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Bounds on Ricci curvature for doubly warped products pointwise bi-slant submanifolds and applications to Physics

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Abstract. In this article, we obtain bounds for Ricci curvature for doubly warped products pointwise bi-slant submanifolds in generalized complex space forms and discuss the equality case of the inequality. We also derive the non-existence of such immersions. Finally, we construct some applications of the result in terms of the Harmonic function, Hessian tensor, and Dirchilet energy function.

1. Introduction

In 2000, B. Unal [22] introduced doubly warped products as a generalization of warped products and according to him: let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively. Further, let us suppose that σ_1 be positive differentiable functions on M_1 and σ_2 be positive differentiable functions on M_2. Then, the doubly warped product $M = {}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ [22] of dimension n is defined on the basis of the product manifold $M_1 \times M_2$ equipped with the warped metric $g = \sigma_2^2 g_1 + \sigma_1^2 g_2$.

In a meticulous manner, if $\iota_1 : M_1 \times M_2 \to M_1$ and $\iota_2 : M_1 \times M_2 \to M_2$ be natural projections, then the metric g is given by

$$g(X, Y) = (\sigma_2 \circ \iota_2)^2 g_1(\iota_1^* X, \iota_1^* Y) + (\sigma_1 \circ \iota_1)^2 g_2(\iota_2^* X, \iota_2^* Y), \tag{1}$$

for any vector fields X, Y on M, where $*$ denotes the symbol for tangent maps and σ_1 and σ_2 are the warping functions on M_1 and M_2, respectively.

It is important to note that on a double warped product manifold $M = {}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ if either σ_1 or σ_2 is constant on M, but not both then M is a single warped product. Furthermore, if both σ_1 and σ_2 are constant function on M, then M is locally a Riemannian product. A doubly warped product manifold is said to be *proper* if both σ_1 and σ_2 are non-constant functions on M.

On the other hand, the immersibility/non-immersibility of a Riemannian manifold in a space form is one of the most fundamental problem in the theory of submanifold which started with the most celebrated Nash [] embedding theorem. In this theorem, actually Nash was aiming to take extrinsic help. However, due to the lack of control of the extrinsic properties of the submanifolds by the known intrinsic invariant,

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the aim cannot be reached. Motivated by this and to overcome the difficulties, Chen introduced new types of Riemannian invariants and established general optimal relationship between extrinsic invariants and intrinsic invariants on the submanifold. The same author obtained the inequalities for submanifolds between the k-Ricci curvature, the squared mean curvature, and the shape operator in the real space form with arbitrary codimension [7]. After that many research articles has been published in this area [6, 8–12, 15, 21].

Motivated by this, we obtain bounds for Ricci curvature for doubly warped product pointwise bi-slant submanifolds in generalized complex space forms and discuss the equality case of the inequality. We also derive non-existence of such immersions. Finally, we construct some applications of the result in terms of Harmonic function, Hessian tensor and Dirchilet energy function.

2. Preliminaries

Let $\tilde{\mathcal{M}}$ be a $2m$-dimensional almost Hermitian manifold with an almost complex structure J and a Riemannian metric g. An almost Hermitian manifold is said to be a nearly Kaehler manifold if $(\tilde{\nabla}_X J)X = 0$ and becomes a Kaehler manifold if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g.

Tricerri and Vanhecke [17] introduced the concept of generalized complex space form as a generalization of the complex space form.

An almost Hermitian manifold $\tilde{\mathcal{M}}$ is called the generalized complex space form, denoted by $\tilde{\mathcal{M}}(f_1, f_2)$, if the Riemannian curvature tensor $\overline{R}$ satisfies

$$\overline{R}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}, \tag{2}$$

for all $X, Y, Z \in T\tilde{\mathcal{M}}$, where f_1 and f_2 are smooth function on $\tilde{\mathcal{M}}(f_1, f_2)$.

Let $\tilde{\mathcal{M}}^{2m}$ be an almost Hermitian manifold and $\mathcal{M}^n$ be a submanifold $\tilde{\mathcal{M}}^{2m}$ with induced metric g. Let ∇ be an induced connection on the tangent bundle $T\mathcal{M}$ and $\nabla^{\perp}$ be an induced connection on the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$. Then, the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi,$$

where $X, Y \in T\mathcal{M}$, $N \in T^{\perp}\mathcal{M}$ and h, A_N are second fundamental form and the shape operator, respectively.

The relation between the shape operator and the second fundamental form is given by

$$g(h(X,Y), \mathcal{N}) = g(A_{\mathcal{N}} X, Y),$$

for vector fields $X, Y \in TM$ and $\mathcal{N} \in T^{\perp}\mathcal{M}$.
Let $\overline{R}$ and R be the curvature tensors of $\overline{M}(c)$ and M, respectively. Then, the Gauss equation is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z), h(Y,W)) - g(h(X,W), h(Y,Z)), \tag{3}$$

for any $X, Y, Z, W \in TM$.

In this context, we shall define another important Riemannian intrinsic invariant called curvature of $\tilde{\mathcal{M}}^m$, and denoted by $\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m)$, which at some x in $\tilde{\mathcal{M}}$ is given as follows"

$$\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m) = \sum_{1\le\alpha<\beta\le m} \tilde{\mathcal{K}}_{\alpha\beta}, \tag{4}$$

where $\tilde{\mathcal{K}}_{\alpha\beta} = \tilde{\mathcal{K}}(e_\alpha \wedge e_\beta)$.
From (4) it follows that

$$2\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m) = \sum_{1\le\alpha<\beta\le m} \tilde{\mathcal{K}}_{\alpha\beta}, \quad 1 \le \alpha, \beta \le n. \tag{5}$$

Similarly, the scalar curvature $\tilde{\tau}(\mathcal{L}_x)$ of the $\mathcal{L}$ plan is given by

$$\tilde{\tau}(\mathcal{L}_x) = \sum_{1\leq\alpha<\beta\leq m} \tilde{\mathcal{K}}_{\alpha\beta}. \tag{6}$$

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space $\mathcal{T}_x\mathcal{M}$ and $e_r = \{e_{n+1}, ..., e_m\}$ belong to an orthonormal basis of the normal space $\mathcal{T}^{\perp}\mathcal{M}$, then we have

$$\|h\|^2 = \sum_{\alpha,\beta=1}^{n} g(h(e_\alpha, e_\beta), h(e_\alpha, e_\beta)). \tag{7}$$

Let $\mathcal{K}_{\alpha\beta}$ and $\tilde{\mathcal{K}}_{\alpha\beta}$ denote the sectional curvature of the plane section in the submanifold $\mathcal{M}^n$ and the Riemannian space form $\tilde{\mathcal{M}}^m(c)$, respectively. Thus, $\mathcal{K}_{\alpha\beta}$ and $\tilde{\mathcal{K}}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvature of the span $\{e_\alpha, e_\beta\}$ at x. From the Gausss equation, we have

$$\begin{aligned} 2\tau(\mathcal{T}_x\tilde{\mathcal{M}}^n) = \mathcal{K}_{\alpha\beta} &= 2\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^n) + \sum_{r=n+1}^{m} (h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) \\ &= \tilde{\mathcal{K}}_{\alpha\beta} + \sum_{r=n+1}^{m} (h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2). \end{aligned} \tag{8}$$

Further, assuming that for a local field of the orthonormal frame $\{e_1, ..., e_n\}$ on $\mathcal{M}^n$, the global tensor is defined as

$$\tilde{\mathcal{S}}(X, Y) = \sum_{i=1}^{m} \left\{\tilde{g}(\tilde{R}(e_\alpha, X)Y, e_\alpha)\right\}, \quad X, Y \in \mathcal{T}_x\mathcal{M}^m, \tag{9}$$

is called Ricci tensor. If we fix a distinct integer from $\{e_1, ..., e_n\}$ on $\mathcal{M}^n$ by e_A, which is governed by X, then the Ricci curvature is given by

$$Ric(X) = \sum_{\alpha=1,\alpha\neq A} \mathcal{K}(e_\alpha \wedge e_A). \tag{10}$$

Now, we define an important Riemannian intrinsic invariant called the scalar curvature of $\mathcal{M}^m$ and it is denoted by $\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m)$, that is

$$\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m) = \sum_{1\leq\alpha<\beta\leq n} \mathcal{K}(e_\alpha \wedge e_\beta) = \frac{1}{2}\sum_{A=1}^{m} Ric(e_A). \tag{11}$$

It is clear that the above the inequality is congruent to the following equation which will be frequently used in next study:

$$2\tilde{\tau}(\mathcal{T}_x\tilde{\mathcal{M}}^m) = \sum_{1\leq\alpha<\beta\leq n} \mathcal{K}(e_\alpha \wedge e_\beta) = \frac{1}{2}\sum_{A=1}^{m} Ric(e_A). \tag{12}$$

For a k-plan $\mathcal{L}$ of $\mathcal{T}_x\mathcal{M}^m$, let $\{e_1, ..., e_n\}$ be an orthonormal basis of $\mathcal{L}_k$, then for each $A \in \{1, ..., k\}$ the k-Ricci curvature $\tilde{Ric}_{\mathcal{L}_k}(e_A)$ of $\mathcal{L}_k$ is defined by

$$\tilde{Ric}_{\mathcal{L}_k}(e_A) = \sum_{\alpha=1,\alpha\neq A} \mathcal{K}(e_\alpha \wedge e_A). \tag{13}$$

Similarly, a Riemannian invariant $\Phi_k x$ for $2 \leq k \leq m$ is defined as

$$\Phi_k(x) = (\frac{1}{k-1})_{\mathcal{L}_k, e_A} = inf\tilde{Ric}_{\mathcal{L}_k}(e_A) \quad , x \in \mathcal{M}. \tag{14}$$

The following consequences can be obtain from (3) and (8) as:

$$\tau(\mathcal{T}_x\mathcal{N}^{n_1}_{\theta_1}) = \sum_{r=n+1}^{m}\sum_{1\le i<j\le n_1}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) - \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_1}_{\theta_1}). \tag{15}$$

Similarly, we obtain

$$\tau(\mathcal{T}_x\mathcal{N}^{n_2}_{1}) = \sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}(h^r_{aa}h^r_{bb} - (h^r_{ab})^2) - \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_2}_{1}). \tag{16}$$

Let $\{e_1, ..., e_n\}$ be an orthonormal frame of vector field $\mathcal{M}^n$, thus the squared norm of gradient of the positive differentiable function φ for an orthonormal frame $\{e_1, ..., e_n\}$ is defined as follows:

$$\|\nabla_\varphi\|^2 = \sum_{i}^{n}(e_i(\varphi))^2. \tag{17}$$

According to B. Unal [22] for the unit vector fields X and Z tangent to M_1 and M_2, respectively, we have

$$K(X\wedge Z) = \frac{1}{\sigma_1}\{(\nabla^1_X X)\sigma_1 - X^2\sigma_1\} + \frac{1}{\sigma_2}\{(\nabla^2_Z Z)\sigma_2 - Z^2\sigma_2\}. \tag{18}$$

Let us assume a local orthonormal frame $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ such that $e_1, e_2, \ldots, e_{n_1}$ are tangent to M_1 and $e_{n_1+1}, \ldots, e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H. Then

$$\begin{aligned}\sum_{1\le i\le n_1}\sum_{n_1+1\le j\le n} K(e_i\wedge e_j) &= n_2\frac{\Delta_1\sigma_1}{\sigma_1} + n_1\frac{\Delta_2\sigma_2}{\sigma_2}\\ &= n_2\Big(\|\nabla^1(\ln\sigma_1)\|^2 - \Delta_1(\ln\sigma_1)\Big) + n_1\Big(\|\nabla^2(\ln\sigma_2)\|^2 - \Delta_2(\ln\sigma_2)\Big),\end{aligned} \tag{19}$$

where Δ_1, Δ_2 are the Laplacian operators and $\nabla^1(\ln\sigma_1)$ and $\nabla^2(\ln\sigma_2)$ are the gradient vectors on $M_1^{n_1}$ and $M_2^{n_2}$, respectively.

Finally, we conclude the section with the following definitions of pointwise slant and pointwise bi-slant submanifolds.

Definition 2.1. *[12] Let $\overline{M}^{2m}$ be an almost Hermitian manifold. Then, a submanifold M^n of $\overline{M}^{2m}$ is said a pointwise slant, if for each given point $x\in M^n$ and for any non-zero vector $X\in T_xM$, the angle $\theta(X)$ between JX and T_xM is free from the choice of X.*

In [12], Chen and Garay obtained necessary and sufficient condition for a submanifold to be a pointwise slant submanifold.They proved that a submanifold M of an almost Hermitian manifold $\tilde{M}$ is *pointwise slant* if and only if

$$T^2 = -(\cos^2\theta)I, \tag{20}$$

for some real-valued function θ defined on M, where I is the identity transformation of the tangent bundle TM of M.

On the other hand, Chen and Siraj Uddin generalized the above concept for a pointwise bi-slant submanifold as follows [11].

Definition 2.2. *Let $\overline{M}^{2m}$ be an almost hermitian manifold. Then, a submanifold M^n of $\overline{M}^{2m}$ is said a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$, such that*

(i) $TM^n = \mathfrak{D}_1\oplus\mathfrak{D}_2$,

(ii) $J\mathfrak{D}_1\perp\mathfrak{D}_2$ *and* $J\mathfrak{D}_2\perp\mathfrak{D}_1$;

(iii) Each distribution $\mathfrak{D}_i$ is a pointwise slant with a slant function $\theta_i : T^*M \to \mathbb{R}$ for $i = 1, 2$, where T^*M is a set of all non-zero vectors on M

Indeed, pointwise bi-slant submanifold englobe not only slant submanifold but also semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds.

Since M^n is a pointwise bi-slant submanifold, we define an adapted orthonormal frame fileds of the tangent space. If $n = 2d_1 + 2d_2$, where $d_1 = \frac{1}{2}\dim \mathfrak{D}_1$ and $d_2 = \frac{1}{2}\dim \mathfrak{D}_2$, the, we have the following frame field of the tangent space TM

$$\{e_1, e_2 = \sec\theta_1 Te_1, \cdots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 Te_{2d_1-1}, \cdots, e_{2d_1+1}, e_{2d_1+2} = \sec\theta_2 Te_{2d_1+1}, \cdots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 Te_{2d_1+2d_2-1}\}.$$

Then, by setting $g(e_1, Je_2) = -g(Je_1, e_2) = -g(Je_1, \sec\theta_1 Te_1)$, one can obtain $g(e_1, Je_2) = -\sec\theta_1 g(Te_1, Te_2)$. Following (20), we get $g(e_1, Je_2) = \cos\theta_1 g(e_1, e_2)$. This implies

$$g^2(e_i, Je_j) = \begin{cases} \cos^2\theta_1, & \forall\ i = 1, \cdots, 2d_1 - 1, \\ \cos^2\theta_2, & \forall\ j = 2d_1 + 1, \cdots, 2d_1 + 2d_2 - 1. \end{cases}$$

Hence, we have

$$\sum_{i,j=1}^{n} g^2(Je_i, e_j) = (n_1\cos^2\theta_1 + n_2\cos^2\theta_2). \tag{21}$$

3. Main theorem

This section is dedicated to the study of main result of the article.

Theorem 3.1. *Let $M = {}_{\sigma_2}N_{\theta_1}^{n_1} \times_{\sigma_1} N_{\theta_2}^{n_2} \to \tilde{M}^{2m}(c)$ be a D_{θ_i}-minimal isometric immersion of an n-dimensional doubly warped products pointwise bi-slant submanifold M into a generalized complex space form $\tilde{M}(c)$ for $i = 1$ or $i = 2$, where N_{θ_1} and N_{θ_2} are pointwise slant submanifolds. Then for each unit vector $X \in T_xM$, we have*

$$\begin{aligned} Ric(X) \leq\ & \frac{1}{4}n^2\|H\|^2 + n_2\|\nabla^1(ln\sigma_1)\|^2 - n_2\Delta(ln\sigma_1) + n_1\|\nabla^2(ln\sigma_2)\|^2 - n_1\Delta(ln\sigma_2) \\ +\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big). \end{aligned} \tag{22}$$

1. *If $\vec{H}(x) = 0$, then at each point $x \in M$ there is a unit tangent vector X satisfies the equality case in* (22) *if and only if M is mixed totally geodesic and X lies in the relative null space $\mathcal{N}_x$ at x.*
2. *For the equality cases, we have*
   - *(a) the equality case of* (22) *holds identically for all unit tangent vectors N_{θ_1} at each $x \in M$ if and only if M is mixed totally geodesic and D_{θ_1}-totally geodesic doubly warped product pointwise bi-slant submanifold in $\tilde{M}(c)$.*
   - *(b) the equality case of* (22) *holds identically for all unit tangent vectors N_{θ_2} at each $x \in M$ if and only if M is mixed totally geodesic and either D_{θ_2}-totally geodesic doubly warped product pointwise bi-slant submanifold or M is a D_{θ_2}-totally umbilical in $\tilde{M}(c)$ with $dimN_{\theta_2} = 2$.*
3. *the equality case of* (22) *holds identically for all unit tangent vectors M at each $x \in M$ if and only if M is mixed totally geodesic submanifold, or M is a mixed totally geodesic, totally umbilical and D_{θ_1}-totally geodesic warped product pointwise bi-slant submanifolds with $dimN_{\theta_2} = 2$.*

Proof. *From the Gauss equation, we have*

$$n^2\|H\|^2 = 2\tau(\mathcal{T}_x\mathcal{M}^n) + \|h\|^2 - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n). \tag{23}$$

Now, we assume that $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_n\}$ *to be a local orthonormal frame fields of* $\chi(\tilde{\mathcal{M}}^{2m}(c))$ *such that* $\{e_1, ..., e_{n_1}\}$ *are tangent to* $\mathcal{N}^{n_1}_{\theta_1}$ *and* $\{e_{n_1+1}, ..., e_n\}$ *are tangent to* $\mathcal{N}^{n_2}_{\theta_2}$*. So, for the unit tangent vector* $X = e_A \in \{e_1, ..., e_n\}$ *from (23), we get*

$$n^2\|H\|^2 = 2\tau(\mathcal{T}_x\mathcal{M}^n) + \frac{1}{2}\sum_{r=n+1}^{2m}\left\{(h^r_{11} + ... + h^r_{nn} - h^r_{AA})\right\} - \sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n} h^r_{\alpha\alpha}h^r_{\beta\beta} - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n). \tag{24}$$

We see that we can write equation (24) *as*

$$\begin{aligned} n^2\|H\|^2 &= 2\tau(\mathcal{T}_x\mathcal{M}^n) + \sum_{r=n+1}^{2m}\left\{(h^r_{11} + ... + h^r_{nn}) + (2h^r_{AA} - (h^r_{11} + ... + h^r_{nn}))^2\right\} \\ &+ 2\sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n}(h^r_{\alpha\beta})^2 - 2\sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n} h^r_{\alpha}h^r_{\beta} - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n). \end{aligned}$$

Since $\mathcal{M}^n$ *is a* $\mathcal{D}_{\theta_1}$*-minimal doubly warped product pointwise bi-slant submanifold, we obtain*

$$\begin{aligned} n^2\|H\|^2 &= \sum_{r=n+1}^{2m}\left\{(h^r_{n_1+1n_1+1} + ... + h^r_{nn}) + (2h^r_{AA} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2\right\} \\ &+ 2\tau(\mathcal{T}_x\mathcal{M}^n) + \sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n}(h^r_{\alpha\beta})^2 - \sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n} h^r_{\alpha\alpha}h^r_{\beta\beta} - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n) \\ &+ \sum_{r=n+1}^{2m}\sum_{\substack{a=1\\a\neq A}}^{n}(h^r_{aA})^2 + \sum_{r=n+1}^{2m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n}(h^r_{\alpha\beta})^2 - \sum_{r=n+1}^{2m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} h^r_{\alpha\alpha}h^r_{\beta\beta}. \end{aligned} \tag{25}$$

It follows from (8) that

$$\sum_{r=n+1}^{2m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n}(h^r_{\alpha\beta})^2 - \sum_{r=n+1}^{2m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} h^r_{\alpha\alpha}h^r_{\beta\beta} = \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n}\tilde{K}_{\alpha\beta} - \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} K_{\alpha\beta}. \tag{26}$$

Moreover, $\mathcal{D}_{\theta_1}$*-minimality of* $\mathcal{M}^n$ *implies*

$$n^2\|H\|^2 = \sum_{r=n+1}^{2m}(h^r_{n_1+1n_1+1} + ... + h^r_{nn})^2. \tag{27}$$

Using (26) and (27) in (25), we derive

$$\begin{aligned} \frac{1}{2}n^2\|H\|^2 &= 2\tau(\mathcal{T}_x\mathcal{M}^n) + \frac{1}{2}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2 \\ &+ \sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n}(h^r_{\alpha\beta})^2 - \sum_{r=n+1}^{2m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} h^r_{\alpha\alpha}h^r_{\beta\beta} - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n) \\ &+ \sum_{r=n+1}^{2m}\sum_{\substack{a=1\\a\neq A}}^{n}(h^r_{aA})^2 + \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n}\tilde{K}_{\alpha\beta} - \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} K_{\alpha\beta}. \end{aligned} \tag{28}$$

On the other hand from (4) and (7), we find

$$\begin{aligned}
\frac{1}{2}n^2\|H\|^2 &= \frac{n_2\Delta^1\sigma_1}{\sigma_1} + \frac{n_1\Delta^2\sigma_2}{\sigma_2} - 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n) \\
&+ \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} \tilde{K}_{\alpha\beta} + \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_1}_{\theta_1}) + \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_2}_{\theta_2}) \\
&+ \sum_{r=n+1}^{2m}\Big\{\sum_{1\le\alpha<\beta\le n}(h^r_{\alpha\beta})^2 - \sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}^{n} h^r_{\alpha\alpha}h^r_{\beta\beta}\Big\} \\
&+ \sum_{r=n+1}^{2m}\sum_{\substack{a=1\\ a\neq A}}^{n}(h^r_{aA})^2 + \sum_{r=n+1}^{2m}\sum_{1\le i\neq j\le n_1}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2). \\
&+ \sum_{r=n+1}^{2m}\sum_{n_1+1\le s\neq t\le n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\
&+ \frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2. \qquad (29)
\end{aligned}$$

Now, we remark that, for the unit tangent unit vector e_A, we have two choices: it is either tangent to base manifold $\mathcal{N}^{n_1}_{\theta_1}$ or fiber to $\mathcal{N}^{n_2}_{\theta_2}$. Next, we will prove for the first case.

Case 1. *If e_A tangent to $\mathcal{N}^{n_1}_{\theta_1}$, then we fix a unit tangent vector from $\{e_1, ..., e_{n_1}\}$ to be e_A and consider $X = e_A = e_1$. Then from (10) and (29), we obtain*

$$\begin{aligned}
\frac{1}{2}n^2\|H\|^2 &\ge Ric(X) + \frac{n_2\Delta^1\sigma_1}{\sigma_1} + \frac{n_1\Delta^2\sigma_2}{\sigma_2} \\
&- 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n) + \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_1}_{\theta_1}) + \tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_2}_{\theta_2}) + \sum_{2\le\alpha<\beta\le n}\tilde{K}_{\alpha\beta} \\
&+ \frac{1}{2}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2\} \\
&+ \sum_{r=n+1}^{2m}\sum_{1\le\alpha<\beta\le n}(h^r_{\alpha\beta})^2 - \sum_{r=n+1}^{2m}\Big\{\sum_{1\le i\neq j\le n_1}(h^r_{ij})^2 + \sum_{n_1+1\le s\neq n}(h^r_{st})^2\Big\} \\
&+ \sum_{r=n+1}^{2m}\Big\{\sum_{1\le i<j\le n_1}(h^r_{ii}h^r_{jj} + \sum_{n_1+1\le s\neq t\le n}h^r_{ss}h^r_{tt} - 2\sum_{2\le\alpha<\beta\le n}h^r_{\alpha\alpha}h^r_{\beta\beta}\Big\}.
\end{aligned} \qquad (30)$$

Combining (2) *and* (3)*, putting $X = Z = e_\alpha$, $Y = W = e_\beta$ and summing over $1 \le \alpha, \beta \le n$, we deduce that*

$$\sum_{\alpha,\beta}^{n}\tilde{R}(e_\alpha, e_\beta, e_\alpha, e_\beta) = f_1\Big\{n(n-1)\Big\} + 3f_2\sum_{i,j=1}^{n}g^2(Je_i, e_j). \qquad (31)$$

From (21) and (30), it is easy to see that

$$
\begin{aligned}
Ric(X) \leq\ & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta^1\sigma_1}{\sigma_1} - \frac{n_1\Delta^2\sigma_2}{\sigma_2} \\
+\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) \\
-\ & \frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \ldots + h^r_{nn}))^2\} + \sum_{r=n+1}^{2m}\Big\{\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 + \sum_{n_1+1\leq s\neq t\leq n}(h^r_{st})^2\Big\} \\
-\ & \sum_{r=n+1}^{2m}\Big\{\sum_{1\leq i<j\leq n_1}(h^r_{ii}h^r_{jj} + \sum_{n_1+1\leq s\neq t\leq n}h^r_{ss}h^r_{tt}\Big\} + \sum_{r=n+1}^{2m}\sum_{2\leq\alpha<\beta\leq n}h^r_{\alpha\alpha}h^r_{\beta\beta} - \sum_{r=n+1}^{2m}\sum_{1\leq\alpha<\beta\leq n}(h^r_{\alpha\beta})^2. \quad (32)
\end{aligned}
$$

But from the last two terms of (32) we see that

$$
\sum_{r=n+1}^{2m}\Big\{\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 + \sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2\Big\} - \sum_{r=n+1}^{2m}\sum_{1\leq\alpha<\beta\leq n}(h^r_{\alpha\beta})^2 = \sum_{r=n+1}^{2m}\sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+}^{n}(h^r_{\alpha\beta})^2. \quad (33)
$$

Similarly, we have

$$
\sum_{r=n+1}^{2m}\Big\{\sum_{1\leq i<j\leq n_1}h^r_{ii}h^r_{jj} + \sum_{n_1+1\leq s\neq t\leq n}h^r_{ss}h^r_{tt} - \sum_{2\leq\alpha<\beta\leq n}h^r_{\alpha}h^r_{\beta}\Big\} = \sum_{r=n+1}^{2m}\Big(\sum_{j=2}^{n_1}h^r_{11}h^r_{jj} - \sum_{\alpha=2}^{n_1}\sum_{\beta=n_1+1}^{n}h^r_{\alpha\alpha}h^r_{\beta\beta}\Big). \quad (34)
$$

Furthermore, (34) and (30) gives

$$
\begin{aligned}
Ric(X) \leq\ & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta^1\sigma_1}{\sigma_1} - \frac{n_1\Delta^2\sigma_2}{\sigma_2} \\
+\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) \\
-\ & \sum_{r=n+1}^{2m}\Big(\sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2 + \sum_{b=2}^{n_1}h^r_{11}h^r_{bb} - \sum_{\alpha=2}^{n_1}\sum_{\beta=n_1+1}^{n}h^r_{\alpha\alpha}h^r_{\beta\beta}\Big) \\
-\ & \frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \ldots + h^r_{nn}))^2\}. \quad (35)
\end{aligned}
$$

Taking into account $\mathcal{M}^n$ is $\mathcal{D}_{\theta_1}$ -minimality doubly warped product pointwise bi-slant submanifold, we compute

$$
\begin{aligned}
\sum_{r=n+1}^{2m}\sum_{\alpha=2}^{n_1}\sum_{\beta=n_1+1}^{n}h^r_{\alpha\alpha}h^r_{\beta\beta} &= \sum_{r=n+1}^{2m}\sum_{\beta=n_1+}^{n}\Big\{g(h(e_2,e_2),e_r) + \ldots + g(h(e_{n_1},e_{n_1}),e_r)\Big\}h^r_{\beta\beta} \\
&= \sum_{r=n+1}^{2m}\sum_{\beta=n_1+}^{n}\Big\{g(h(e_1,e_1),e_r) + \ldots + g(h(e_{n_1},e_{n_1}),e_r) - g(h(e_1,e_1),e_r)\Big\}h^r_{\beta\beta} \\
&= -\sum_{r=n+1}^{2m}\sum_{\beta=n_1+}^{n}h^r_{11}h^r_{\beta\beta}. \quad (36)
\end{aligned}
$$

Similarly, we have

$$
\sum_{r=n+1}^{2m}\sum_{b=2}^{n_1}h^r_{11}h^r_{bb} = -\sum_{r=n+1}^{2m}(h^r_{11})^2. \quad (37)
$$

Further, by $\mathcal{D}_{\theta_1}$-minimality, it follows that

$$\begin{aligned}
&\frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{11}-(h^r_{n_1+1n_1+1}+...+h^r_{nn}))^2\}+\sum_{r=n+1}^{2m}\sum_{\beta=n_1+1}^{n}h^r_{11}h^r_{\beta\beta}\\
&\qquad=2\sum_{r=n+1}^{2m}(h^r_{11})^2+\frac{1}{2}\sum_{r=n+1}^{2m}(h^r_{n_1+1n_1+1}+...+h^r_{nn}))^2\}\\
&\qquad-\sum_{r=n+1}^{2m}h^r_{11}(h^r_{n_1+1n_1+1}+...+h^r_{nn}))\}+\sum_{r=n+1}^{2m}\sum_{\beta=n_1+1}^{n}h^r_{11}h^r_{\beta\beta}\\
&\qquad=2\sum_{r=n+1}^{2m}(h^r_{11})^2+\frac{1}{2}n^2\|H\|^2. \qquad (38)
\end{aligned}$$

Equations (36), (37), (35) and (38) yield the following relation

$$\begin{aligned}
Ric(X) \;\leq\;& \frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta^1\sigma_1}{\sigma_1}-\frac{n_1\Delta^2\sigma_2}{\sigma_2}\\
+\;& f_1\Big(n_1(n_2+n-1\Big)+\frac{3}{2}f_2\Big(cos^2\theta_1+cos^2\theta_2\Big)\\
-\;& \sum_{r=n+1}^{2m}\Big\{(h^r_{11})^2-\sum_{\beta=n_1+1}^{n}h^r_{11}h^r_{\beta\beta}+\frac{1}{4}\sum_{r=n+1}^{2m}(h^r_{n_1+1n_1+1}+...+h^r_{nn}))^2\Big\}\\
-\;& \frac{1}{4}\sum_{r=n+1}^{2m}(h^r_{n_1+1n_1+1}+...+h^r_{nn})^2\}. \qquad (39)
\end{aligned}$$

Making use of (27) *in (39), we conclude*

$$\begin{aligned}
Ric(X) \;\leq\;& \frac{1}{4}n^2\|H\|^2-\frac{n_2\Delta^1\sigma_1}{\sigma_1}-\frac{n_1\Delta^2\sigma_2}{\sigma_2}\\
+\;& f_1\Big(n_1n_2+n-1\Big)+\frac{3}{2}f_2\Big(cos^2\theta_1+cos^2\theta_2\Big)-\frac{1}{2}\sum_{r=n+1}^{2m}\Big(2h^r_{11}-\sum_{\beta=n_1+1}^{n}h_{\beta\beta}\Big)^2, \qquad (40)
\end{aligned}$$

which together with (19) yield (22).
For the other case, we have
Case 2. *If e_A tangent to $\mathcal{N}^{n_2}_{\theta_2}$, then we fix a unit tangent vector from $\{e_{n_1+1},...,e_n\}$ such that $X=e_A=e_n$. From (10) to (30) and using similar analogue to* **Case 1**, *we obtain*

$$\begin{aligned}
\frac{1}{4}n^2\|H\|^2 \;\geq\;& Ric(X)+\frac{n_2\Delta^1\sigma_1}{\sigma_1}+\frac{n_1\Delta^2\sigma_2}{\sigma_2}\\
-\;& 2\tilde{\tau}(\mathcal{T}_x\mathcal{M}^n)+\tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_1}_{\theta_1})+\tilde{\tau}(\mathcal{T}_x\mathcal{N}^{n_2}_{\theta_2})\\
+\;& \sum_{1\leq\alpha<\beta\leq n}\tilde{K}_{\alpha\beta}+\frac{1}{2}(2h^r_{nn}-(h^r_{n_1+1n_1+1}+...+h^r_{nn}))^2\\
+\;& \sum_{r=n+1}^{2m}\sum_{\beta=1}^{n-1}h^r_{nn}h^r_{\beta\beta}+\sum_{r=n+1}^{2m}\sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2-\sum_{r=n+1}^{2m}\sum_{\alpha=2}^{n_1}\sum_{\beta=n_1+1}^{n}h^r_{\alpha\alpha}h^r_{\beta\beta}. \qquad (41)
\end{aligned}$$

By using (21) in the last equation, we find that

$$\begin{aligned}
Ric(X) \leq\ & \frac{1}{2}n^2\|H\|^2 + n_2\|\nabla^1(ln\sigma_1)\|^2 - n_2\Delta^1(ln\sigma_1) \\
+\ & n_1\|\nabla^2(ln\sigma_2)\|^2 - n_1\Delta^2(ln\sigma_2) \\
+\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) \\
-\ & \frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{nn} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2\} \\
-\ & \sum_{r=n+1}^{2m}\sum_{\beta=n_1+1}^{n-1} h^r_{nn}h_{\beta\beta} - \sum_{r=n+1}^{2m}\sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2 \\
-\ & \sum_{r=n+1}^{2m}\sum_{\alpha=2}^{n_1}\sum_{\beta=n_1+1}^{n} h^r_{\alpha\alpha}h^r_{\beta\beta}. \qquad (42)
\end{aligned}$$

Application of $\mathcal{D}_{\theta_1}$-minimality of $\mathcal{M}^n$ in (42) gives

$$\begin{aligned}
Ric(X) \leq\ & \frac{1}{2}n^2\|H\|^2 + n_2\|\nabla^1(ln\sigma_1)\|^2 - n_2\Delta(ln\sigma_1) + n_1\|\nabla^2(ln\sigma_2)\|^2 - n_1\Delta(ln\sigma_2) \\
+\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) \\
-\ & \frac{1}{2}\sum_{r=n+1}^{2m}(2h^r_{nn} - (h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2 \\
-\ & \sum_{r=n+1}^{2m}\sum_{\beta=1}^{n-1} h^r_{nn}h_{\beta\beta} - \sum_{r=n+1}^{2m}\sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2. \qquad (43)
\end{aligned}$$

On the other hand, by a straight forward but lengthy computation we obtain that

$$\begin{aligned}
&\sum_{r=n+1}^{2m}\Big\{\frac{1}{2}((h^r_{n_1+1n_1+1} + ... + h^r_{nn})) - 2h^r_{nn})^2 + \sum_{\beta=n+1}^{n-1} h^r_{nn}h^r_{\beta\beta}\Big\} \\
&= \sum_{r=n+1}^{2m}\frac{1}{2}(h^r_{n_1+1n_1+1} + ... + h^r_{nn})^2 + \sum_{r=n+1}^{2m}2(h^r_{nn})^2 - \sum_{r=n+1}^{2m}\sum_{\beta=1}^{n_1} h_{nn}h_{\beta\beta} - \sum_{r=n+1}^{2m}\sum_{\beta=n_1+1}^{n} h^r_{nn}h^r_{\beta\beta} \\
&= \sum_{r=n+1}^{2m}\frac{1}{2}(h^r_{n_1+1n_1+1} + ... + h^r_{nn})^2 + \sum_{r=n+1}^{2m}2(h^r_{nn})^2 - \sum_{r=n+1}^{2m}(h^r_{nn})^2 \\
&= \sum_{r=n+1}^{2m}\Big\{\frac{1}{2}(h^r_{n_1+1n_1+1} + ... + h^r_{nn}) + h^r_{nn})^2 - \sum_{\beta=n+1}^{n-1} h^r_{nn}h^r_{\beta\beta}\Big\}. \qquad (44)
\end{aligned}$$

Thus, by using (44) in (43), we deduce that

$$\begin{aligned}
Ric(X) \leq\ & \frac{1}{2}n^2\|H\|^2 + n_2\|\nabla^1(ln\sigma_1)\|^2 - n_2\Delta(ln\sigma_1) + n_1\|\nabla^2(ln\sigma_2)\|^2 - n_1\Delta(ln\sigma_2) \\
+\ & f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) - \frac{1}{4}\sum_{r=n+1}^{2m}(h^r_{n_1+1n_1+1} + ... + h^r_{nn})^2 \\
-\ & \sum_{r=n+1}^{2m}\Big\{(h^r_{nn})^2 - \sum_{\beta=n_1+1}^{n} h^r_{nn}h^r_{\beta\beta} + \frac{1}{4}(h^r_{n_1+1n_1+1} + ... + h^r_{nn}))^2\Big\}. \qquad (45)
\end{aligned}$$

Finally, it follows from $\mathcal{D}_{\theta_1}$-minimality of $\mathcal{M}^n$ that

$$\begin{aligned} Ric(X) &\leq \frac{1}{2}n^2\|H\|^2 + n_2\|\nabla^1(ln\sigma_1)\|^2 - n_2\Delta(ln\sigma_1) + n_1\|\nabla^2(ln\sigma_2)\|^2 - n_1\Delta(ln\sigma_2) \\ &+ f_1\Big(n_1n_2 + n - 1\Big) + \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big) - \frac{n^2}{4}\|H\|^2 - \sum_{r=n+1}^{2m}\Big(2h^r_{nn} - \frac{1}{2}\sum_{\beta=n_1+1}^{n} h^r_{\beta\beta}\Big)^2. \end{aligned} \tag{46}$$

This concludes the proof of the inequality (22). *To derive the inequality* (22), *when warped product pointwise bi-slant submanifold $\mathcal{M}^n$ is $\mathcal{D}_{\theta_1}$-minimal, we will use a similar techniches as in* **Case 1**. *Hence, we conclude that the inequality* (22) *holds for the both $\mathcal{D}_{\theta_1}$-minimal isometric immersion for $i = 1$ or 2. Now, we will verify the equality cases in the inequality* (22). *Let us consider the relative null space $\mathcal{N}_x$ of the warped product pointwise bi-slant submanifold $\mathcal{M}^n$ in a complex spaces $\tilde{\mathcal{M}}^{2m}(c)$. For $A \in \{e_1, ..., e_n\}$ a unit tangent vector e_A to $\mathcal{M}^n$ at x satisfies the equality sign of* (22), *if and only if the following conditions hold:*

$$\begin{cases} (i) \quad \sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2 = 0, \\ (ii) \quad \sum_{b=1, b\neq A}^{2m}(h^r_{bA})^2 = 0, \\ (iii) \quad 2h^r_{AA} = \sum_{\beta=n_1+1}^{n}(h_{\beta\beta}), \end{cases} \tag{47}$$

such that $r \in \{\ e_{n+1}, ..., e_{2m}\}$. The first condition (i) implies that $\mathcal{M}^n$ is a mixed totally geodesic warped product pointwise bi-slant submanifold. Using the fact of minimality and combining (ii) and (iii) of (47), it can be easily seen that the unit tangent vector $X = e_{AQ}$ lies in the relative null space $\mathcal{N}_x$ at x. The converse part is straightforward and hence, we complete the proof of (1) of the inequality (22), *Moreover for $\mathcal{D}_{\theta_1}$-minimal isometric warped product pointwise bi-slant submanifold, the equality condition in* (22) *hold if and only if*

$$\begin{cases} (i) \quad \sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2 = 0, \\ (ii) \quad \sum_{b=1}^{n}\sum_{A=1, b\neq A}^{n_1}(h^r_{bA})^2 = 0, \\ (iii) \quad 2h^r_{\alpha\alpha} = \sum_{\beta=n_1+1}^{n}(h_{\beta\beta}), \end{cases} \tag{48}$$

where $\alpha \in \{1, ..., n_1\}$ and $r \in \{n+1, ..., 2m\}$. As $\mathcal{M}^n$ is a $\mathcal{D}_{\theta_1}$-minimal, then the third term of (48) implies that $h^r_{\alpha\alpha} = 0$, $\alpha \in \{1, ..., n_1\}$. So combining these condition with the second term (ii) of (48), we find that $\mathcal{M}^n$ is a $\mathcal{D}_{\theta_1}$-totally geodesic warped product pointwise bi-slant submanifold in a complex space form $\tilde{\mathcal{M}}^{2m}(c)$. This proves the statement (a) of (2).
As we assume that $\mathcal{M}^n$ is a $\mathcal{D}_{\theta_1}$-minimal, the equality sign hold in (22) *for all unit tangent vectors to $\mathcal{N}^{n_2}_{\theta_2}$ at x if and only if the following condition are satisfied:*

$$\begin{cases} (i) \quad \sum_{\alpha=1}^{n_1}\sum_{\beta=n_1+1}^{n}(h^r_{\alpha\beta})^2 = 0, \\ (ii) \quad \sum_{b=1}^{n}\sum_{A=n_1+1, b\neq A}^{n_1}(h^r_{bA})^2 = 0, \\ (iii) \quad 2h^r_{LL} = \sum_{\beta=n_1+1}^{n}(h_{\beta\beta}), \end{cases} \tag{49}$$

such that $L \in \{n_1 + 1, ..., n\}$ and $r \in \{n + 1, ..., 2m\}$. There are two cases which arises from third condition (iii) of (49), that is

$$\begin{aligned} h^r_{LL} = 0, \quad \forall L \in \{n_1 + 1, ..., n\} \quad and \quad r \in \{n + 1, ..., 2m\}, \\ or \quad dim\mathcal{N}^{n_2}_{\theta_2} = 2. \end{aligned} \tag{50}$$

If the first part of (50) holds, then in the light of second condition in (49), we get that $\mathcal{M}^n$ is a $\mathcal{D}_{\theta_2}$ is totally geodesic warped product pointwise bi-slant submanifold in complex space form $\tilde{\mathcal{M}}^{2m}(c)$. This is the first statement of part (b) of (2) of the theorem. For the other part, we consider that $\mathcal{M}^n$ is a $\mathcal{D}_{\theta_1}$ is not totally geodesic warped product *submanifold and* $dim\mathcal{N}_{\theta_2}^{n_2} = 2$*, then from the (ii) of (49), we hypothesize that* $\mathcal{M}^n$ *is a* $\mathcal{D}_{\theta_2}$ *umbilical warped product pointwise bi-slant submanifold in* $\tilde{\mathcal{M}}^{2m}(c)$*. Hence, the part (b) of (2) is proved completely.*

Now, to prove (3), we emerge (48) and (49), together and we use the part (a) and (b) of (2). thus, let consider that $dim\mathcal{N}_{\theta_2}^{n_2} \neq 2$*. Since from part (a) and (b) of statements (3), respectively imply that* $\mathcal{M}^n$ *is a* $\mathcal{D}_{\theta_1}$ *is totally geodesic and* $\mathcal{D}_{\theta_2}$ *totally geodesic submanifold in* $\tilde{\mathcal{M}}^{2m}(c)$*, this means that* $\mathcal{M}^n$ *totally geodesic warped product pointwise bi-slant submanifold in* $\tilde{\mathcal{M}}^{2m}(c)$*. Moreover, for the other case, we assume that the previous does not hold then from part (a) and (b) of statements (2) gives that* $\mathcal{M}^n$ *is mixed totally geodesic and* $\mathcal{D}_{\theta_1}$*-totally geodesic warped product pointwise bi-slant submanifold in* $\tilde{\mathcal{M}}^{2m}(c)$ *with* $dim\mathcal{N}_{\theta_2}^{n_2} = 2$*. As for last assertion to show that* $\mathcal{M}^n$ *is a totally umbilical warped product pointwise bi-slant submanifold into complex space form* $\tilde{\mathcal{M}}^{2m}(c)$*, it is sufficient to prove that* $\mathcal{M}^n$ *is* $\mathcal{D}_{\theta_2}$ *and* $\mathcal{D}_{\theta_1}$*-totally geodesic warped product pointwise bi-slant submanifold in* $\tilde{\mathcal{M}}^{2m}(c)$ *which comes directly from (b) and (a), respectively. this gives the complete proof of part (3). By similar technique as in the above case, we can prove the theorem when* $\mathcal{M}^n$ *is* $\mathcal{D}_{\theta_2}$*-minimal warped product pointwise bi-slant submanifold in a complex space form* $\tilde{\mathcal{M}}^{2m}(c)$*. This completes the proof of the theorem.* □

4. Application of the results in Physics

In this section, we obtain physical applications of the main result.

4.1. Results on doubly warped product poimtwise bi-slant submanifolds with harmonic function

Theorem 4.1. *Let* $M = {}_{\sigma_2}N_{\theta_1}^{n_1} \times_{\sigma_1} N_{\theta_2}^{n_2} \to \tilde{M}^{2m}(c)$ *be a* D_{θ_i}*-minimal isometric immersion of an n-dimensional doubly warped products poimtwise bi-slant submanifold M into a generalized complex space form* $\tilde{M}(c)$ *for* $i = 1$ *or* 2*. Then for each unit vector* $X \in T_xM$*, if the warping functions* σ_1 *and* σ_2 *are harmonic functions, we have*

$$\frac{1}{4}n^2|H||^2 \geq Ric(X) - f_1\Big(n_1n_2 + n - 1\Big) - \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big). \tag{51}$$

Proof. If σ_1 and σ_2 are harmonic functions, then $\Delta^1\sigma_1 = 0$ and $\Delta^2\sigma_2 = 0$. Using this fact in (22) yields the results. □

4.2. Results on doubly warped product poimtwise bi-slant submanifolds related to Hessian functions

Let ϕ be a positive differentiable C^∞-differentiable function. Then the Hessian tensor of function ϕ is a symmetric 2-covariant tensor field on M^n defined by

$$\mathcal{H}^\phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M) \tag{52}$$

such that

$$\mathcal{H}^\phi(X, Y) = \mathcal{H}_{ij}^\phi X^iY^j, \tag{53}$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathcal{H}_{ij}^\phi$ can be expressed as

$$\mathcal{H}_{ij}^\phi = \frac{\partial^2\phi}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial\phi}{\partial x_k}. \tag{54}$$

Let us assume that $\phi = ln\sigma_1 = ln\sigma_2$. Then as a consequence of the Theorem 3.1 and the above relation, we conclude the following result.

Theorem 4.2. *Let* $M = {}_{\sigma_2}N_{\theta_1}^{n_1} \times_{\sigma_1} N_{\theta_2}^{n_2} \to \tilde{M}^{2m}(c)$ *be a* D_{θ_i}*-minimal isometric immersion of an n-dimensional doubly warped products poimtwise bi-slant submanifold M into a generalized complex space form* $\tilde{M}(c)$*. Then for each unit vector* $X \in T_xM$*, we have*

$$\frac{1}{4}n^2|H||^2 \geq Ric(X) + n_2\frac{trace\mathcal{H}^\phi}{\sigma_1} + n_1\frac{trace\mathcal{H}^\phi}{\sigma_2} - f_1\Big(n_1n_2 + n - 1\Big) - \frac{3}{2}f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big). \tag{55}$$

4.3. Results on doubly warped product poimtwise bi-slant submanifolds related to Dirichlet energy functions

A great motivation of bound of Ricci curvature is to express the Dirichlet energy of the warping functions σ_1 and σ_2, which is a useful tool in physics. The Dirichlet energy of any function ς on a compact manifold M is defined as:

$$E(\varsigma) = \frac{1}{2}\int_M \|\nabla\varsigma\|^2 dV, \tag{56}$$

where $\nabla\varsigma$ is the gradient of ς and dV is the volume element.

Theorem 4.3. *Let $M = {}_{\sigma_2}N^{n_1}_{\theta_1} \times_{\sigma_1} N^{n_2}_{\theta_2} \to \tilde{M}^{2m}(c)$ be a D_{θ_i}-minimal isometric immersion of an n-dimensional compact oriented doubly warped products pointwise bi-slant submanifold M into a generalized complex space form $\tilde{M}(c)$ for $i = 1$ or $i = 2$, then*

$$\begin{aligned} n_2E(ln\sigma_1) + n_1E(ln\sigma_2) \geq & \frac{1}{2}\int_M \Big(Ric(X) - \frac{1}{4}n^2|H||^2\Big)dV \\ & - \frac{1}{2}\Big[\int_M f_1\Big(n_1n_2 + n - 1\Big)dV + \frac{3}{2}\int_M f_2\Big(cos^2\theta_1 + cos^2\theta_2)dV\Big]. \end{aligned} \tag{57}$$

where $vol(N_{\theta_1})$ is the volume N_{θ_1}.

Proof. Taking integration along M with respect to volume element dV, we get

$$\begin{aligned} \int_M Ric(X)dV \leq\ & \frac{1}{4}n^2\int_M \|H\|^2 dV + n_2\int_M \|\nabla^1(ln\sigma_1)\|^2 dV \\ -\ & n_2\int_M \Delta(ln\sigma_1)dV + n_1\int_M \|\nabla^2(ln\sigma_2)\|^2 dV \\ -\ & n_1\int_M \Delta(ln\sigma_2)dV + \int_M f_1\Big(n_1n_2 + n - 1\Big)dV \\ +\ & \frac{3}{2}\int_M f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big)dV. \end{aligned} \tag{58}$$

Since M is compact, (58) implies

$$\begin{aligned} \int_M Ric(X)dV \leq\ & \frac{1}{4}n^2\int_M \|H\|^2 dV + n_2\int_M \|\nabla^1(ln\sigma_1)\|^2 dV + n_1\int_M \|\nabla^2(ln\sigma_2)\|^2 dV \\ +\ & \int_M f_1\Big(n_1n_2 + n - 1\Big)dV + \frac{3}{2}\int_M f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big)dV. \end{aligned} \tag{59}$$

Now, Making use of (56) in (60), we find

$$\begin{aligned} \int_M Ric(X)dV \leq\ & \frac{1}{4}n^2\int_M \|H\|^2 dV + 2n_2E(ln\sigma_1) + 2n_1E(ln\sigma_2) \\ +\ & \int_M f_1\Big(n_1n_2 + n - 1\Big)dV + \frac{3}{2}\int_M f_2\Big(cos^2\theta_1 + cos^2\theta_2\Big)dV, \end{aligned} \tag{60}$$

which is the desired inequality. □

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