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Blow up and growth of solutions to a viscoelastic parabolic type Kirchhoff equation

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Abstract. In this article, we study a system of viscoelastic parabolic type Kirchhoff equation with multiple nonlinearities. We obtain a finite time blow up of solutions and exponential growth of solution with negative initial energy.

1. Introduction

This article deals with the following initial value problem

$$\begin{cases} u_t - M(\|\nabla u\|^2)\Delta u + \int_0^t \omega_1(t-s)\Delta u(s)ds + |u|^{q-2}\, u_t = f_1(u,v), & x \in \Omega,\ t > 0, \\ v_t - M(\|\nabla v\|^2)\Delta v + \int_0^t \omega_2(t-s)\Delta v(s)ds + |v|^{q-2}\, v_t = f_2(u,v), & x \in \Omega,\ t > 0, \\ u\,(x,t) = v\,(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u\,(x,0) = u_0\,(x)\,,\ \ v\,(x,0) = v_0\,(x)\,, & x \in \Omega, \end{cases} \tag{1}$$

where $q > 2$ is a real number and Ω is a bounded domain in R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$. For the relaxation functions ω_1 and ω_2, we suppose that $\omega_i : R^+ \to R^+$ $(i = 1,2)$ are of $C^1\,(R^+)$, and represent the kernel of memory term. $f_i(u,v)$ $(i = 1,2)$ will be given later. $M(s)$ is a nonnegative C^1 function for $s \geq 0$ satisfying

$$M(s) = 1 + s^\gamma,\ \gamma > 0.$$

To motivate our problem (1) can trace back to the initial boundary value problem for the single parabolic equation

$$u_t - \Delta u + \int_0^t \omega(t-s)\Delta u(s)ds + |u|^{q-2}\, u_t = f(u). \tag{2}$$

This type problem appears a variety of mathematical models in applied science. For instance heat transfer, population dynamics, chemical reactions and so on (see [1, 7] and references therein).

We now state some existing results in the literature: Firstly, we mention the pioneer work of Hu et al. [1] where the authors proved a blow up result of solution with vanishing initial energy for problem (2). Truong

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and Van Y [2] showed the local and global existence of solutions by using the Faedo-Galerkin method for problem (2). In addition, they proved the finite time blow-up and the decay of the weak solutions. Later, Truong and Van Y [3] investigated the result of exponentional growth of the weak solutions under the sufficient conditions for problem (2). Dang et al. [4] also studied problem (2) and they showed the exponential growth of solution.

Pişkin and Ekinci [5] investigated the following problem

$$u_t - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t \omega(t-s)\Delta u(s)ds + |u|^{q-2} u_t = |u|^{p-2} u.$$

The authors studied the local and global existence of weak solutions by using the Faedo Galerkin method. Also they discussed the finite time blow up of the weak solution with positive initial energy and the general decay of the solution. Finally, they showed the exponential growth of the solution with sufficient conditions. Also, in [6], the authors considered

$$u_t - \Delta u_t - M\left(\|\nabla u\|^2\right)\Delta u + |u|^{q-2} u_t = |u|^{p-2} u.$$

They gave appropriate conditions in order to have nonexistence of global solutions or exponential growth incase of global existence.

In the absence of the diffusion term $|u|^{q-2} u_t$, equation (2) reduced to following equation

$$u_t - \Delta u + \int_0^t \omega(t-s)\Delta u(s)ds = f(u). \tag{3}$$

Problem (3) has been studied by various authors and several results concerning blow-up of solutions, both the lower and the upper bounds for blow-up time when the blow-up occurs, see [7–9].

When we examine system problems in the literature, we mention work of Pişkin and Ekinci [10] as follows

$$\begin{cases} u_t - div(|\nabla u|^{p-2}\nabla u) + |u|^{q-2} u_t = f_1(u,v), \\ v_t - div(|\nabla v|^{p-2}\nabla v) + |v|^{q-2} v_t = f_2(u,v), \end{cases} \tag{4}$$

where $p, q > 2$. They considered nonexistence of solutions and exponential growth of solution with negative initial energy. Also, Pang and Qiao [11] studied the blow up properties of the problem (4) with negative and positive initial energy for $p = 2$. Furthermore, in [12], the authors investigated the blow up and exponential growth of solution with negative initial energy for the single equation of problem (4) with Δu_t term.

Another system problem is Braik et al.'s work [13] and they proved a finite-time blow-up result for a large class of solutions of the problem (1) with positive initial energy for the case $M \equiv 1$ and without the multiple nonlinearities terms $\left(|u|^{q-2} u_t \text{ and } |v|^{q-2} v_t\right)$.

In the absence of the multiple nonlinearities terms $\left(|u|^{q-2} u_t \text{ and } |v|^{q-2} v_t\right)$, and $p = 2$, system (4) becomes

$$\begin{cases} u_t - \Delta u = f_1(u,v), \\ v_t - \Delta v = f_2(u,v). \end{cases} \tag{5}$$

Systems like form (5) naturally appears in studying non-linear phenomena in biology, chemistry, medicine and physics and so on (see [14–25]). In [26], the authors obtained the global existence solutions, blow-up of solutions in a finite time and asymptotic behavior of solutions in subcritical energy level and critical energy level, which are divided from potential well theory, respectively. Furthermore, they showed the sufficient conditions of global well posedness with supercritical energy level by combining with comparison principle and semigroup theory for (5).

Recently, in [27] the author also investigated the problem (5). He studied the global existence of solutions by combining the energy method with the Faedo-Galerkin's procedure. Moreover, he discussed the asymptotic stability by using Nakao's technique. Finally, he got blow up of solution when initial energy is negative.

We extent the prior works to coupled system where we associate the viscoelastic parabolic Kirchhoff-type equations with memory term to merged a more applications area.

This paper is organized as follows: In Section 2, we establish some notations and stament of assumptions. In Section 3-4, we discuss the blow up and the growth of solutions, respectively.

2. Preliminaries

In the present section, we shall give some assumptions for the proof of our results. Let $\|.\|$, $\|.\|_p$ and $(u,v)=\int_\Omega u(x)v(x)dx$ denote the usual $L^2(\Omega)$ norm, $L^p(\Omega)$ norm and inner product of $L^2(\Omega)$, respectively. Throughout this paper, we accept that C is a general positive constant.

To state and prove our results, we need some assumptions:

For the numbers m and q, we suppose that

$$\begin{cases} 2<q<m\leq \frac{2(n-1)}{n-2} \text{ if } n>2, \\ 2<q<m<+\infty \text{ if } n=1,2. \end{cases} \tag{6}$$

Regarding the functions $f_1(u,v)$, $f_2(u,v)\in C^1$ such that

$$f_1(u,v)=\frac{\partial F(u,v)}{\partial u},\quad f_2(u,v)=\frac{\partial F(u,v)}{\partial v}$$

and

$$\begin{cases} l_0(|u|^m+|v|^m)\leq F(u,v)\leq l_1(|u|^m+|v|^m), \\ uf_1(u,v)+vf_2(u,v)=(m+1)F(u,v) \end{cases} \tag{7}$$

where l_0, l_1 are positive constants.

The relaxation functions ω_i $(i=1,2)$ are C^1 nonnegative functions such that

$$\omega_i(0)>0,\quad 1-\int_0^\infty \omega_i(s)ds=\eta_i>0, \tag{8}$$

There exist two positive differentiable functions ξ_1 and ξ_2 such that

$$\omega_i(t)>0,\ \omega_i'(t)\leq 0,\ \omega_i'(t)\leq -\xi_i(t)\omega_i(t),\ \int_0^\infty \xi_i(t)dt=\infty,\ (i=1,2),\ \text{ for } t\geq 0.$$

Also, we introduce following notation:

$$(\omega_i\circ\nabla w)(t)=\int_0^t \omega_i(t-s)\,\|\nabla w(t)-\nabla w(s)\|^2\,ds.$$

Combining arguments of [1, 7, 24], $(u(x,t),v(x,t))$ is called a solution of problem (1) on $\Omega\times[0,T)$ if

$$\begin{cases} u,v\in C(0,T;W_0^{1,2(\gamma+1)}(\Omega))\cap C^1(0,T;L^2(\Omega)), \\ |u|^{q-2}u_t,\ |v|^{q-2}v_t\in L^2(\Omega\times[0,T)) \end{cases} \tag{9}$$

satisfying the initial condition $u(x,0)=u_0(x)$, $v(x,0)=v_0(x)$ and

$$(u_t,w)+\left(\left(1+\left(\int_\Omega |\nabla u|^2\,dx\right)^\gamma\right)\nabla u,\nabla w\right)-\left(\int_0^t \omega_1(t-s)\nabla u(s)\,ds,\nabla w\right)+\left(|u|^{q-2}u_t,w\right) = (f_1(u,v),w), \tag{10}$$

$$(v_t,\varphi)+\left(\left(1+\left(\int_\Omega |\nabla v|^2\,dx\right)^\gamma\right)\nabla v,\nabla\varphi\right)-\left(\int_0^t \omega_2(t-s)\nabla v(s)\,ds,\nabla\varphi\right)+\left(|v|^{q-2}v_t,\varphi\right) = (f_2(u,v),\varphi) \tag{11}$$

for all $w, \varphi \in C(0,T; W_0^{1,2(\gamma+1)}(\Omega))$.

The energy functional associated with problem (1) is

$$\begin{aligned} E(t) &= \frac{1}{2}\left(1-\int_0^t \omega_1(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{2}\left(1-\int_0^t \omega_2(s)\,ds\right)\|\nabla v\|^2 \\ &+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)} \\ &+\frac{1}{2}\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] - \int_\Omega F(u,v)dx, \end{aligned} \tag{12}$$

where $u, v \in W_0^{1,2(\gamma+1)}(\Omega)$.

Multiplying the first equation in (1) by u_t and second by v_t, integrating over Ω, since $\omega_i'(s) \le 0$, we have

$$\begin{aligned} E'(t) &= -\|u_t\|^2 - \|v_t\|^2 - \int_\Omega |u|^{q-2}u_t^2dx - \int_\Omega |v|^{q-2}v_t^2dx \\ &+\frac{1}{2}\left(\omega_1' \circ \nabla u\right)(t) + \frac{1}{2}\left(\omega_2' \circ \nabla v\right)(t) \\ &-\frac{1}{2}\omega_1(t)\|\nabla u\|^2 - \frac{1}{2}\omega_2(t)\|\nabla v\|^2 \\ &\le 0. \end{aligned} \tag{13}$$

3. Blow up of solutions

In this section, we deal with the blow up results of the solution for the system (1).

Theorem 3.1. *Suppose that (6) and (8) hold, $u_0, v_0 \in W_0^{1,2(\gamma+1)}(\Omega)$ and (u,v) is a local solution of the system (1). Assume further that*

$$E(0) < 0$$

and

$$\int_0^t \omega_i(s)\,ds \ge \frac{\gamma}{\gamma + 1/4}.$$

Then, the solution of the system (1) blows up in finite time.

Proof. We set

$$H(t) = -E(t). \tag{14}$$

From (14) and (13)

$$H'(t) = -E'(t) \ge 0. \tag{15}$$

Since $E(0) < 0$, we get

$$H(0) = -E(0) > 0. \tag{16}$$

By integrating (15), we obtain

$$0 < H(0) \le H(t). \tag{17}$$

By using (14) and (12)

$$\begin{aligned} H(t)-\int_\Omega F(u,v)dx &= -\frac{1}{2}\left(1-\int_0^t \omega_1(s)\,ds\right)\|\nabla u\|^2-\frac{1}{2}\left(1-\int_0^t \omega_2(s)\,ds\right)\|\nabla v\|^2 \\ &\quad -\frac{1}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &\quad -\frac{1}{2}\left[(\omega_1\circ\nabla u)(t)+(\omega_2\circ\nabla v)(t)\right] \\ &< 0. \end{aligned} \tag{18}$$

Then, by using (7), we have

$$0<H(0)\le H(t)\le\int_\Omega F(u,v)dx\le l_1\left(\|u\|_m^m+\|v\|_m^m\right). \tag{19}$$

Then, we define

$$\Psi(t)=H^{1-\sigma}(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|v\|^2, \tag{20}$$

where $\varepsilon>0$ small to be chosen later and $0<\sigma\le(m-2)/m$ since $2<m$. By differentiating (20) and by using (1) and (7), we get

$$\begin{aligned} \Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\int_\Omega uu_t dx+\varepsilon\int_\Omega vv_t dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\|\nabla u\|^{2(\gamma+1)}-\varepsilon\|\nabla v\|^{2(\gamma+1)} \\ &\quad -\varepsilon\left(1-\int_0^t\omega_1(t-s)\,ds\right)\|\nabla u\|^2-\varepsilon\left(1-\int_0^t\omega_2(t-s)\,ds\right)\|\nabla v\|^2 \\ &\quad -\varepsilon\int_\Omega\nabla u(t)\int_0^t\omega_1(t-s)\left(\nabla u(t)-\nabla u(s)\right)dsdx \\ &\quad -\varepsilon\int_\Omega\nabla v(t)\int_0^t\omega_2(t-s)\left(\nabla v(t)-\nabla v(s)\right)dsdx \\ &\quad +\varepsilon(m+1)\int_\Omega F(u,v)dx-\varepsilon\int_\Omega|u|^{q-2}uu_t^2dx-\varepsilon\int_\Omega|v|^{q-2}vv_t^2dx \\ &\ge (1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\left[\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right] \\ &\quad -\varepsilon\left(1-\int_0^t\omega_1(t-s)\,ds\right)\|\nabla u\|^2-\varepsilon\left(1-\int_0^t\omega_2(t-s)\,ds\right)\|\nabla v\|^2 \\ &\quad -\varepsilon\int_0^t\omega_1(t-s)\int_\Omega\nabla u(t)\left(\nabla u(\tau)-\nabla u(t)\right)dxds \\ &\quad -\varepsilon\int_0^t\omega_2(t-s)\int_\Omega\nabla v(t)\left(\nabla v(\tau)-\nabla v(t)\right)dxds \\ &\quad +\varepsilon(m+1)\left(\|u\|_m^m+\|v\|_m^m\right)-\varepsilon\int_\Omega|u|^{q-2}uu_t^2dx-\varepsilon\int_\Omega|v|^{q-2}vv_t^2dx \end{aligned} \tag{21}$$

In order to estimate the last terms in (21), we use the following Young's inequality

$$ab\le\delta^{-1}a^2+\delta b^2,$$

for $\delta > 0$, with $a = |u|^{\frac{q-2}{2}} u_t$ and $b = |u|^{\frac{q-2}{2}} u$, we have

$$\begin{aligned}\int_\Omega |u|^{q-2} uu_t dx &\leq \int_\Omega |u|^{\frac{q-2}{2}} u_t |u|^{\frac{q-2}{2}} udx \\ &\leq \delta^{-1} \int_\Omega |u|^{q-2} u_t^2 dx + \delta \int_\Omega |u|^q dx.\end{aligned}$$

In the same way for $a = |v|^{\frac{q-2}{2}} v_t$ and $b = |v|^{\frac{q-2}{2}} v$, we get

$$\int_\Omega |v|^{q-2} vv_t dx \leq \delta^{-1} \int_\Omega |v|^{q-2} v_t^2 dx + \delta \int_\Omega |v|^q dx.$$

The Cauchy-Schwarz and Young's inequalities allows us to estimate the sixth and seventh term in the right-hand side of (21) as follows

$$\begin{aligned}\int_0^t \omega_1 (t-s) \int_\Omega \nabla u(t) (\nabla u(\tau) - \nabla u(t)) dxds &\leq \int_0^t \omega_1 (t-s) \|\nabla u(t)\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2 d\tau \\ &\leq (\omega_1 \circ \nabla u)(t) + \frac{1}{4}\left(\int_0^t \omega_1(s) ds\right) \|\nabla u\|^2.\end{aligned}$$

Similarly

$$\int_0^t \omega_2 (t-s) \int_\Omega \nabla v(t) (\nabla v(\tau) - \nabla v(t)) dxds \leq (\omega_2 \circ \nabla v)(t) + \frac{1}{4}\left(\int_0^t \omega_2(s) ds\right) \|\nabla v\|^2.$$

So, (21) becomes

$$\begin{aligned}\Psi'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] \\ & -\varepsilon\left(1 - \frac{3}{4}\int_0^t \omega_1(s) ds\right)\|\nabla u\|^2 - \varepsilon\left(1 - \frac{3}{4}\int_0^t \omega_2(s) ds\right)\|\nabla v\|^2 \\ & -\varepsilon\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] + \varepsilon(m+1)\left(\|u\|_m^m + \|v\|_m^m\right) \\ & -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2} u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2} v_t^2 dx \\ \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] \\ & -\varepsilon\left(\frac{1 - \frac{3}{4}\int_0^t \omega_1(s) ds}{1 - \int_0^t \omega_1(s) ds}\right)\eta_1 \|\nabla u\|^2 - \varepsilon\left(\frac{1 - \frac{3}{4}\int_0^t \omega_2(s) ds}{1 - \int_0^t \omega_2(s) ds}\right)\eta_2 \|\nabla v\|^2 \\ & -\varepsilon\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] + \varepsilon(m+1)\left(\|u\|_m^m + \|v\|_m^m\right) \\ & -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2} u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2} v_t^2 dx. \end{aligned} \tag{22}$$

We denote $k_i = \frac{1 - \frac{3}{4}\int_0^t \omega_i(s) ds}{1 - \int_0^t \omega_i(s) ds}$, $i = 1, 2$.

$$\begin{aligned}\Psi'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] \\ & -\varepsilon k_1 \eta_1 \|\nabla u\|^2 - \varepsilon k_2 \eta_2 \|\nabla v\|^2 \\ & -\varepsilon\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] + \varepsilon(m+1)\left(\|u\|_m^m + \|v\|_m^m\right) \\ & -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2} u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2} v_t^2 dx\end{aligned}$$

$$\begin{aligned}
&\geq (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] \\
&\quad -\varepsilon \min_{i=1,2}\{k_i\}\left(\eta_1\|\nabla u\|^2 + \eta_2\|\nabla v\|^2\right) \\
&\quad -\varepsilon\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] + \varepsilon(m+1)\left(\|u\|_m^m + \|v\|_m^m\right) \\
&\quad -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2}u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2}v_t^2 dx
\end{aligned}$$

Then, from the definition $H(t)$, we have

$$\begin{aligned}
\Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right] \\
&\quad -\varepsilon \min_{i=1,2}\{k_i\}\left[-2H(t) - \frac{1}{(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \frac{1}{(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}\right. \\
&\quad \left. - (\omega_1 \circ \nabla u)(t) - (\omega_2 \circ \nabla v)(t) + 2\int_\Omega F(u,v)dx\right] \\
&\quad -\varepsilon\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] + \varepsilon(m+1)\left(\|u\|_m^m + \|v\|_m^m\right) \\
&\quad -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2}u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2}v_t^2 dx \\
&\geq (1-\sigma)H^{-\sigma}(t)H'(t) + 2\varepsilon \min_{i=1,2}\{k_i\}H(t) \\
&\quad +\varepsilon\left[\min_{i=1,2}\{k_i\}\frac{1}{(\gamma+1)} - 1\right]\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\
&\quad +\varepsilon\left(\min_{i=1,2}\{k_i\} - 1\right)\left[(\omega_1 \circ \nabla u)(t) + (\omega_2 \circ \nabla v)(t)\right] \\
&\quad +\varepsilon\left(m+1-2\min_{i=1,2}\{k_i\}\right)\left(\|u\|_m^m + \|v\|_m^m\right) \\
&\quad -\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2}u_t^2 dx - \varepsilon\delta^{-1}\int_\Omega |v|^{q-2}v_t^2 dx \qquad (23)
\end{aligned}$$

As the embedding $L^m(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^2(\Omega)$, (since $m > q > 2$), we have

$$\begin{cases} \|u\|_q^q \leq C\|u\|_m^q \leq C(\|u\|_m^m)^{\frac{q}{m}}, \\ \|v\|_q^q \leq C\|v\|_m^q \leq C(\|v\|_m^m)^{\frac{q}{m}}. \end{cases} \qquad (24)$$

Since $0 < \frac{q}{m} < 1$, now applying the following inequality

$$x^l \leq (x+1) \leq (1+\frac{1}{z})(x+z), \qquad (25)$$

which holds for all $x \geq 0$, $0 \leq l \leq 1$, $z > 0$, especially, taking $x = \|u\|_m^m$, $l = \frac{q}{m}$, $z = H(0)$, we get

$$C(\|u\|_m^m)^{\frac{q}{m}} \leq (1+\frac{1}{H(0)})\left(\|u\|_m^m + H(0)\right),$$

similarly

$$C(\|v\|_m^m)^{\frac{q}{m}} \leq (1+\frac{1}{H(0)})\left(\|v\|_m^m + H(0)\right).$$

Then, from (19) and (24), we get

$$\begin{aligned}
\|u\|_q^q + \|v\|_q^q &\leq C\left(\|u\|_m^q + \|v\|_m^q\right) \\
&\leq C_1(\|u\|_m^m + \|v\|_m^m). \qquad (26)
\end{aligned}$$

Insert (26) into (23), it follows that

$$\begin{aligned}\Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t)+2\varepsilon\min_{i=1,2}\{k_i\}H(t)+\varepsilon c'\left(\|u\|_m^m+\|v\|_m^m\right)\\ &\quad -\varepsilon\delta^{-1}\int_\Omega |u|^{q-2}u_t^2dx-\varepsilon\delta^{-1}\int_\Omega |v|^{q-2}v_t^2dx, \end{aligned} \tag{27}$$

where we pick δ small enough such that $c' = m+1-2\min_{i=1,2}\{k_i\}-C_1\delta > 0$ and taking $\delta^{-1} = \beta H^{-\sigma}(t)$ (27) follows that

$$\begin{aligned}\Psi'(t) &\geq (1-\sigma-\beta\varepsilon)H^{-\sigma}(t)H'(t)+2\varepsilon\min_{i=1,2}\{k_i\}H(t)+\varepsilon c'\left(\|u\|_m^m+\|v\|_m^m\right)\\ &\geq \alpha(H(t)+\|u\|_m^m+\|v\|_m^m), \end{aligned} \tag{28}$$

where $\alpha = \min\left\{2\varepsilon\min_{i=1,2}\{k_i\}, \varepsilon c'\right\}$ and we pick ε small enough such that $1-\sigma-\beta\varepsilon \geq 0$.

We now estimate $\Psi^{\frac{1}{1-\sigma}}(t)$. From definition of $\Psi(t)$

$$\Psi^{\frac{1}{1-\sigma}}(t) = \left(H^{1-\sigma}(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|v\|^2\right)^{\frac{1}{1-\sigma}}. \tag{29}$$

As the embedding $L^m(\Omega) \hookrightarrow L^2(\Omega)$, $m > 2$, we have

$$\Psi^{\frac{1}{1-\sigma}}(t) \leq C\left(H(t)+\|u\|_m^{2/(1-\sigma)}+\|v\|_m^{2/(1-\sigma)}\right). \tag{30}$$

Now, by the inequality $x^l \leq (x+1) \leq (1+\frac{1}{z})(x+z)$ for $x = \|u\|_m^m$, $l = 2/m(1-\sigma) < 1$, since $\sigma < (m-2)/m$, $z = H(0)$, we get

$$\begin{aligned}\|u\|_m^{2/(1-\sigma)} &\leq (\|u\|_m^m)^{2/m(1-\sigma)} \leq (1+\frac{1}{H(0)})\left(\|u\|_m^m+H(0)\right)\\ &\leq C\|u\|_m^m. \end{aligned} \tag{31}$$

In the same way, we get

$$\|v\|_m^{2/(1-\sigma)} \leq C\|v\|_m^m. \tag{32}$$

Therefore, (30) becomes that

$$\Psi^{\frac{1}{1-\sigma}}(t) \leq C\left(H(t)+\|u\|_m^m+\|v\|_m^m\right). \tag{33}$$

By associatining of (28) and (33) we reach

$$\Psi'(t) \geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \tag{34}$$

where $\xi > 0$ is a constant. A simple integration (34) from 0 to t yields that $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

4. Growth of solutions

In this section, we state and prove exponential growth result.

Theorem 4.1. *Suppose that (6) and (8) hold, $u_0, v_0 \in W_0^{1,2(\gamma+1)}(\Omega)$ and (u,v) is a solution of the system (1). Furthermore, we assume that*

$$E(0) < 0$$

and

$$\int_0^t \omega_i(s)\,ds \geq \frac{\gamma}{\gamma + 1/2}.$$

Then the solution of the system (1) grows exponentially.

Proof. Let us define the functional

$$F(t) = H(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2, \tag{35}$$

where $H(t) = -E(t)$. By differentiating (35) and using Eq.(1), we get

$$\begin{aligned}
F'(t) &= H'(t) + \varepsilon\left(\int_\Omega u u_t dx + \int_\Omega v v_t dx\right) \\
&= \|u_t\|^2 + \|v_t\|^2 + \int_\Omega |u|^{q-2} u_t^2 dx + \int_\Omega |v|^{q-2} v_t^2 dx \\
&\quad -\frac{1}{2}\left(\omega_1' \circ \nabla u\right)(t) - \frac{1}{2}\left(\omega_2' \circ \nabla v\right)(t) \\
&\quad +\frac{1}{2}\omega_1(s)\|\nabla u(t)\|^2 + \frac{1}{2}\omega_2(s)\|\nabla v(t)\|^2 \\
&\quad -\varepsilon\int_\Omega |u|^{q-2} u u_t dx - \varepsilon\int_\Omega |v|^{q-2} v v_t^2 dx \\
&\quad -\varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla v\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)} - \varepsilon\|\nabla v\|^{2(\gamma+1)} \\
&\quad +\varepsilon\int_\Omega [u f_1(u,v) + v f_2(u,v)]\,dx \\
&\quad +\varepsilon\int_\Omega\int_0^t \omega_1(t-s)\nabla u(t)\nabla u(s)\,ds dx \\
&\quad +\varepsilon\int_\Omega\int_0^t \omega_2(t-s)\nabla v(t)\nabla v(s)\,ds dx.
\end{aligned} \tag{36}$$

$$\begin{aligned}
F'(t) &\geq \|u_t\|^2 + \|v_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla v\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)} - \varepsilon\|\nabla v\|^{2(\gamma+1)} \\
&\quad +\varepsilon(m+1)\int_\Omega F(u,v)dx + \varepsilon\int_\Omega\int_0^t \omega_1(t-s)\nabla u(t)\nabla u(s)\,ds dx \\
&\quad +\varepsilon\int_\Omega\int_0^t \omega_2(t-s)\nabla v(t)\nabla v(s)\,ds dx \\
&\quad +\int_\Omega |u|^{q-2} u_t^2 dx + \int_\Omega |v|^{q-2} v_t^2 dx - \varepsilon\int_\Omega |u|^{q-2} u u_t dx - \varepsilon\int_\Omega |v|^{q-2} v v_t^2 dx.
\end{aligned} \tag{37}$$

In order to estimate the last two terms in the right-hand side of (37), we use the following Young's inequality,

$$ab \leq \delta^{-1}a^2 + \delta b^2,$$

so we have

$$\begin{aligned}\int_\Omega |u|^{q-2} uu_t dx &\leq \int_\Omega |u|^{\frac{q-2}{2}} u_t |u|^{\frac{q-2}{2}} udx \\ &\leq \delta^{-1} \int_\Omega |u|^{q-2} u_t^2 dx + \delta \int_\Omega |u|^q dx.\end{aligned}$$

Similarly,

$$\int_\Omega |v|^{q-2} vv_t dx \leq \delta^{-1} \int_\Omega |v|^{q-2} v_t^2 dx + \delta \int_\Omega |v|^q dx.$$

Therefore, combining with

$$\begin{aligned}&\int_\Omega \int_0^t \omega_1 (t-s) \nabla u(t) \nabla u(s) dsdx \\ = &\int_0^t \omega_1 (s) ds \|\nabla u\|^2 + \int_0^t \omega_1 (t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t)) dxds \\ \geq &-\frac{1}{2} (\omega_1 \circ \nabla u)(t) + \frac{1}{2} \left(\int_0^t \omega_1 (s) ds \right) \|\nabla u\|^2 .\end{aligned}$$

Similarly

$$\int_\Omega \int_0^t \omega_2 (t-s) \nabla v(t) \nabla v(s) dsdx \leq -\frac{1}{2} (\omega_2 \circ \nabla v)(t) + \frac{1}{2} \left(\int_0^t \omega_2 (s) ds \right) \|\nabla v\|^2 .$$

Then, (37) becomes

$$\begin{aligned}F'(t) \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(1 - \frac{1}{2}\int_0^t \omega_1 (s) ds\right) \|\nabla u\|^2 - \varepsilon \left(1 - \frac{1}{2}\int_0^t \omega_2 (s) ds\right) \|\nabla v\|^2 \\ & -\varepsilon \left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right] + \varepsilon (m+1) \left(\|u\|_m^m + \|v\|_m^m \right) \\ & -\frac{\varepsilon}{2} (\omega_1 \circ \nabla u)(t) - \frac{\varepsilon}{2} (\omega_2 \circ \nabla v)(t) \\ & -\varepsilon\delta \left(\|u\|_q^q + \|v\|_q^q \right) + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |v|^{q-2} v_t^2 dx \\ \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(\frac{1 - \frac{1}{2}\int_0^t \omega_1 (s) ds}{1 - \int_0^t \omega_1 (s) ds} \right) \eta_1 \|\nabla u\|^2 - \varepsilon \left(\frac{1 - \frac{1}{2}\int_0^t \omega_2 (s) ds}{1 - \int_0^t \omega_2 (s) ds} \right) \eta_2 \|\nabla v\|^2 \\ & -\varepsilon \left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right] + \varepsilon (m+1) \left(\|u\|_m^m + \|v\|_m^m \right) \\ & -\frac{\varepsilon}{2} (\omega_1 \circ \nabla u)(t) - \frac{\varepsilon}{2} (\omega_2 \circ \nabla v)(t) \\ & -\varepsilon\delta \left(\|u\|_q^q + \|v\|_q^q \right) + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |v|^{q-2} v_t^2 dx. \qquad (38)\end{aligned}$$

We denote $\lambda_i = \frac{1-\frac{1}{2}\int_0^t \omega_i(s)ds}{1-\int_0^t \omega_i(s)ds}$, $i = 1, 2$.

$$\begin{aligned}F'(t) \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon\lambda_1\eta_1 \|\nabla u\|^2 - \varepsilon\lambda_2\eta_2 \|\nabla v\|^2 \\ & -\varepsilon \left[\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right] + \varepsilon (m+1) \left(\|u\|_m^m + \|v\|_m^m \right) \\ & -\frac{\varepsilon}{2} (\omega_1 \circ \nabla u)(t) - \frac{\varepsilon}{2} (\omega_2 \circ \nabla v)(t) \\ & -\varepsilon\delta \left(\|u\|_q^q + \|v\|_q^q \right) + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \left(1 - \varepsilon\delta^{-1}\right) \int_\Omega |v|^{q-2} v_t^2 dx\end{aligned}$$

$$\begin{aligned}
&\geq \|u_t\|^2+\|v_t\|^2-\varepsilon\min_{i=1,2}\lambda_i\left(\eta_1\|\nabla u\|^2+\eta_2\|\nabla v\|^2\right)\\
&-\varepsilon\left[\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right]+\varepsilon(m+1)\left(\|u\|_m^m+\|v\|_m^m\right)\\
&-\frac{\varepsilon}{2}\left(\omega_1\circ\nabla u\right)(t)-\frac{\varepsilon}{2}\left(\omega_2\circ\nabla v\right)(t)\\
&-\varepsilon\delta\left(\|u\|_q^q+\|v\|_q^q\right)+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|u|^{q-2}u_t^2dx+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|v|^{q-2}v_t^2dx. \qquad (39)
\end{aligned}$$

From $H(t)$ definition, (39) becomes

$$\begin{aligned}
F'(t) \geq\ & 2\varepsilon\min_{i=1,2}\{\lambda_i\}H(t)+\|u_t\|^2+\|v_t\|^2+\varepsilon(m+1-2\min_{i=1,2}\{\lambda_i\})\left(\|u\|_m^m+\|v\|_m^m\right)\\
&-\varepsilon\left(1-\frac{1}{\gamma+1}\min_{i=1,2}\{\lambda_i\}\right)\left[\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right]\\
&+\varepsilon\left(\min_{i=1,2}\{\lambda_i\}-\frac{1}{2}\right)\left[\left(\omega_1\circ\nabla u\right)(t)+\left(\omega_2\circ\nabla v\right)(t)\right]\\
&-\varepsilon\delta\left(\|u\|_q^q+\|v\|_q^q\right)+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|u|^{q-2}u_t^2dx+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|v|^{q-2}v_t^2dx. \qquad (40)
\end{aligned}$$

Then, from (26) we obtain

$$\begin{aligned}
F'(t) \geq\ & \varepsilon 2\min_{i=1,2}\{\lambda_i\}H(t)+\|u_t\|^2+\|v_t\|^2+\varepsilon a_1\left(\|u\|_m^m+\|v\|_m^m\right)\\
&+\varepsilon a_2\left[\left(\omega_1\circ\nabla u\right)(t)+\left(\omega_2\circ\nabla v\right)(t)\right]\\
&+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|u|^{q-2}u_t^2dx+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega|v|^{q-2}v_t^2dx,
\end{aligned}$$

where δ small enough such that $a_1=m+1-2\min_{i=1,2}\{\lambda_i\}-\delta C_1>0$ and $a_2=\min_{i=1,2}\{\lambda_i\}-\frac{1}{2}>0$, then taking ε and δ small enough such that $1-\varepsilon\delta^{-1}>0$, we have

$$F'(t)\geq C(H(t)+\|u_t\|^2+\|v_t\|^2+\|u\|_m^m+\|v\|_m^m+\left(\omega_1\circ\nabla u\right)(t)+\left(\omega_2\circ\nabla v\right)(t)). \qquad (41)$$

On the other hand, by definition of $F(t)$, we get

$$F(t)=H(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|v\|^2.$$

As the embedding $L^m(\Omega)\hookrightarrow L^2(\Omega)$, $m>2$, and then using the inequality $x^l\leq(x+1)\leq(1+\frac{1}{z})(x+z)$ for $x=\|u\|_m^m$, $l=2/m<1$, $z=H(0)$, we get

$$\begin{aligned}
F(t) &\leq C\left(H(t)+\|u\|_m^m+\|v\|_m^m\right) \qquad (42)\\
&\leq C\left(H(t)+\|u\|_m^m+\|v\|_m^m+\|u_t\|^2+\|v_t\|^2+\left(\omega_1\circ\nabla u\right)(t)+\left(\omega_2\circ\nabla v\right)(t)\right). \qquad (43)
\end{aligned}$$

From (41) and (43), we arrive at

$$F'(t)\geq rF(t), \qquad (44)$$

where r is a positive constant.

Integration of (44) over $(0,t)$ gives us

$$F(t)\geq F(0)\exp(rt).$$

From (42) and (19), we get

$$F(t)\leq H(t)\leq\|u\|_m^m+\|v\|_m^m.$$

Consequently, we show that the solution in the L_m-norm growths exponentially. □

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