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Positive solutions for integral boundary value problems of nonlinear fractional differential equations with delay

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Abstract. In this study, we consider integral boundary value problems of nonlinear fractional differential equations with finite delay. Existence results of positive solutions for the problems are obtained on the basis of the Guo-Krasnoselskii theorem and the Leggett-Williams fixed point theorem. Comprehensive examples follow the main results in the respective sections.

Keywords:

1. Introduction

A wide interest in fractional differential equations has emerged of late, following advances in fractional calculus theories resulting in its applications in engineering, mechanics, chemistry, physics among other fields [8]-[10], [18]-[32], [34]-[37], [40]- [42].

There are a fair number of approaches dedicated to the existence of positive solutions for fractional boundary value problems such as the Leggett-Williams theorem [11]-[12], the fixed point theorem on cones [13]-[14] and [17]. Using some of these approaches, various studies entail integral boundary value conditions, for example, [15]-[16], which have diverse applicability in the field of thermo-elasticity, population dynamics and so on.

Comprehensive details on integral boundary value conditions can be seen in [33] and references entailed therein. Although, Caputo and Riemann-Liouville derivatives are usually considered separately in many instances, hardly any work covers a fusion of the two fractional-order derivatives, the few includes [5]-[7].

Inasmuch as the previous literature included integral boundary value problems (IBVP), the combination of IBVP with mixed fractional-order derivatives for delayed fractional differential equations is scarce. Consequently, we address this scarcity by considering a finite delayed IBVP with mixed fractional order derivatives which presents a unique and rare contribution in this field of study. In this study, we consider

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the following IBVP

$$\begin{cases} D^{\beta}(\varphi_p({}^cD^{\alpha}y(t))) + f(t, y_t) = 0, & t \in [0,1], \\ y(t) = \phi(t), & t \in [-\tau, 0], \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds, & \\ \varphi_p({}^cD^{\alpha}y(0)) = [\varphi_p({}^cD^{\alpha}y(0))]' = 0, & \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $0 < k < 2$, ${}^cD^{\alpha}$ and D^{β} are the Caputo and Riemann-Liouville derivatives respectively, $f : [0,1] \times C_{\tau} \to [0, +\infty)$ is a continuous function such that $\tau \in \mathbb{R}^+$, $y_t(\theta) = y(t+\theta)$ for $t \in [0,1]$ and $\theta \in [-\tau, 0]$, $\phi \in C_{\tau}(:= C[-\tau, 0])$ and C_{τ} is a Banach space with $\|\phi\|_{[-\tau,0]} = \max_{\theta\in[-\tau,0]} |\phi(\theta)|$ that $C_{\tau}^{+} = \{y \in C[-\tau, 0] | y(t) \geq 0,\ t \in [-\tau, 0]\}$, $\varphi_p(y) = |y|^{p-2}y$ such that $p > 1$, $\varphi_p^{-1} = \varphi_q$ and $1/p + 1/q = 1$.

This paper is organized in such a manner, in section 2, we present some background material, definitions and lemmas. Section 3 deals with the existence of single and multiple positive solutions for the functional differential equation with fractional order and finite delay. Section 4 focuses on existence of multiple positive solutions for fractional differential equation finite delay.

2. Basic Definitions and Preliminaries

In this section, we introduce some necessary definitions and lemmas.

Definition 2.1. *[33] The integral*

$$I^{\beta}g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds, \tag{2.1}$$

where $\beta > 0$ and Γ is the Euler gamma function, is the fractional integral of order β for a function $g(t)$.

Definition 2.2. *[33] For a function $g(t)$ the expression*

$$D_{0^+}^{\beta}g(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} g(s)ds, \tag{2.2}$$

is called the Riemann-Liouville fractional derivative of order β, where $n = [\beta] + 1$, and $[\beta]$ denotes the integer part of number β.

Definition 2.3. *[33] The α order Caputo fractional derivatives for a function $f(t)$ is defined as follows:*

$${}^cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha < n. \tag{2.3}$$

Definition 2.4. *[2] Let $P \subseteq K$ be a nonempty, convex closed set and K a real Banach space. Then P is called a cone in K provided that*

1. *$\lambda y \in P$, for all $y \in P$ and $\lambda \geq 0$,*
2. *$y, -y \in P$ implies that $y = 0$.*

Definition 2.5. *[2] Let P be a cone in real Banach space K. If the map $\Upsilon : P \to [0, \infty)$ is continuous and satisfies*

$$\Upsilon(tx + (1-t)y) \geq t\Upsilon(x) + (1-t)\Upsilon(y), \quad x, y \in P,\ t \in [0,1],$$

then Υ is called a nonnegative continuous concave functional on P.
In a similar way, the map ν is a nonnegative continuous convex function on a cone P of a real Banach space K provided that $\nu : P \to [0, \infty)$ is continuous and

$$\nu(tx + (1-t)y) \leq t\nu(x) + (1-t)\nu(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Lemma 2.6. [1] *Assume that $g \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^\beta D^\beta g(t) = g(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_N t^{\beta-N}, \tag{2.4}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \cdots, N$, where N is the smallest integer greater than or equal to β.

Lemma 2.7. [2] *Assume that $\alpha > 0$ and $n = [\alpha] + 1$. If the function $y \in L[0,1] \cap C[0,1]$, then there exists $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n$, such that*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - c_1 - c_2 t \cdots - c_n t^{n-1}. \tag{2.5}$$

Lemma 2.8. *The IBVP* (1.1) *has a unique solution as follows:*

$$y(t) \;=\; \begin{cases} \int_0^1 G(t,s)\varphi_q(I^\beta f(s, y_s))ds, & t \in [0,1], \\ \phi(t), & t \in [-\tau, 0], \end{cases} \tag{2.6}$$

where

$$G(t,s) \;=\; \begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))-\alpha(2-k)(t-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}, & 0 \le s \le t \le 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Proof. Let $u(t) = \varphi_p({}^cD^\alpha y(t))$, we now show that IBVP (1.1) can be expressed as the following IBVPs:

$$\begin{cases} D^\beta u(t) + f(t, y_t) = 0, \\ u(0) = u'(0) = 0 \end{cases} \tag{2.8}$$

and

$$\begin{cases} {}^cD^\alpha y(t) = \varphi_q(u(t)), & t \in (0,1) \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds. \end{cases} \tag{2.9}$$

Using Lemma 2.6 and (2.8), we get

$$u(t) = -I^\beta f(t, y_t) + c_1 t^{\beta-1} + c_2 t^{\beta-2}.$$

Since $u(0) = u'(0) = 0$, then $c_1 = c_2 = 0$ and we have

$$\begin{aligned} u(t) &= -\, I^\beta f(t, y_t) \\ &= \frac{-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_s)ds. \end{aligned} \tag{2.10}$$

Also, from (2.9) and Lemma 2.7

$$y(t) = -I^\alpha \varphi_q(I^\beta f(t, y_t)) + c_0 + c_1 t + c_2 t^2.$$

Since $y(0) = y''(0) = 0$, then $c_0 = c_2 = 0$, so

$$y(t) = -I^\alpha \varphi_q(I^\beta f(t, y_t)) + c_1 t. \tag{2.11}$$

From the condition $y(1) = k\int_0^1 y(s)ds$ of (2.9), we get

$$y(1) = k\int_0^1 y(s)ds = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_q(I^\beta f(s, y_s))ds + c_1,$$

then

$$c_1 = k\int_0^1 y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds.$$

Substituting for c_1 into (2.11) implies that

$$y(t) = -I^\alpha\varphi_q(I^\beta f(t,y_t)) + kt\int_0^1 y(s)ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds. \tag{2.12}$$

Let $H = \int_0^1 y(t)dt$, then from (2.12), we have

$$\begin{aligned} H =& -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))dsdt + \int_0^1 ktHdt \\ &+ \int_0^1 t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))dsdt \\ =& -\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds + \frac{k}{2}H + \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds \end{aligned}$$

Thus we get

$$\begin{aligned} \frac{(2-k)}{2}H =& -\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds + \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds \\ H =& \frac{-2}{2-k}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds + \frac{1}{2-k}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we get

$$\begin{aligned} y(t) =& -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds - \frac{2kt}{2-k}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds \\ &+ \frac{kt}{2-k}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q(I^\beta f(s,y_s))ds \\ =& \int_0^t \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s)) - \alpha(t-s)^{\alpha-1}(2-k)}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds \\ &+ \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}\varphi_q(I^\beta f(s,y_s))ds \\ =& \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y_s))ds. \end{aligned}$$

This completes the proof. □

Lemma 2.9. [2] *The function $G(t,s)$ defined in (2.7) satisfies the following properties:*

1. $0 < G(t,s) \leq \frac{2}{(2-k)\Gamma(\alpha)}$, *for $t,s \in (0,1)$ if and only if* $0 < k < 2$.
2. $tG(1,s) \leq G(t,s) \leq \frac{2\alpha}{k(\alpha-2)}G(1,s)$, *for all $t,s \in (0,1)$, $2 < \alpha < 3$ and $0 < k < 2$.*

Lemma 2.10. [3] *Let K be a Banach space and let $X \subset K$ be a cone in K. Assume that Ω_1 and Ω_2 are open subsets of K with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : X \cap (\overline{\Omega}_2 \setminus \Omega_1) \to X$ be completely continuous operator. In addition, suppose that either*

1. $\|Ty\| \leq \|y\|$, *for all* $y \in X \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|$, *for all* $y \in X \cap \partial\Omega_2$ *or*
2. $\|Ty\| \leq \|y\|$, *for all* $y \in X \cap \partial\Omega_2$ *and* $\|Ty\| \geq \|y\|$, *for all* $y \in X \cap \partial\Omega_1$ *holds.*

Then T *has a fixed point in* $X \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.11. [2] *If* $\rho \in (0, \frac{1}{2})$ *is a fixed number, then for each* $y \in P$ *and* $s \in [\rho, 1-\rho]$ *there exists a constant* $\mu \in (0,1)$ *that satisfies*

$$\|y_s\|_{[-\tau,0]} \geq \mu\|y\|_{[0,1]}, \qquad \|y\|_{[0,1]} = \sup_{t\in[0,1]} |y(t)|,$$

where P *is defined in Lemma* 3.1.

3. Main results

We introduce some notations and hypotheses for convenience as follows:

$$f_\infty = \lim_{y\in C_\tau^+, \|y\|_{[-\tau,0]}\to+\infty} \frac{f(t,y)}{\varphi_p(\|y\|_{[-\tau,0]})}, \qquad f_0 = \lim_{y\in C_\tau^+, \|y\|_{[-\tau,0]}\to 0^+} \frac{f(t,y)}{\varphi_p(\|y\|_{[-\tau,0]})},$$

$$B_0 = \frac{\mu}{2(\Gamma(\beta+1))^{q-1}} \int_\rho^{1-\rho} s^{\beta(q-1)} G(1,s)ds, \qquad \mu \in (0,1), \quad \rho \in (0, \frac{1}{2}),$$

$$B_1 = \frac{2\alpha}{k(\alpha-2)(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s)ds;$$

(H_1) $\phi \geq 0$ on $[-\tau, 0]$;
(H_2) $f(t,y) \geq 0$ for $t \in [0,1]$ and $y \in C_\tau^+$;
(H_3) $f_0 = f_\infty = +\infty$;
(H_4) $f_0 = f_\infty = 0$;
(H_5) if there exists a constant $m_1 \geq \|\phi\|_{[-\tau,0]} > 0$, then

$$f(t,y) \leq \varphi_p\left(\frac{m_1}{B_1}\right), \quad \|y\|_{[-\tau,0]} \in [0, m_1], \ t \in [0,1],$$

(H_6) if there exists a constant $m_0 \geq \|\phi\|_{[-\tau,0]} > 0$, then

$$f(t,y) \geq \varphi_p\left(\frac{\mu m_0}{B_0}\right), \quad \|y\|_{[-\tau,0]} \in [\mu m_0, m_0], \ t \in [\rho, 1-\rho].$$

On $C[-\tau, 1]$, let we define an operator T

$$Ty(t) = \begin{cases} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y_s))ds, & t \in [0,1], \\ \phi(t), & t \in [-\tau, 0]. \end{cases}$$

Lemma 3.1. *Suppose that* $(H_1), (H_2)$ *hold and* P *is a cone in Banach space* $K = C[-\tau,1]$ *with norm* $\|y\|_{[-\tau,1]} = \max_{t\in[-\tau,1]} |y(t)|$ *as follows;*

$$P = \{y \in K \mid y \geq 0, y \text{ is concave down on } [0,1]\},$$

then the followings hold:

1. $T(P) \subseteq P$,
2. $T : P \to P$ *is completely continuous.*

Proof. Trivially, Part 1 holds and T is continuous. We proceed to prove the validity of Part 2. If J is a bounded subset in P, it implies that there exists $b > 0$ such that $\|y\| \leq b$ for all $y \in J$. We let

$$M_1 = \sup_{t\in[0,1], y\in[0,b]} |f(t, y_t)| + 1.$$

Then, for $y \in J$, we get

$$|Ty(t)| = \left| \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y_s))ds \right| \leq \frac{2M_1^{q-1}}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}.$$

Thus, $T(J)$ is uniformly bounded.

We let $y \in J$ and $t_1 < t_2,\ t_1, t_2 \in [-\tau, 1]$. If $0 \leq t_1 \leq t_2 \leq 1$, then

$$\begin{aligned}
|(Ty)'(t)| =& \left| \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} \varphi_q(I^\beta f(s,y_s))ds - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_q(I^\beta f(s,y_s))ds \right| \\
\leq& \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} |\varphi_q(I^\beta f(s,y_s))|ds - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |\varphi_q(I^\beta f(s,y_s))|ds \\
\leq& \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} \left| \varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, y_\tau)d\tau \right) \right| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| \varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, y_\tau)d\tau \right) \right| ds \\
\leq& \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} \varphi_q\left(\frac{M_1 s^\beta}{\Gamma(\beta+1)} \right) ds + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_q\left(\frac{M_1 s^\beta}{\Gamma(\beta+1)} \right) ds \\
\leq& \frac{2\alpha M_1^{q-1}}{(2-k)\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \int_0^1 (1-s)^{\alpha-1} ds + \frac{M_1^{q-1}}{\Gamma(\alpha-1)(\Gamma(\beta+1))^{q-1}} \int_0^t (t-s)^{\alpha-2} ds \\
\leq& \left[\frac{4-k}{(2-k)\Gamma(\alpha)} \right] \frac{M_1^{q-1}}{(\Gamma(\beta+1))^{q-1}} := M_0.
\end{aligned}$$

Therefore,

$$|Ty(t_2) - Ty(t_1)| \leq \int_{t_1}^{t_2} |(Ty)'(s)| ds \leq M_0(t_2 - t_1).$$

Suppose $-\tau \leq t_1 < 0 < t_2 \leq 1$, then

$$\begin{aligned}
|Ty(t_2) - Ty(t_1)| =& |Ty(t_2) - Ty(0)| + |Ty(0) - Ty(t_1)| \\
\leq& \int_0^1 |G(t_2,s) - G(0,s)||\varphi_q(I^\beta f(s,y_s))|ds + |\phi(0) - \phi(t_1)| \\
\leq& \frac{2M_1^{q-1}}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} t_2 + |\phi(0) - \phi(t_1)| \\
\leq& \frac{2M_1^{q-1}}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} |t_2 - t_1| + |\phi(0) - \phi(t_1)|.
\end{aligned}$$

Thus, $T(J)$ is equicontinuous. We can conclude that $T(J)$ is relatively compact from the Ascoli-Arzela theorem. $\square$

Theorem 3.2. *Suppose that* $(H_1), (H_2), (H_3)$ *and* (H_5) *hold, then the IBVP* (1.1) *has at least two positive solutions* y_1 *and* y_2 *with*

$$0 \leq \|y_1\|_{[-\tau,1]} < m_1 < \|y_2\|_{[-\tau,1]}.$$

Proof. Suppose (H_5) holds, we let $\Omega_{m_1} = \{y \in P : \|y\|_{[-\tau,1]} < m_1\}$, for any $y \in P \cap \partial\Omega_{m_1}$, we get

$$\begin{aligned} Ty(t) &= \begin{cases} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y_s))ds, & t \in [0,1], \\ \phi(t), & t \in [-\tau,0]. \end{cases} \\ &\leq \begin{cases} \frac{2\alpha m_1}{B_1 k(\alpha-2)(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(t,s)ds, & t \in [0,1], \\ \|\phi\|_{[-\tau,0]}, & t \in [-\tau,0], \end{cases} \\ &\leq \begin{cases} m_1, & t \in [0,1], \\ \|\phi\|_{[-\tau,0]}, & t \in [-\tau,0], \end{cases} \\ &\leq \|y\|_{[-\tau,1]}, \end{aligned}$$

which gives

$$\|Ty\|_{[-\tau,1]} \leq \|y\|_{[-\tau,1]}, \text{ for } y \in P \cap \partial\Omega_{m_1}.$$

Suppose that (H_3) holds. Since $f_0 = \infty$, we choose $\|\phi\|_{[-\tau,0]} < n_1 < m_1$, such that $f(t,y) \geq \varphi_p(A_0\|y\|_{[-\tau,0]})$ for $0 \leq \|y\|_{[-\tau,0]} \leq n_1$, where $A_0 > 0$ satisfies $A_0 B_0 \geq 1$. Let $\Omega_{n_1} = \{y \in P : \|y\|_{[-\tau,1]} < n_1\}$, for any $y \in P \cap \partial\Omega_{n_1}$, we get

$$\begin{aligned} (Ty)\left(\frac{1}{2}\right) &\geq \int_\rho^{1-\rho} G\left(\frac{1}{2},s\right)\varphi_q(I^\beta f(s,y_s))ds \\ &\geq \frac{A_0}{(\Gamma(\beta+1))^{q-1}} \int_\rho^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2},s\right)\varphi_q(\varphi_p(\|y_s\|_{[-\tau,0]}))ds \\ &\geq \frac{A_0\mu}{2(\Gamma(\beta+1))^{q-1}} \int_\rho^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2},s\right)\|y\|_{[0,1]}ds \\ &= \frac{A_0\mu}{2(\Gamma(\beta+1))^{q-1}} \int_\rho^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2},s\right)\|y\|_{[-\tau,1]}ds \\ &\geq \|y\|_{[-\tau,1]}, \end{aligned}$$

which gives

$$\|Ty\|_{[-\tau,1]} \geq \|y\|_{[-\tau,1]}, \text{ for } y \in P \cap \partial\Omega_{n_1}.$$

Also, since $f_\infty = \infty$, we choose $n_2 > m_1 > \|\phi\|_{[-\tau,0]}$, such that $f(t,y) \geq \varphi_p(A_1\|y\|_{[-\tau,0]})$ for $\|y\|_{[-\tau,0]} \geq \mu n_2$, where $A_1 > 0$ satisfies $A_1 B_0 \geq 1$. We let $\Omega_{n_2} = \{y \in P : \|y\|_{[-\tau,1]} < n_2\}$, for any $y \in P \cap \partial\Omega_{n_2}$, we get

$$\begin{aligned} (Ty)\left(\frac{1}{2}\right) &\geq \int_\rho^{1-\rho} G\left(\frac{1}{2},s\right)\varphi_q(I^\beta f(s,y_s))ds \\ &\geq \frac{A_1}{(\Gamma(\beta+1))^{q-1}} \int_\rho^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2},s\right)\varphi_q(\varphi_p(\|y_s\|_{[-\tau,0]}))ds \end{aligned}$$

$$\geq \frac{A_1\mu}{2(\Gamma(\beta+1))^{q-1}} \int_{\rho}^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2}, s\right) ||y||_{[0,1]} ds$$
$$= \frac{A_1\mu}{2(\Gamma(\beta+1))^{q-1}} \int_{\rho}^{1-\rho} s^{\beta(q-1)} G\left(\frac{1}{2}, s\right) ||y||_{[-\tau,1]} ds$$
$$\geq ||y||_{[-\tau,1]},$$

which gives

$$||Ty||_{[-\tau,1]} \geq ||y||_{[-\tau,1]}, \text{ for } y \in P \cap \partial\Omega_{n_2}.$$

By Parts 1 and 2 of Lemma 3.1, the conclusion is proved. □

Theorem 3.3. *If* $(H_1), (H_2), (H_4)$ *and* (H_6) *are satisfied, then the IBVP* (1.1) *has at least two positive solutions* y_1 *and* y_2 *with*

$$0 \leq ||y_1||_{[-\tau,1]} < m_0 < ||y_2||_{[-\tau,1]}.$$

Proof. Suppose (H_6) holds, we let $\Omega_{m_0} = \{y \in P : ||y||_{[-\tau,1]} < m_0\}$, for any $y \in P \cap \partial\Omega_{m_0}$, we get

$$(Ty)\left(\frac{1}{2}\right) \geq \int_{\rho}^{1-\rho} G\left(\frac{1}{2}, s\right) \varphi_p(I^\beta f(s, y_s)) ds \geq \frac{1}{2} \int_{\rho}^{1-\rho} G(1, s) \varphi_p(I^\beta f(s, y_s)) ds$$
$$\geq \frac{m_0\mu}{2B_0(\Gamma(\beta+1))^{q-1}} \int_{\rho}^{1-\rho} s^{\beta(q-1)} G(1, s) ds$$
$$= m_0 = ||y||_{[-\tau,1]},$$

which gives

$$||Ty||_{[-\tau,1]} \geq ||y||_{[-\tau,1]} \text{ for } y \in P \cap \partial\Omega_{m_0}.$$

If (H_4) holds and also since $f_0 = 0$, we choose $||\phi||_{[-\tau,0]} < m_1 < m_0$, such that $f(t, y) \leq \varphi_p(N||y||_{[-\tau,0]})$, for $0 \leq ||y||_{[-\tau,0]} \leq m_1$, where $N > 0$ satisfies $NB_1 \leq 1$. We let $\Omega_{m_1} = \{y \in P : ||y||_{[-\tau,1]} < m_1\}$, for any $y \in P \cap \partial\Omega_{n_2}$, we get

$$Ty(t) \leq \begin{cases} \int_0^1 G(t, s)\varphi_q(I^\beta f(s, y_s)) ds, & t \in [0, 1], \\ \phi(t), & t \in [-\tau, 0]. \end{cases}$$
$$\leq \begin{cases} \frac{2\alpha N}{k(\alpha-2)(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(t, s) ||y_s||_{[-\tau,0]} ds, & t \in [0, 1], \\ ||\phi||_{[-\tau,0]}, & t \in [-\tau, 0], \end{cases}$$
$$\leq ||y||_{[-\tau,1]},$$

which gives

$$||Ty||_{[-\tau,1]} \leq ||y||_{[-\tau,1]}, \text{ for } y \in P \cap \partial\Omega_{m_1}.$$

Furthermore, since $f_\infty = 0$, there exists $Q > m_o$, such that $f(t, y) \leq X||y||_{[-\tau,0]}$, for $||y||_{[-\tau,0]} > Q$, where $X > 0$ satisfies $(X + 1)B_1 \leq 1$. We choose a constant $m_2 > 0$, such that $m_2 > \max\{m_0, ||\phi||_{[-\tau,0]}, \max\{f(s, y_s) \mid 0 \leq \varphi_p(||y_s||_{[-\tau,0]}) \leq Q\}X\}$.
We let $\Omega_{m_2} = \{y \in P : ||y||_{[-\tau,1]} < m_2\}$, for any $y \in P \cap \partial\Omega_{m_2}$, we get

$$Ty(t) \leq \begin{cases} \int_{||y_s||_{[-\tau,0]} > Q} \frac{2\alpha Q}{k(\alpha-2)} G(1, s)\varphi_q(I^\beta f(s, y_s)) ds \\ + \int_{0 \leq ||y_s||_{[-\tau,0]} \leq Q} \frac{2\alpha Q}{k(\alpha-2)} G(1, s)\varphi_q(I^\beta f(s, y_s)) ds, \quad t \in [0, 1], \\ \phi(t), \quad t \in [-\tau, 0]. \end{cases}$$

$$
\leq \begin{cases} \{Xm_2 + \max\{f(s,y(s)) \mid 0 \leq \varphi_p(\|y_s\|_{[-\tau,0]}) \leq Q\}\}B_1, & t \in [0,1], \\ \|\phi\|_{[-\tau,0]}, & t \in [-\tau,0], \end{cases}
$$
$$
\leq \begin{cases} m_2, & t \in [0,1], \\ \|\phi\|_{[-\tau,0]}, & t \in [-\tau,0], \end{cases}
$$
$$
\leq m_2 = \|y\|_{[-\tau,1]},
$$

which gives

$$
\|Ty\|_{[-\tau,1]} \leq \|y\|_{[-\tau,1]}, \text{ for } y \in P \cap \partial\Omega_{m_2}.
$$

Thus, by Parts 1 and 2 of Lemma 2.10, the conclusion has been proved. □

Similarly, from the proofs of Theorem 3.2 and Theorem 3.3, we get Theorem 3.4 and Theorem 3.5.

Theorem 3.4. *If* $(H_1), (H_2)$ *are satisfied and the conditions* $f_0 = \infty, f_\infty = 0$ *hold, then IBVP* (1.1) *has at least one positive solution.*

Theorem 3.5. *If* $(H_1), (H_2)$ *are satisfied and the conditions* $f_0 = 0, f_\infty = \infty$ *hold, then IBVP* (1.1) *has at least one positive solution.*

Example 3.1. *Consider the functional differential equation:*

$$
\begin{cases} D^{\frac{9}{5}}(\varphi_2({}^cD^{\frac{7}{3}}y(t))) = -(y^{\frac{1}{4}}(t-\frac{1}{5}) + y^{\frac{1}{3}}(t-\frac{1}{5})), & t \in [0,1], \\ y(t) = t^8, & t \in [-\frac{1}{5}, 0], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{3}{4}\int_0^1 y(s)ds, \\ \varphi_2({}^cD^{\frac{7}{3}}y(0)) = [\varphi_2({}^cD^{\frac{7}{3}}y(0))]' = 0, \end{cases} \tag{3.1}
$$

where $\alpha = \frac{7}{3}$, $\beta = \frac{9}{5}$, $\tau = \frac{1}{5}$, $p = 2$, $k = \frac{3}{4}$, $f(t,y) = y^{\frac{1}{4}}(-\frac{1}{5}) + y^{\frac{1}{3}}(-\frac{1}{5})$ *thus*

$$
\frac{f(t,y)}{\varphi_p(\|y\|_{[-\tau,0]})} = \frac{y^{\frac{1}{4}}(-\frac{1}{5}) + y^{\frac{1}{3}}(-\frac{1}{5})}{\varphi_2(\|y\|_{[-\frac{1}{5},0]})} \leq \frac{\|y\|^{\frac{1}{4}}_{[-\frac{1}{5},0]} + \|y\|^{\frac{1}{3}}_{[-\frac{1}{5},0]}}{\|y\|^{2-1}_{[-\frac{1}{5},0]}} = \|y\|^{-\frac{3}{4}}_{[-\frac{1}{5},0]} + \|y\|^{-\frac{2}{3}}_{[-\frac{1}{5},0]} \to 0,
$$

as $\|y\|_{[-\frac{1}{5},0]} \to +\infty$, *we get* $f_\infty = 0$.
Furthermore, there exists a constant $a > 0$ *such that* $y(t) \geq a\|y\|_{[-\tau,0]}$,

$$
\frac{f(t,y)}{\varphi_p(\|y\|_{[-\tau,0]})} \geq a\left[\|y\|^{-\frac{3}{4}}_{[-\frac{1}{5},0]} + \|y\|^{-\frac{2}{3}}_{[-\frac{1}{5},0]}\right] \to +\infty, \text{ as } \|y\|_{[-\frac{1}{5},0]} \to 0.
$$

Therefore, $f_0 = +\infty$, *which implies that the problem (3.1) has at least one positive solution by Theorem 3.4.*

Example 3.2. *Consider the functional differential equation:*

$$
\begin{cases} D^{\frac{3}{2}}(\varphi_{\frac{4}{3}}({}^cD^{\frac{5}{2}}y(t))) = -(y^{\frac{1}{3}}(t-\frac{1}{4}) + y^{\frac{1}{5}}(t-\frac{1}{4})), & t \in [0,1], \\ y(t) = t^8, & t \in [-\frac{1}{4}, 0], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2}\int_0^1 y(s)ds, \\ \varphi_{\frac{4}{3}}({}^cD^{\frac{5}{2}}y(0)) = [\varphi_{\frac{4}{3}}({}^cD^{\frac{5}{2}}y(0))]' = 0, \end{cases} \tag{3.2}
$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\tau = \frac{1}{4}$, $p = \frac{4}{3}$, $k = \frac{1}{2}$, $f(t,y) = y^{\frac{1}{3}}(-\frac{1}{4}) + y^{\frac{1}{5}}(-\frac{1}{4})$, *and there exists a constant a such that* $y(t) = a\|y\|_{[-\tau,0]}$ *thus*

$$
\frac{f(t,y)}{\varphi_p(\|y\|_{[-\tau,0]})} = \frac{y^{\frac{1}{3}}(-\frac{1}{4}) + y^{\frac{1}{5}}(-\frac{1}{4})}{\varphi_{\frac{4}{3}}(\|y\|_{[-\frac{1}{4},0]})} \geq a\frac{\|y\|^{\frac{1}{3}}_{[-\frac{1}{4},0]} + \|y\|^{\frac{1}{5}}_{[-\frac{1}{4},0]}}{\|y\|^{\frac{4}{3}-1}_{[-\frac{1}{4},0]}} = a\left[\|y\|_{[-\frac{1}{4},0]} + \|y\|^{-\frac{2}{15}}_{[-\frac{1}{4},0]}\right] \to +\infty,
$$

as $||y||_{[-\frac{1}{4},0]} \to +\infty$*, also*

$$\frac{f(t,y)}{\varphi_p(||y||_{[-\tau,0]})} \geq a\frac{||y||^{\frac{1}{3}}_{[-\frac{1}{4},0]} + ||y||^{\frac{1}{5}}_{[-\frac{1}{4},0]}}{||y||^{\frac{4}{3}-1}_{[-\frac{1}{4},0]}} = a\left[||y||_{[-\frac{1}{4},0]} + ||y||^{-\frac{2}{15}}_{[-\frac{1}{4},0]}\right] \to +\infty, \text{ as } ||y||_{[-\frac{1}{4},0]} \to 0,$$

we get $f_0 = +\infty$ *and* $f_\infty = +\infty$*. Therefore,* (H_3) *holds. In addition,*

$$\begin{aligned} B_1 =& \frac{2\alpha}{k(\alpha-2)(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)}G(1,s)ds \\ =& \frac{2\alpha}{k(\alpha-2)(\Gamma(\beta+1))^{q-1}}\int_0^1 \frac{s^{\beta(q-1)}(1-s)^{\alpha-1}(k[\alpha-2(1-s)])}{(2-k)\Gamma(\alpha+1)}ds \\ =& \frac{5}{72\pi}. \end{aligned}$$

We set m_1=2*, then if* $0 \leq ||y||_{[-\frac{1}{4},0]} \leq 2$ *we get,* $f(t,y) \leq 2^{\frac{1}{3}} + 2^{\frac{1}{5}} \leq \left(\frac{144}{5}\pi\right)^{p-1}$*, which means that condition* (H_5) *holds. Thus, by Theorem 3.2, the problem (3.2) has at least two possible solutions* y_1 *and* y_2 *with* $0 < ||y_1||_{[-\frac{1}{4},1]} < 2 < ||y_2||_{[-\frac{1}{4},1]}$.

4. Multiplicity result for BVP (1.1)

Suppose $f : \mathcal{J} \times \mathcal{B} \to [0, +\infty)$ is a specified function which satisfies certain assumptions to be stated in the next subsections and $\phi \in \mathcal{B}$ where $\mathcal{B}$ is a *phasespace*. For a function y and any $t \in [0,1]$, y_t denotes the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\tau, 0]$, we assume that y_t are the histories belonging to $\mathcal{B}$. We let $P \subset K$ be a cone in K and $(K, ||.||)$ be a Banach space. We show a continuous mapping

$$\psi : P \to [0, \infty)$$

by a concave, positive and continuous functional ψ on P with
$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y)$ for all $x, y \in P$ and $\lambda \in [0,1]$. For $\mathcal{K}, \mathcal{L}, r \geq 0$ constants with P and ψ as shown above, we let

$$P_{\mathcal{K}} = \{y \in P : ||y|| < \mathcal{K}\}$$

and

$$P(\psi, \mathcal{L}, \mathcal{K}) = \{y \in P : \psi(y) \geq \mathcal{L} \text{ and } ||y|| \leq \mathcal{K}\}.$$

The presented work is based on the fixed point theorem as presented by Leggett and Williams [39], see also [38].

Theorem 4.1. *Let* $P \subset K$ *be a cone in K, which is a Banach space and* $\mathcal{R} > 0$ *a constant. Suppose there exists a concave positive continuous functional on P with* $\psi(y) \leq ||y||$ *for* $y \in \overline{P_{\mathcal{R}}}$ *and let* $\mathcal{N} : \overline{P_{\mathcal{R}}} \to \overline{P_{\mathcal{R}}}$ *be a continuous compact map. Assume that there are numbers* $r, \mathcal{L}$ *and* $\mathcal{K}$ *with* $0 < r < \mathcal{L} < \mathcal{K} \leq \mathcal{R}$*:*

(A_1) $\{y \in P(\psi, \mathcal{L}, \mathcal{K}) : \psi(y) > \mathcal{L}, ||y|| \leq \mathcal{K}\} \neq \emptyset$ *and* $\psi(\mathcal{N}(y)) > \mathcal{L}$ *for* $y \in P(\psi, \mathcal{L}, \mathcal{K})$*;*

(A_2) $||\mathcal{N}(y)|| < r$ *for* $y \in \overline{P}_r$*;*

(A_3) $\psi(\mathcal{N}(y)) > \mathcal{L}$ *for* $y \in P(\psi, \mathcal{L}, \mathcal{K})$ *with* $||\mathcal{N}(y)|| > \mathcal{K}$.

Then $\mathcal{N}$ *has at least three fixed point* y_1, y_2, y_3 *in* $\overline{P}_{\mathcal{R}}$*. Also we get*

$$y_1 \in P_r, \ y_2 \in \{y \in P(\psi, \mathcal{L}, \mathcal{R}) : \psi(y) > \mathcal{L}\}$$

and

$$y_3 \in \overline{P}_{\mathcal{R}} - \{P(\psi, \mathcal{L}, \mathcal{R}) \cup \overline{P}_r\}.$$

A solution to IBVP (1.1) is obtained by setting

$$\mathcal{B}_1 = \{y : (-\tau, 1] \to R : y|_{(-\tau,0]} \in \mathcal{B}, y|_{\mathcal{J}} \in C^2(\mathcal{J}, R)\},$$

and let $\|.\|_1$ the semi norm in $\mathcal{B}_1$ defined by:

$$\|y\|_1 = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)| : 0 \leq t \leq 1\}, y \in \mathcal{B}_1.$$

Definition 4.2. *IBVP (1.1) has a solution y, which is a function $y \in \mathcal{B}_1$ that satisfies the equation $D^\beta(\varphi_p({}^cD^\alpha y(t))) = f(s, y_s)$ on $\mathcal{J}$ and conditions $y(0) = 0,\ y(0) = y''(0) = 0$, $y(1) = k\int_0^1 y(s)ds,\ \varphi_p({}^cD^\alpha y(0)) = [\varphi_p({}^cD^\alpha y(0))]' = 0$ and $y(t) = \phi(t),\ t \in (-\tau, 0]$.*

We denote the Banach space of all continuous functions from $\mathcal{J}$ into R by $C(\mathcal{J}, R)$, with the norm:

$$\|y\|_\infty := \sup\{|y(t)| : t \in \mathcal{J}\}.$$

Now, we present axioms for definition of the phase space $\mathcal{B}$.

(C_1) For every $t \in [0, 1]$, if $y : (-\tau, 1) \to R,\ y_0 \in \mathcal{B}$, then the following conditions hold:
  (a) $y_t \in \mathcal{B}$,
  (b) There exists a positive constant $\mathcal{H} : |y(t)| \leq \mathcal{H}\|y_t\|_{\mathcal{B}}$;
  (c) There exist two functions $\mathcal{K}(.), \mathcal{M}(.) : R_+ \to R_+$, independent of y, with $\mathcal{K}$ continuous and $\mathcal{M}$ locally bounded:
  $$\|y_t\|_{\mathcal{B}} \leq \mathcal{K}(t)\sup\{|y(s)| : 0 \leq s \leq t\} + \mathcal{M}(t)\|y_0\|_{\mathcal{B}}.$$
(C_2) y_t is a $\mathcal{B}$-valued continuous function on $[0, 1]$ for the function $y(.)$ in (A_1).
(C_3) The space $\mathcal{B}$ is complete. Denoted by

$$\mathcal{K} = \sup\{\mathcal{K}(t) : t \in [0, 1]\} \text{ and } \mathcal{M} = \sup\{\mathcal{M}(t) : t \in [0, 1]\}.$$

The following assumptions are necessary for the underlying theorem:

(N_1) f is a continuous function.
(N_2) There exists a function $q^* : [0, \infty) \to [0, \infty)$ which is continuous and non-decreasing and a function $h^* : [0, \infty) \to [0, \infty)$ which is continuous and non-increasing, $p_1 \in C(\mathcal{J}, R_+)$ and $p_2 \in C(\mathcal{J}, R_+)$ such that

$$p_2(t)h^*(\|u\|) \leq f(t, u) \leq p_1(t)q^*(\|u\|),$$

for each $(t, u) \in \mathcal{J} \times \mathcal{B}$.
(N_3) There exists a constant $r > 0$ such that

$$\frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\left[q^*(\mathcal{K}r + \mathcal{M}\|\phi\|_{\mathcal{B}})\|p_1\|_\infty\right]^{q-1} \leq r.$$

(N_4) There exists a constant $\mathcal{L} > r$ such that

$$\left[h^*(\mathcal{K}\mathcal{L} + \mathcal{M}\|\phi\|_{\mathcal{B}})\ \|p_2\|_\infty\right]^{q-1} \times \frac{1}{4(\Gamma(\beta+1))^{q-1}}\int_{s\in I} s^{\beta(q-1)}G(1, s)ds \geq \mathcal{L}.$$

(N_5) There exists a constant $\mathcal{R}$ such that $0 < r < \mathcal{L} \leq \sigma\mathcal{R}$, where $\sigma = \frac{k(\alpha-2)}{8\alpha}$ and

$$\frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\left[q^*(\mathcal{K}\mathcal{R} + \mathcal{M}\|\phi\|_B)\|p_1\|_\infty\right]^{q-1} \leq \mathcal{R}.$$

Theorem 4.3. *If* (N_1) − (N_5) *are satisfied. IBVP* (1.1) *has at least three positive solutions.*

Proof. Relying on Leggett-William fixed point theorem, we transform IBVP (1.1) into a fixed point problem. Considering the operator

$$T : \mathcal{B}_1 \to \mathcal{B}_1$$

defined as the following:

$$T(y)(t) = \begin{cases} \phi(t), & t \in (-\tau, 0], \\ \int_0^1 G(t,s)\varphi_q(I^\beta f(s, y_s))ds, & t \in [0,1]. \end{cases}$$

$G(t,s)$ is defined in (2.7). Obviously, the fixed points of the operator T are solutions of problem (1.1). We define $x(.) : (-\tau, 1] \to R$ be the function defined as:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\tau, 0], \\ 0, & \text{if } t \in [0,1]. \end{cases}$$

Then, $x_0 = \phi$. For each $z \in \mathcal{B}$ with $z_0 = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \in (-\tau, 0], \\ z(t), & \text{if } t \in [0,1]. \end{cases}$$

Let $y(.)$ satisfy the integral equation

$$y(t) = \int_0^1 G(t,s)\varphi_q(I^\beta f(s, y_s))ds.$$

We partition $y(.)$ into $y(t) = \bar{z}(t) + x(t)$, $0 \le t \le 1$, which makes $y_t = \bar{z}_t + x_t$, for every $t \in [0,1]$, and the function z(.) satisfies

$$z(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds.$$

Let $\mathcal{B}_0 = \{z \in C([0,1], R) : z_0 = 0\}$ and $\|.\|_1$ be the seminorm in $\mathcal{B}_0$ defined by

$$\|z\|_1 = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \le s \le 1\} = \|z\|_0.$$

$\mathcal{B}_0$ is a Banach space with the norm $\|.\|_0$. We let the operator $\mathcal{N} : \mathcal{B}_0 \to \mathcal{B}_0$ be defined by

$$\mathcal{N}(z)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds. \tag{4.1}$$

It is easily seen that the operator T has a fixed point that is equivalent to the one $\mathcal{N}$ has, so we must prove that $\mathcal{N}$ has a fixed point. From Lemma 3.1, $\mathcal{N}$ is completely continuous.
Let

$$\varrho = \{z \in \mathcal{B}_0 : z(t) \ge 0 \quad \min_{t \in I} z(t) \ge \frac{\sigma}{3}\|z\|_0 \text{ for } t \in \mathcal{J}\}$$

be a cone in $\mathcal{B}_0$. We show that $\mathcal{N} : \varrho \to \varrho$ is well defined. Let $z \in \varrho$, then it follows from Lemma 2.9 and (4.1) that

$$\begin{aligned} \|\mathcal{N}(z)\|_0 \le & \frac{2\alpha}{k(\alpha-2)} \int_0^1 G(1,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds \\ \le & 3\left[\frac{2\alpha}{k(\alpha-2)} \int_{s\in I} G(1,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds\right]. \end{aligned}$$

Also, considering Lemma 2.9, this means that for any $t \in I$

$$\begin{aligned}(\mathcal{N}z)(t) =& \int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\\ \geq& \frac{1}{4}\int_0^1 G(1,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\\ \geq& \sigma\left[\frac{2\alpha}{k(\alpha-2)}\int_{s\in I} G(1,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\right]\\ \geq& \frac{\sigma}{3}\|\mathcal{N}z\|_0.\end{aligned}$$

This implies that $\mathcal{N} : \varrho \to \varrho$ is well defined. Using the assumptions $(N_1) - (N_2)$ and (N_5) $\mathcal{N} : \overline{P}_{\mathcal{R}} \to \overline{P}_{\mathcal{R}}$ is well defined and completely continuous. Let $\psi : \varrho \to [0,\infty)$ is defined by

$$\psi(z) = \min_{t\in[\vartheta,\delta]} z(t).$$

It is evident that ψ is a non-negative concave continuous functional and

$$\psi(z) \leq \|z\|_0 \text{ for } z \in \overline{P}_{\mathcal{R}}.$$

We are left to show that the hypotheses of Theorem 4.1 to be stated are satisfied.
We note that condition (A_2) of Theorem 4.1 is valid for $z \in \overline{P}_r$, and from (N_2) and (N_3), we get

$$\begin{aligned}\|\mathcal{N}(z)\| =& \max_{0\leq t\leq 1}|\mathcal{N}(z)(t)|\\ \leq& \max_{0\leq t\leq 1}\left\{\int_0^1 |G(t,s)|\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} q^*(\|\overline{z}_\tau + x_\tau\|)|p_1(\tau)|d\tau\right)ds\right\}\\ \leq& \max_{0\leq t\leq 1}\left\{\int_0^1 |G(t,s)|\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} q^*(\mathcal{K}\|z\|_0 + \mathcal{M}\|\phi\|_{\mathcal{B}})p_1(\tau)d\tau\right)ds\right\}\\ \leq& \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\left[q^*(\mathcal{K}r + \mathcal{M} + \|\phi\|_{\mathcal{B}})\|p_1\|_\infty\right]^{q-1}\\ \leq& r,\end{aligned}$$

where

$$\begin{aligned}\|\overline{z}_\tau + x_\tau\|_{\mathcal{B}} \leq& \|\overline{z}_\tau\|_{\mathcal{B}} + \|x_\tau\|_{\mathcal{B}}\\ \leq& \mathcal{K}(s)\sup\{|z(\tau)| : 0 \leq \tau \leq s\} + \mathcal{M}(s)\|z_0\|_{\mathcal{B}}\\ &+ \mathcal{K}(s)\sup\{|x(\tau)| : 0 \leq \tau \leq s\} + \mathcal{M}(s)\|x_0\|_{\mathcal{B}}\\ \leq& \mathcal{K}\sup\{|z(\tau)| : 0 \leq \tau \leq s\} + \mathcal{M}\|\phi\|_{\mathcal{B}}\\ \leq& \mathcal{K}r + \mathcal{M}\|\phi\|_{\mathcal{B}}.\end{aligned} \tag{4.2}$$

We now proceed to show that condition (A_1) of Theorem 4.1 is satisfied.
Evidently, if $z \in P(\psi, \mathcal{L}, \frac{\mathcal{L}}{\sigma})$ then $\mathcal{L} \leq z(s) \leq \frac{\mathcal{L}}{\sigma}$, $s \in I$, and then $\{z \in P(\psi, \mathcal{L}, \frac{\mathcal{L}}{\sigma}), \psi(z) > \mathcal{L}\} \neq \emptyset$. By condition (N_4) we have

$$\begin{aligned}\psi(\mathcal{N}(z)) =& \min_{0\leq t\leq 1}\left\{\int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau,\overline{z}_\tau + x_\tau)d\tau\right)ds\right\}\\ \geq& \min_{t\in I}\left\{\int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h^*(\|\overline{z}_\tau + x_\tau\|)p_2(\tau)d\tau\right)ds\right\}\end{aligned}$$

$$\geq \left[h^*(\mathcal{K}\mathcal{L} + \mathcal{M}\|\phi\|_{\mathcal{B}})\,\|p_2\|_\infty\right]^{q-1} \times \frac{1}{4(\Gamma(\beta+1))^{q-1}} \int_{s\in I} s^{\beta(q-1)}G(1,s)ds.$$

Thus, condition (A_1) of Theorem 4.1 is satisfied.

We also show that condition (A_3) of Theorem 4.1 is satisfied. If $z \in P(\psi, \mathcal{L}, \mathcal{R})$ and $\|\mathcal{N}z\| > \frac{\mathcal{L}}{\sigma}$, we get

$$\begin{aligned}\psi(\mathcal{N}(z)) &= \min_{t\in I}\left\{\int_0^1 G(t,s)\varphi_q\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds\right\}\\ &\geq \sigma\|\mathcal{N}z\|\\ &\geq \mathcal{L}.\end{aligned}$$

Thus, condition (A_3) holds. By the Leggett and William fixed point theorem this implies that $\mathcal{N}$ has at least three fixed points z_1, z_2, z_3 such that

$$z_1 \in P_r,\ z_2 \in \{z \in P(\psi, \mathcal{L}, \mathcal{R}) : \psi(z) > \mathcal{L}\},\ z_3 \in P_{\mathcal{R}} - \{(\psi, \mathcal{L}, \mathcal{R}) \cup P_r\}.$$

Once more, condition (A_3) of Theorem 4.1 is satisfied. By Theorem 4.1, there exist three positive solutions z_1, z_2, z_3 such that $\|z_1\| < r$, $\mathcal{L} < \alpha(z_2(t))$, and $\|z_3\| > r$, with $\alpha(z_3(t)) < \mathcal{L}$.

Finally, IBVP (1.1) has three positive solutions y_1, y_2, y_3 such that

$$y_i(t) = \begin{cases}\phi(t), & if\ t \in (-\tau, 0],\\ z_i(t), & if\ \ t\in[0,1],\end{cases} \qquad \text{for} \quad i \in \{1,2,3\}.$$

□

Example 4.1. *Consider the functional differential equation:*

$$\begin{cases} D^{\frac{9}{5}}(\varphi_2({}^cD^{\frac{7}{3}}y(t))) = \frac{10\|y_t\|e^{t-\|y_t\|}}{(1+t^2)}, & t \in \mathcal{J} = [0,1],\\ y(t) = \phi(t), & t\in(-\tau, 0],\\ y(0) = y''(0) = 0, \quad y(1) = \frac{3}{4}\int_0^1 y(s)ds, & \\ \varphi_2({}^cD^{\frac{7}{3}}y(0)) = [\varphi_2({}^cD^{\frac{7}{3}}y(0))]' = 0, & \end{cases} \tag{4.3}$$

where $\alpha = \frac{7}{3},\ \beta = \frac{9}{5},\ \tau = \frac{1}{5},\ p = 2,\ k = \frac{3}{4},\ f(t,y) = \frac{10\|y_t\|e^{t-\|y_t\|}}{(1+t^2)}$.
We set ϕ *such that* $\|\phi\| = \frac{1}{10}$, $\mathcal{B}_\gamma$ *to be defined by:*

$$\mathcal{B}_\gamma = \{u \in C((-\tau, 0], R) : \lim_{\theta\to-\tau} e^{\gamma\theta}u(\theta)\ exists\}$$

with the norm

$$\|u\|_\gamma = \sup_{\theta\in(-\tau,0]} e^{\gamma\theta}|u(\theta)|.$$

Let $u : (-\tau, 1] \to R$ *be such that* $u_0 \in \mathcal{B}_\gamma$. *Then*

$$\begin{aligned}\lim_{\theta\to-\tau} e^{\gamma\theta}u(\theta) &= \lim_{\theta\to-\tau} e^{\gamma\theta}u(t+\theta)\\ &= \lim_{\theta\to-\tau} e^{\gamma(\theta-t)}u(\theta)\\ &= e^{\gamma t}\lim_{\theta\to-\tau} e^{-\gamma\theta}u_0(\theta) < +\infty.\end{aligned}$$

Therefore, $u_t \in \mathcal{B}_\gamma$. *We now prove that*

$$\|u_t\| \leq \mathcal{K}(t)\sup\{|u(s)| : 0 \leq s \leq t\} + \mathcal{M}(t)\|y_0\|_\gamma,$$

where $\mathcal{K} = \mathcal{M} = 1$ and $\mathcal{H} = 1$. we get $u(t) = u(t + \phi)$.
If $t + \theta \leq 0$ we have

$$\|u_t(\theta)\| \leq \sup\{|u(s)| : 0 \leq s \leq t\}.$$

Hence, for all $t + \theta \in [0, 1]$, we get

$$\|u_t(\theta)\| \leq \sup\{|u(s)| : -\tau \leq s \leq 0\} + \sup\{|u(s)| : 0 \leq s \leq t\}.$$

Therefore,

$$\|u_t\|_\gamma \leq \|u\|_0 + \sup\{|u(s)| : 0 \leq s \leq t\}.$$

It is evident that $(\mathcal{B}_\gamma, \|u\|_\gamma)$ is a Banach space, we conclude that $\mathcal{B}_\gamma$ is a phase space. Since

$$f(t, y) = \frac{10\|y_t\|e^{t-\|y_t\|}}{(1 + t^2)}, \quad (t, y) \in \mathcal{J} \times \mathcal{B}_\gamma.$$

We choose

$$q^*(y) = \frac{y}{10}, \quad p_1(t) = e^t, \quad h^*(y) = 100e^{-y}, \quad p_2(t) = \frac{1}{1 + t^2}, \quad y \geq 0, \quad t \in [0, 1].$$

By the definitions of f, q^*, p_1, h^*, p_2, it follows that:

$$p_2(t)h^*(\|y\|) \leq f(t, y) \leq p_1(t)q^*(\|y\|).$$

By calculations, we obtain

$$\frac{1}{4(\Gamma(\beta + 1))^{q-1}} \int_{s\in I} s^{\beta(q-1)} G(1, s)ds = 0.0022194.$$

Also,

$$\left[h^*(\mathcal{K}\mathcal{L} + \mathcal{M}\|\phi\|_{\mathcal{B}})\ \|p_2\|_\infty\right]^{q-1} = h^*\left(\mathcal{L} + \frac{1}{100}\right)$$

and then

$$\left[h^*(\mathcal{K}\mathcal{L} + \mathcal{M}\|\phi\|_{\mathcal{B}})\ \|p_2\|_\infty\right]^{q-1} \times \frac{1}{4(\Gamma(\beta + 1))^{q-1}} \int_{s\in I} s^{\beta(q-1)} G(1, s)ds \geq \mathcal{L},$$

which gives

$$100e^{\left(-\frac{1}{100}-\mathcal{L}\right)} \times 0.0022194 \geq \mathcal{L} \quad \textit{and we choose} \quad \mathcal{L} = 0.10.$$

Also,

$$\left[q^*(\mathcal{K}r + \mathcal{M}\|\phi\|_{\mathcal{B}})\|p_1\|_\infty\right]^{q-1} = \frac{1}{10}\left(r + \frac{1}{100}\right)e^1$$

and

$$\frac{2}{(2 - k)\Gamma(\alpha)(\Gamma(\beta + 1))^{q-1}} = 0.80159,$$

then

$$\frac{2}{(2 - k)\Gamma(\alpha)(\Gamma(\beta + 1))^{q-1}} \left[q^*(\mathcal{K}r + \mathcal{M}\|\phi\|_{\mathcal{B}})\|p_1\|_\infty\right]^{q-1} \leq r,$$

which gives

$$\frac{1}{10}\left(r+\frac{1}{100}\right)e^{1}\times 0.80159 \leq r \text{ and we choose } r=0.15.$$

Also,

$$\frac{1}{10}\left(\mathcal{R}+\frac{1}{100}\right)e^{1}\times 0.80159 \leq \mathcal{R} \text{ and we choose } \mathcal{R}=0.18.$$

Since all assumptions of Theorem 4.3 are satisfied, Problem (4.3) has three positive solutions $y_1,\ y_2$ and y_3.

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