma 3.4 is employed instead of Lemma 3.3. □

4. Matrix mappings

In the present section, we determine necessary and sufficient conditions for a matrix to define mapping from the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $e_c^{\alpha,\beta}(q)$ to the space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. The following theorem is fundamental in our investigation.

Theorem 4.1. *Let T be any arbitrary subset of w. Then $\Phi = (\phi_{rv}) \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \mathsf{T})$ (or respectively $(\mathsf{e}_c^{\alpha,\beta}(q) \to \mathsf{T})$) if and only if $\Theta^{(r)} = (\theta_{mv}^{(r)}) \in (c_0 \to c)$ (or respectively $(c \to c)$) for each $r \in \mathbb{N}_0$, and $\Theta = (\theta_{rv}) \in (c_0 \to \mathsf{T})$ (or respectively $(c \to \mathsf{T})$) where*

$$\theta_{mv}^{(r)} = \begin{cases} 0 & (v > m), \\ \sum\limits_{l=v}^{m} (-1)^{l-v} \frac{\left[{l \atop v}\right]_q q^{\binom{l-v}{2}} \beta^{l-v} (\alpha+\beta)_q^v}{\alpha^l q^{\binom{l}{2}}} \phi_{rl} & (0 \le v \le m), \end{cases}$$

and

$$\theta_{rv} = \sum_{l=v}^{\infty} (-1)^{l-v} \frac{\left[{l \atop v}\right]_q q^{\binom{l-v}{2}} \beta^{l-v} (\alpha+\beta)_q^v}{\alpha^l q^{\binom{l}{2}}} \phi_{rl} \tag{9}$$

for all $r, v \in \mathbb{N}_0$.

Proof. The details of the proof are omitted since it is similar to the proof of Theorem 4.1 of [28]. □

Now, by using the results presented in the Stielglitz and Tietz [41] together with Theorem 4.1, we obtain the following results:

Corollary 4.2. *The following statements hold:*

1. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_\infty)$ *if and only if*

$$\sup_{m\in\mathbb{N}_0}\sum_{v=0}^{\infty}\left|\theta_{mv}^{(r)}\right| < \infty, \tag{10}$$

$$\lim_{m\to\infty}\theta_{mv}^{(r)} \text{ exists for all } v\in\mathbb{N}_0 \tag{11}$$

hold and

$$\sup_{r\in\mathbb{N}_0}\sum_{v=0}^{\infty}|\theta_{rv}| < \infty, \tag{12}$$

also holds.

2. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c)$ *if and only if (10) and (11) hold, and*

$$\sup_{r\in\mathbb{N}_0}\sum_{v=0}^{\infty}|\theta_{rv}| < \infty, \tag{13}$$

$$\lim_{r\to\infty}\theta_{rv} \text{ exists for all } v\in\mathbb{N}_0, \tag{14}$$

also hold.

3. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c_0)$ *if and only if (10) and (11) hold, and (12) and*

$$\lim_{r\to\infty}\theta_{rv} = 0 \text{ for all } v\in\mathbb{N}_0, \tag{15}$$

also hold.

4. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_1)$ *if and only if (10) and (11) hold, and*

$$\sup_{R\in\mathcal{R}}\sum_{v=0}^{\infty}\left|\sum_{r\in R}\theta_{rv}\right| < \infty, \tag{16}$$

also holds.

5. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to bs)$ *if and only if (10) and (11) hold, and*

$$\sup_{r\in\mathbb{N}_0}\sum_{v=0}^{\infty}\left|\sum_{m=0}^{r}\theta_{mv}\right| < \infty, \tag{17}$$

also holds.

6. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs)$ *if and only if (10) and (11) hold, and (17) and*

$$\sum_{r=0}^{\infty}\theta_{rv} \text{ converges for all } v\in\mathbb{N}_0, \tag{18}$$

also hold.

7. $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs_0)$ *if and only if (10) and (11) hold, and (17) and*

$$\sum_{r=0}^{\infty} \theta_{rv} = 0 \text{ for all } v \in \mathbb{N}_0, \tag{19}$$

also hold.

Corollary 4.3. *The following statements hold:*

1. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to \ell_\infty)$ *if and only if (10), (11) and*

$$\lim_{m\to\infty} \sum_{v=0}^{\infty} \theta_{mv}^{(r)}, \tag{20}$$

hold, and (13) also holds.

2. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to c)$ *if and only if (10), (11) and (20) hold, and (12), (14) and*

$$\lim_{r\to\infty} \sum_{v=0}^{r} \theta_{rv} \text{ exists} \tag{21}$$

also hold.

3. $\Phi \in (e_c^{pq} \to c_0)$ *if and only if (10), (11) and (20) hold, and (12), (15) and*

$$\lim_{r\to\infty} \sum_{v=0}^{r} \theta_{rv} = 0 \tag{22}$$

also hold.

4. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to \ell_1)$ *if and only if (10), (11) and (20) hold, and (16) also holds.*
5. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to bs)$ *if and only if (10), (11) and (20) hold, and (17) also holds.*
6. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to cs)$ *if and only if (10), (11) and (20) hold, and (17), (18) and*

$$\sum_{r=0}^{\infty}\sum_{v=0}^{\infty} \theta_{rv} \text{ converges,}$$

also hold.

7. $\Phi \in (\mathsf{e}_c^{\alpha,\beta}(q) \to cs_0)$ *if and only if (10), (11) and (20) hold, and (17), (18) and*

$$\sum_{r=0}^{\infty}\sum_{v=0}^{\infty} \theta_{rv} = 0$$

also hold.

We recall a basic lemma due to Başar and Altay [8] that will help in characterizing certain classes of matrix mappings from the spaces $\mathsf{e}_0^{\alpha,\beta}(q)$ and $\mathsf{e}_c^{\alpha,\beta}(q)$ to any arbitrary space T.

Lemma 4.4. *[8] Let S and T be any two sequence spaces, Φ be an infinite matrix and Θ be a triangle. Then, $\Phi \in (\mathsf{S} \to \mathsf{T}_\Theta)$ if and only if $\Theta\Phi \in (\mathsf{S} \to \mathsf{T})$.*

Now, by combining Lemma 4.4 with Corollaries 4.2 and 4.3, we define following classes of matrix mappings:

Corollary 4.5. *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $C^q = (c^q_{rv})$ by*

$$c^q_{rv} = \sum_{m=0}^{r} \frac{q^{m-1}}{(r+1)[q]} \phi_{mv}, \ (0 < q < 1)$$

for all $r, v \in \mathbb{N}$, where $r[q]$ is the q-analog of $r \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(\mathsf{e}_0^{\alpha,\beta}(q) \to X_0^q)$, $(\mathsf{e}_0^{\alpha,\beta}(q) \to X_c^q)$, $(\mathsf{e}_c^{\alpha,\beta}(q) \to X_0^q)$ and $(\mathsf{e}_c^{\alpha,\beta}(q) \to X_c^q)$ is determined from the respective ones in Corollaries 4.2 and 4.3, by replacing the elements of the matrix Φ by those of matrix C^q, where X_0^q and X_c^q are q-Cesàro sequence spaces defined by Demiriz and Şahin [18].

Corollary 4.6. *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $\tilde{C} = (C_{rv})$ by*

$$\tilde{C}_{rv} = \sum_{m=0}^{r} \frac{C_m C_{r-1}}{C_{r+1}} \phi_{mv}, \ (r, v \in \mathbb{N}_0)$$

where (C_r) are sequence of Catalan numbers. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(\mathsf{e}_0^{\alpha,\beta}(q) \to c_0(\tilde{C}))$, $(\mathsf{e}_0^{\alpha,\beta}(q) \to c(\tilde{C}))$, $(\mathsf{e}_c^{\alpha,\beta}(q) \to c_0(\tilde{C}))$ and $(\mathsf{e}_c^{\alpha,\beta}(q) \to c(\tilde{C}))$ is determined from the respective ones in Corollaries 4.2 and 4.3, by replacing the elements of the matrix Φ by those of matrix $\tilde{C}$, where $c(\tilde{C})$ and $c_0(\tilde{C})$ are Catalan sequence spaces defined by İlkhan [19].

Corollary 4.7. *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $F = (f_{rv})$ by*

$$f_{rv} = \sum_{m=0}^{r} \frac{f_m^2}{f_r f_{r+1}} \phi_{mv}, \ (r, v \in \mathbb{N}_0)$$

where (f_r) are sequence of Fibonacci numbers. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_\infty(F))$, $(\mathsf{e}_0^{\alpha,\beta}(q) \to c(F))$, $(\mathsf{e}_0^{\alpha,\beta}(q) \to c_0(F))$, $(\mathsf{e}_c^{\alpha,\beta}(q) \to \ell_\infty(F))$, $(\mathsf{e}_c^{\alpha,\beta}(q) \to c(F))$ and $(\mathsf{e}_c^{\alpha,\beta}(q) \to c_0(F))$, is determined from the respective ones in Corollaries 4.2 and 4.3, by replacing the elements of the matrix Φ by those of matrix F, where $\ell_\infty(F)$, $c(F)$ and $c_0(F)$ are Fibonacci sequence spaces defined by Kara and Başarır [23].

5. Compactness by Hmnc

It is known from Theorem 3.2.4 (a) of [33] that if S and T are any two *BK*-spaces, then every matrix $\Phi \in (\mathsf{S} \to \mathsf{T})$ defines a linear operator $\mathsf{C}_\Phi \in B(\mathsf{S} \to \mathsf{T})$, where $\mathsf{C}_\Phi s = \Phi s$ for all $s \in \mathsf{S}$. Moreover, if $\mathsf{S} \supset \sigma$ is a *BK*-space and $\Phi \in (\mathsf{S} \to \mathsf{T})$ then $\|\mathsf{C}_\Phi\| = \|\Phi\|_{(\mathsf{S}\to\mathsf{T})} = \sup_{r\in\mathbb{N}_0} \|\|\Phi_r\|_{\mathsf{S}}^\dagger < \infty$ (see [29, Theorem 1.23]), where σ represents the set all sequences that terminate in zeroes. The following Lemmas are essential for our investigation:

Lemma 5.1. $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$. *Further, if $\mathsf{S} \in \{\ell_\infty, c, c_0\}$, then $\|s\|_{\mathsf{S}}^\dagger = \|s\|_{\ell_1}$.*

Lemma 5.2. *[29, Theorem 2.15] Let H be a bounded subset in c_0 and define the operator $\pi_r : c_0 \to c_0$ by $\pi_r(s_0, s_1, s_2 \ldots) = (s_0, s_1, s_2 \ldots, s_r, 0, 0, \ldots)$ for all $s = (s_r) \in c_0$, then*

$$\chi(H) = \lim_{r\to\infty} \left(\sup_{r\in H} \|(I - \pi_r)(s)\| \right),$$

where I is the identity operator on c_0.

Lemma 5.3. *[36, Theorem 3.7] Let $\mathsf{S} \supset \sigma$ be a BK–space. Then, the following statements hold:*

(a) If $\Phi \in (\mathsf{S} \to c_0)$, then $\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty} \|\Phi_r\|_{\mathsf{S}}^\dagger$ and C_Φ is compact if and only if $\lim_{r\to\infty} \|\Phi_r\|_{\mathsf{S}}^\dagger = 0$.

(b) *If* S *has AK and* $\Phi \in (\mathsf{S} \to c)$, *then*

$$\frac{1}{2}\limsup_{r\to\infty}\left\|\Phi_r - \phi\right\|_{\mathsf{S}}^{\dagger} \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left\|\Phi_r - \phi\right\|_{\mathsf{S}}^{\dagger}$$

and C_Φ *is compact if and only if* $\lim_{r\to\infty}\left\|\Phi_r - \phi\right\|_{\mathsf{S}}^{\dagger} = 0$, *where* $\phi = (\phi_v)$ *with* $\phi_v = \lim_{r\to\infty}\phi_{rv}$ *for all* $v \in \mathbb{N}_0$.

(c) *If* $\Phi \in (\mathsf{S} \to \ell_\infty)$, *then* $0 \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\|\Phi_r\|_{\mathsf{S}}^{\dagger}$ *and* C_Φ *is compact if and only if* $\lim_{r\to\infty}\|\Phi_r\|_{\mathsf{S}}^{\dagger} = 0$.

In the rest of the paper, $\mathcal{R}_m$ is the subcollection of $\mathcal{R}$ consisting of subsets of $\mathbb{N}_0$ with elements that are greater than m.

Lemma 5.4. *[36, Theorem 3.11] Let* $\mathsf{S} \supset \sigma$ *be a BK–space. If* $\Phi \in (\mathsf{S} \to \ell_1)$, *then*

$$\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Phi_r\right\|_{\mathsf{S}}^{\dagger}\right) \le \|\mathsf{C}_\Phi\|_\chi \le 4\cdot\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Phi_r\right\|_{\mathsf{S}}^{\dagger}\right)$$

and C_Φ *is compact if and only if* $\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Phi_r\right\|_{\mathsf{S}}^{\dagger}\right) = 0$.

Lemma 5.5. *[36, Theorem 4.4, Corollary 4.5] Let* $\mathsf{S} \supset \sigma$ *be a BK–space and let*

$$\|\Phi\|_{bs}^{[r]} = \left\|\sum_{v=0}^{r}\Phi_v\right\|_{\mathsf{S}}^{\dagger}.$$

Then, the following statements hold:

(a) *If* $\Phi \in (\mathsf{S} \to cs_0)$, *then* $\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty}\|\Lambda\|_{(\mathsf{S}\to bs)}^{[r]}$ *and* C_Φ *is compact if and only if* $\lim_{r\to\infty}\|\Phi\|_{(\mathsf{S}\to bs)}^{[r]} = 0$.

(b) *If* S *has AK and* $\Phi \in (\mathsf{S} \to cs)$, *then*

$$\frac{1}{2}\limsup_{r\to\infty}\left\|\sum_{v=0}^{r}\Phi_v - \tilde{\phi}\right\|_{\mathsf{S}}^{\dagger} \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left\|\sum_{v=0}^{r}\Phi_v - \tilde{\phi}\right\|_{\mathsf{S}}^{\dagger}$$

and C_Φ *is compact if and only if* $\limsup_{r\to\infty}\left\|\sum_{v=0}^{r}\Phi_v - \tilde{\phi}\right\|_{\mathsf{S}}^{\dagger} = 0$, *where* $\tilde{\phi} = (\tilde{\phi}_v)$ *with* $\tilde{\phi}_v = \lim_{r\to\infty}\sum_{m=0}^{r}\phi_{mv}$ *for all* $v \in \mathbb{N}_0$.

(c) *If* $\Phi \in (\mathsf{S} \to bs)$, *then* $0 \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\|\Phi\|_{(\mathsf{S}\to bs)}^{[r]}$ *and* C_Φ *is compact if and only if* $\lim_{r\to\infty}\|\Phi\|_{(\mathsf{S}\to bs)}^{[r]} = 0$.

Lemma 5.6. *Let* S *be a sequence space and* $\Phi = (\phi_{rv})$ *be an infinite matrix. If* $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \mathsf{S})$, *then* $\Theta \in (c_0 \to \mathsf{S})$ *and* $\Phi s = \Theta t$ *for all* $s \in \mathsf{e}_0^{\alpha,\beta}(q)$, *where the matrix* $\Theta = (\theta_{rv})$ *is defined in (9).*

Proof. Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \mathsf{S})$ and $s \in \mathsf{e}_0^{\alpha,\beta}(q)$. Then $\Phi_r = (\phi_{rv})_{v\in\mathbb{N}_0} \in \left[\mathsf{e}_0^{\alpha,\beta}(q)\right]^\beta$ for all $r \in \mathbb{N}_0$. Consider the following equality,

$$\begin{aligned}(\Theta t)_v &= \sum_{v=0}^{\infty}\theta_{rv}t_v \\ &= \sum_{v=0}^{\infty}\left(\sum_{m=v}^{\infty}(-1)^{m-v}\frac{\left[{m \atop v}\right]_q q^{\binom{m-v}{2}}\beta^{m-v}(\alpha+\beta)_q^v}{\alpha^m q^{\binom{m}{2}}}\phi_{rm}\right)\left(\frac{1}{(\alpha+\beta)_q^v}\sum_{l=0}^{v}\begin{bmatrix} v \\ l\end{bmatrix}_q q^{\binom{l}{2}}\alpha^l\beta^{v-l}s_l\right) \\ &= \sum_{v=0}^{\infty}\phi_{rv}s_v = (\Phi s)_v \end{aligned} \tag{23}$$

for all $v \in \mathbb{N}_0$, where the sequence $t = (t_v)$ is the $\mathsf{E}^{\alpha,\beta}(q)$-transform of the sequence $s = (s_v)$. Thus, we realize that Θ_r is absolutely summable for each $r \in \mathbb{N}_0$ and $\Theta t \in \mathsf{S}$. This yields the desired consequence $\Theta \in (c_0 \to \mathsf{S})$. $\square$

Theorem 5.7. *The following statements hold:*

(a) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c_0)$, *then* $\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty} \sum_{v=0}^{\infty} |\theta_{rv}|$.

(b) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c)$, *then*

$$\frac{1}{2}\limsup_{r\to\infty}\sum_{v=0}^{\infty}|\theta_{rv}-\theta| \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\sum_{v=0}^{\infty}|\theta_{rv}-\theta|,$$

where $\theta = (\theta_v)$ *and* $\theta_v = \lim_{r\to\infty}\theta_{rv}$ *for each* $v \in \mathbb{N}_0$.

(c) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_\infty)$, *then* $0 \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty} \sum_{v=0}^{\infty} |\theta_{rv}|$.

(d) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_1)$, *then*

$$\lim_{m\to\infty}\|\Phi\|^{[m]}_{(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)} \le \|\mathsf{C}_\Phi\|_\chi \le 4\lim_{m\to\infty}\|\Phi\|^{[m]}_{(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)},$$

where $\|\Phi\|^{[m]}_{(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)} = \sup_{R\in\mathcal{R}_m}\sum_{v=0}^{\infty}\left|\sum_{r\in R}\theta_{rv}\right|$.

(e) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs_0)$, *then* $\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}\right|\right)$.

(f) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs)$, *then*

$$\frac{1}{2}\limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}-\tilde{\theta}\right|\right) \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}-\tilde{\theta}\right|\right),$$

where $\tilde{\theta} = (\tilde{\theta}_v)$ *with* $\tilde{\theta}_v = \lim_{r\to\infty}\sum_{l=0}^{r}\theta_{lv}$ *for each* $v \in \mathbb{N}_0$.

(g) If $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to bs)$, *then* $0 \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}\right|\right)$.

Proof. (a) Let $\Phi \in \left(\mathsf{e}_0^{\alpha,\beta}(q) \to c_0\right)$. We observe that

$$\|\Phi_r\|^{\dagger}_{\mathsf{e}_0^{\alpha,\beta}(q)} = \|\Theta_r\|^{\dagger}_{c_0} = \|\Theta_r\|_{\ell_1} = \sum_{v=0}^{\infty}|\theta_{rv}|$$

for $r \in \mathbb{N}_0$. We realize on employing Part (a) of Lemma 5.3 that

$$\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}|\theta_{rv}|\right).$$

(b) Notice that

$$\|\Theta_r-\theta\|^{\dagger}_{c_0} = \|\Theta_r-\theta\|_{\ell_1} = \sum_{v=0}^{r}|\theta_{rv}-\theta_v| \tag{24}$$

for each $r \in \mathbb{N}$. Now, let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c)$, then Lemma 5.6 implies that $\Theta \in (c_0 \to c)$. Employing Part (b) of Lemma 5.3, we deduce that

$$\frac{1}{2}\limsup_{r\to\infty}\|\Theta_r-\theta\|^{\dagger}_{c_0} \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\|\Theta_r-\theta\|^{\dagger}_{c_0},$$

which in the light of (24) yields us

$$\frac{1}{2}\limsup_{r\to\infty}\sum_{v=0}^{\infty}|\theta_{rv}-\theta_v| \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\sum_{v=0}^{\infty}|\theta_{rv}-\theta_v|$$

which is the desired result.

(c) The proof is analogous to the proof of Part (a). Hence details are excluded.

(d) We have

$$\left\|\sum_{r\in R}\Theta_r\right\|_{c_0}^{\dagger} = \left\|\sum_{r\in R}\Theta_r\right\|_{\ell_1} = \sum_{v=0}^{\infty}\left|\sum_{r\in R}\theta_{rv}\right|. \tag{25}$$

Let $\Phi\in(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)$. Then Lemma 5.6 implies that $\Theta\in(c_0\to\ell_1)$. Hence, by employing Lemma 5.4, we get

$$\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Theta_r\right\|_{c_0}^{\dagger}\right) \le \|\mathsf{C}_\Phi\|_\chi \le 4\cdot\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Theta_r\right\|_{c_0}^{\dagger}\right)$$

which further reduces on using (25) to

$$\lim_{m\to\infty}\|\Phi\|^{[m]}_{(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)} \le \|\mathsf{C}_\Phi\|_\chi \le 4\lim_{m\to\infty}\|\Phi\|^{[m]}_{(\mathsf{e}_0^{\alpha,\beta}(q)\to\ell_1)},$$

as desired.

(e) Notice that

$$\left\|\sum_{l=0}^{r}\Phi_l\right\|_{\mathsf{e}_0^{\alpha,\beta}(q)}^{\dagger} = \left\|\sum_{l=0}^{r}\Theta_l\right\|_{c_0}^{\dagger} = \left\|\sum_{l=0}^{r}\Theta_l\right\|_{\ell_1} = \sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}\right|,$$

which on using Part (a) of Lemma 5.5 yields

$$\|\mathsf{C}_\Phi\|_\chi = \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}\right|\right).$$

(f) We have

$$\left\|\sum_{l=0}^{r}\Theta_l-\tilde{\theta}\right\|_{c_0}^{\dagger} = \left\|\sum_{l=0}^{r}\Theta_l-\tilde{\theta}\right\|_{\ell_1} = \sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}-\tilde{\theta}_v\right| \tag{26}$$

for each $r\in\mathbb{N}_0$. Let $\Phi\in(\mathsf{e}_0^{\alpha,\beta}(q)\to cs)$. Then Lemma 5.6 implies that $\Theta\in(c_0\to cs)$. Thus with the aid of Part (b) of Lemma 5.5, we deduce that

$$\frac{1}{2}\limsup_{r\to\infty}\left\|\sum_{l=0}^{r}\Theta_l-\tilde{\theta}_v\right\|_{c_0}^{\dagger} \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left\|\sum_{l=0}^{r}\Theta_l-\tilde{\theta}_l\right\|_{c_0}^{\dagger},$$

which on using (26) yields us

$$\frac{1}{2}\limsup_{r\to\infty}\left(\sum_{v=0}\left|\sum_{l=0}^{r}\theta_{lv}-\tilde{\theta}_v\right|\right) \le \|\mathsf{C}_\Phi\|_\chi \le \limsup_{r\to\infty}\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{r}\theta_{lv}-\tilde{\theta}_v\right|\right),$$

as desired.

(g) This proof is analogous to proof of Part (e). Hence details are excluded. □

Now, we have the following corollaries:

Corollary 5.8. *The following statements hold:*

(a) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c_0)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \sum_{v=0}^{\infty} |\theta_{rv}| = 0$.

(b) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to c)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \left(\sum_{v=0}^{\infty} \left| \tilde{\theta}_{ij} - \tilde{\theta}_j \right| \right) = 0$.

(c) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_\infty)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \sum_{v=0}^{\infty} |\theta_{rv}| = 0$.

(d) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to \ell_1)$*, then* C_Φ *is compact if and only if* $\lim_{m\to\infty} \left(\sup_{R\in\mathcal{R}_m} \left(\sum_{v=0}^{\infty} \left| \sum_{r\in R} \theta_{rv} \right| \right) \right) = 0$.

(e) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs_0)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \sup \left(\sum_{v=0}^{\infty} \left| \sum_{l=0}^{r} \theta_{lv} \right| \right) = 0$.

(f) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to cs)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \sup \left(\sum_{v=0}^{\infty} \left| \sum_{l=0}^{r} \theta_{lv} - \tilde{\theta} \right| \right) = 0$.

(g) Let $\Phi \in (\mathsf{e}_0^{\alpha,\beta}(q) \to bs)$*, then* C_Φ *is compact if and only if* $\lim_{r\to\infty} \sup \left(\sum_{v=0}^{\infty} \left| \sum_{l=0}^{r} \theta_{lv} \right| \right) = 0$.

6. Point spectrum of $E^{\alpha,\beta}(q)$ on c (set of convergent sequences)

In the present section, we compute the point spectrum of the q-Euler operator $\mathsf{E}^{\alpha,\beta}(q)$ on the space c of convergent sequences.

Let $\mathsf{S} \neq \{\theta\}$ be a complex normed space and Ψ be any linear operator that maps domain of Ψ to S. By Ψ^* and $B(\mathsf{S})$, we shall denote the adjoint of Ψ and the set of all bounded linear operators on S into itself, respectively. Denote $\Psi_\mu = \Psi - \mu I$, where $\mu \in \mathbb{C}$ and I is the identity operator on the domain of Ψ. Then the operator $\Psi_\mu^{-1} = (\Psi - \mu I)^{-1}$ is called the resolvent operator of Ψ, given that Ψ_μ is invertible. Define the set $\zeta_p(\Psi, \mathsf{S})$ by

$$\zeta_p(\Psi, \mathsf{S}) = \{\mu \in \mathbb{C} : \ \Psi_\mu^{-1} \text{ does not exist}\}.$$

Then the set $\zeta_p(\Psi, \mathsf{S})$ is called point spectrum of Ψ over the space S. Recently, Yıldırım [49] studied the fine spectrum of q-analogue $C(q)$ of Cesàro operator C of order 1 over the space c_0. For more details on spectrum and the fine spectrum of well known operators in literature, one may refer [48] and the references mentioned therein, in which the author has provided a detailed survey of spectrum of well known triangles.

Lemma 6.1. *The matrix* $\Phi = (\phi_{rv})$ *gives rise to a bounded linear operator* $\Psi \in B(c)$ *if and only if* $\lim_{r\to\infty} \phi_{rv} = \phi_v$ *for each* $v \in \mathbb{N}_0$ *and* $\sup_{r\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left|\phi_{rv}\right| < \infty$*. Further,* $\|\Psi\| = \sup_{r\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left|\phi_{rv}\right|$.

Theorem 6.2. $\mathsf{E}^{\alpha,\beta}(q) \in B(c)$ *and* $\left\|\mathsf{E}^{\alpha,\beta}(q)\right\|_{(c\to c)} = 1$.

Proof. We recall that the matrix $\mathsf{E}^{\alpha,\beta}(q)$ is conservative. That is $\lim_{r\to\infty} \mathsf{e}_{rv}^{\alpha,\beta}$ exists, for each $v \in \mathbb{N}_0$. Furthermore

$$\left\|\mathsf{E}^{\alpha,\beta}(q)\right\|_{c\to c} = \sup_{r\in\mathbb{N}} \sum_{v=0}^{\infty} |\mathsf{e}_{rv}^{\alpha,\beta}| = \sup_{r\in\mathbb{N}_0} \left(\sum_{v=0}^{r} \frac{q^{\binom{v}{2}} \left[{r \atop v}\right]_q \alpha^v \beta^{r-v}}{(\alpha+\beta)_q^r} \right) = \sup_{r\in\mathbb{N}} \left(\frac{(\alpha+\beta)_q^r}{(\alpha+\beta)_q^r} \right) = 1.$$

This completes the proof. □

Theorem 6.3. *Let* $0 < q < 1$. *Then* $\zeta_p(\mathsf{E}^{\alpha,\beta}(q), c) = \emptyset$.

Proof. On the contrary, we assume that $\zeta_p(\mathsf{E}^{\alpha,\beta}, c) \neq \emptyset$. Then there exists atleast one non-zero sequence $s = (s_v) \in c$ with $\mathsf{E}^{\alpha,\beta}(q)s = \mu s$. This gives us the following system of equations:

$$\begin{aligned}
s_0 &= \mu s_0 \\
\beta \frac{\begin{bmatrix}1\\0\end{bmatrix}_q}{(\alpha+\beta)_q} s_0 + \alpha \frac{\begin{bmatrix}1\\1\end{bmatrix}_q}{(\alpha+\beta)_q} s_1 &= \mu s_1 \\
\beta^2 \frac{\begin{bmatrix}2\\0\end{bmatrix}_q}{(\alpha+\beta)_q^2} s_0 + \alpha\beta \frac{\begin{bmatrix}2\\1\end{bmatrix}_q}{(\alpha+\beta)_q^2} s_1 + \alpha^2 \frac{q^{\binom{2}{2}}\begin{bmatrix}2\\2\end{bmatrix}_q}{(\alpha+\beta)_q^2} s_2 &= \mu s_2 \\
&\vdots \\
\beta^v \frac{\begin{bmatrix}v\\0\end{bmatrix}_q}{(\alpha+\beta)_q^v} s_0 + \alpha\beta^{v-1} \frac{\begin{bmatrix}v\\1\end{bmatrix}_q}{(\alpha+\beta)_q^v} s_1 + \ldots + \alpha^{v-1}\beta \frac{q^{\binom{v-1}{2}}\begin{bmatrix}v\\v-1\end{bmatrix}_q}{(\alpha+\beta)_q^v} s_{v-1} + \alpha^v \frac{q^{\binom{v}{2}}\begin{bmatrix}v\\v\end{bmatrix}_q}{(\alpha+\beta)_q^v} s_v &= \mu s_v \\
&\vdots
\end{aligned}$$

Let s_v be the first non-zero component of s, then we get $\mu = \alpha^v \frac{q^{\binom{v}{2}}}{(\alpha+\beta)_q^v}$. Taking this in account, the next terms $s_{v+1}, s_{v+2}, \ldots$ are obtained as

$$\begin{aligned}
s_{v+1} &= \begin{bmatrix}v+1\\v\end{bmatrix}_q s_v \\
s_{v+2} &= \begin{bmatrix}v+2\\v\end{bmatrix}_q s_v \\
s_{v+3} &= \begin{bmatrix}v+3\\v\end{bmatrix}_q s_v \\
&\vdots \\
s_r &= \begin{bmatrix}r\\v\end{bmatrix}_q s_v \\
s_{r+1} &= \begin{bmatrix}r+1\\v\end{bmatrix}_q s_v \\
&\vdots
\end{aligned}$$

Thus

$$\frac{s_{r+1}}{s_r} = \frac{(r+1)[q]}{(r+1-v)[q]} \geq 1.$$

Thus we realize that the sequence (s_v) is not a sequence in c, which is a contradiction to our assumption. This completes the proof. □

Acknowldgement

The research of the first author (T. Yaying) is supported by SERB, DST, New Delhi, India under the grant number EEQ/2019/000082.

Declaration

Conflict of Interest/Competing interests: The authors declare that the article is free from conflicts of interest.
Author's Contributions: All the authors have equal contribution for the preparation of the article.

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