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Tangent bundle of unit 2-sphere and slant ruled surfaces

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Abstract. In this paper, an isomorphism between unit dual sphere, DS^2, and the subset of tangent bundle of unit 2-sphere, $T\bar{M}$, is represented. According to E. Study mapping, a ruled surface in IR^3 corresponds to each curve on DS^2. Through this isomorphism, new forms of ruled surfaces called slant ruled surfaces in IR^3 were introduced. Moreover, conditions for these surfaces to be slant ruled surfaces were given. Finally, a unique $\bar{q}-$, $\bar{h}-$ and $\bar{a}-$ slant ruled surfaces in IR^3 were corresponded to each striction curve of natural lift curve on $T\bar{M}$.

1. Introduction

The theory of surfaces and curves has been applied to many research fields in geometry, physics, surface design, etc. Especially, the characterizations of curves and surfaces in Riemannian geometry are given in [5]. The definition of natural lift curve was first encountered in J. A. Thorpe's book, see [8]. The natural lift curve is defined as the curve generated by the endpoints of tangent vectors of main curve. There are some studies about the geometric interpretations and dual spherical curves of natural lift curve, see [4],[6], [7].

The curves which satisfy special conditions for the curvatures have significiant role in differential geometry. The most commonly known of such curves are helices. In IR^3, a general helix is the curve whose tangent vector makes a constant angle with a fixed straight line. Helices have been studied in different spaces by some mathematicians in literature. The definition of slant helix is defined as the curve whose normal vector makes a constant angle with a fixed direction. Characterizations of slant helices have been studied by some authors, see [17-21].

Ruled surface is defined as a surface formed by a one-parameter set of straight lines. Ruled surfaces have been studied in physics, differential geometry, geometric design problems and manufacturing, see [1-3]. In literature, the characterizations and concepts of ruled surfaces have been studied by many authors e.g., [3, 9-15]. Characteristic properties of ruled surfaces, which are associated with the geodesic curvature, the normal curvature and the geodesic torsion, are investigated, see [13]. Some results and the distribution parameters of the ruled surface are presented with special cases, see [14]. Frenet frames and invariants of timelike ruled surfaces are given, see [12]. The kinematic interpretations between timelike ruled surfaces and associated surfaces are introduced, see [15]. Some new types of ruled surfaces called slant ruled surfaces are defined, see [16]. In the same study, conditions for being a slant ruled surface are given in IR^3.

Dual numbers were first introduced by Clifford, see [23]. Then the correspondence between the geometry

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of lines and unit dual sphere, DS^2 is given by E. Study, see [22]. The relation among DS^2, the tangent bundle of unit 2-sphere, TS^2, and ruled surfaces in IR^3 is presented, see [10]. Then through this relation, a unique ruled surface in IR^3 has been corresponded to each curve on TS^2, see [9]. The correspondence between ruled surfaces in IR^3_1 and the curves on tangent bundle of pseudo-sphere is given by same authors, see [11].

Moreover, it is well known that a ruled surface has an orthonormal base along its striction curve. This frame is defined as Frenet frame of ruled surface. In this paper, a one-to-one correspondence is given between DS^2 and TS^2. By using E. Study mapping, a ruled surface in IR^3 corresponds to each curve on DS^2. In this study, we define a new ruled surface called slant ruled surface. That is, a unique $\bar{q}-$, $\bar{h}-$ and $\bar{a}-$ slant ruled surface in IR^3 has been corresponded to each striction curve of natural lift curve. Furthermore, considering the striction curve of natural lift curve and the Frenet frame of a ruled surface, the definitions of some special ruled surfaces whose Frenet vectors make a constant angle with some fixed directions in IR^3 are given.

2. Preliminaries

In this section, tangent bundle of unit 2-sphere, concepts of natural lift curve of a given curve and properties of slant ruled surfaces are considered.

Assume that S^2 is the unit 2-sphere in IR^3. The tangent bundle of S^2 is

$$TS^2 = \{(q, \vartheta) \in IR^3 \times IR^3 : |q| = 1, \ \langle q, \vartheta \rangle = 0\}, \tag{1}$$

where "$\langle , \rangle$" is the inner product and "$|,|$" is the norm in IR^3, respectively, see [9].

Assume that $T\bar{M}$ is a subset of TS^2, defined by

$$T\bar{M} = \{(\bar{q}, \bar{\vartheta}) \in IR^3 \times IR^3 : |\bar{q}| = 1, \ \langle \bar{q}, \bar{\vartheta} \rangle = 0\}, \tag{2}$$

where $\bar{q}$ and $\bar{\vartheta}$ are the derivatives of q and ϑ, respectively, see [4].

Definition 2.1. *Let $\Gamma : I \longrightarrow \bar{M}$ be a smooth curve. Here $\bar{M}$ represents a surface in IR^3. Γ is called an integral curve of X*

$$\frac{d(\Gamma(u))}{du} = X(\Gamma(u)), \tag{3}$$

where X is smooth tangent vector field on $\bar{M}$, see [6].

Definition 2.2. *For the curve Γ, $\bar{\Gamma}$ is called the natural lift of Γ on $T\bar{M}$, which produces in the following equation:*

$$\bar{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}(u)) = (q^{'}(u)|_{\gamma(u)}, \vartheta^{'}(u)|_{\vartheta(u)}). \tag{4}$$

Accordingly, we can write

$$\frac{d\bar{\Gamma}(u)}{du} = \frac{d}{du}(\Gamma^{'}(u)|_{\Gamma(u)}) = D_{\Gamma'(u)}\Gamma^{'}(u).$$

Here D refers the Levi-Civita connection in IR^3. We have

$$T\bar{M} = \bigcup T_p\bar{M}, \ \ p \in \bar{M},$$

where $T_p\bar{M}$ is taken as the tangent space of $\bar{M}$ at p. $\chi(\bar{M})$ is the space of vector fields on $\bar{M}$, see [6].

Given a one-parameter family lines $\{\vec{k}(u), \vec{q}(u)\}$, the parametric represantation of ruled surface ϕ obtained by the family $\{\vec{k}(u), \vec{q}(u)\}$ is

$$\vec{h}(u, v) = \vec{k}(u) + v\vec{q}(u), \ \ u \in I, \ \ v \in IR \tag{5}$$

where $k = \vec{k}(u)$ presents a point and $\bar{q} = \vec{q}(u)$ denotes a non-null vector in IR^3. Also $\vec{k}(u)$ and $\vec{q}(u)$ are the base curve and various of the generating lines for the ruled surface ϕ, respectively, see [5].

For the unit normal vector $\vec{m}$ of the ruled surface, we get

$$m = \frac{\vec{h}_u \times \vec{h}_v}{|\vec{h}_u \times \vec{h}_v|} = \frac{(\vec{k}' + v\vec{q}') \times \vec{q}}{\sqrt{|\langle \vec{k}', \vec{q}\rangle^2 - \langle \vec{q}, \vec{q}\rangle \langle \vec{k}' + v\vec{q}, \vec{k}' + v\vec{q}\rangle|}}. \tag{6}$$

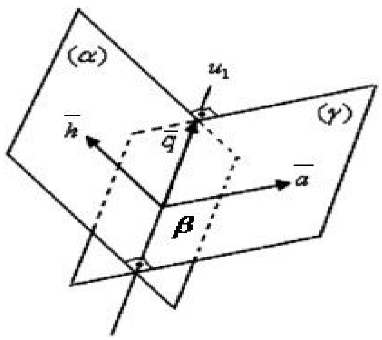

Figure 1: Asymptotic plane and central plane

Along a rulling $u = u_1$, we define

$$\vec{a} = \lim_{v \to \infty} \vec{m}(u_1, v) = \frac{\vec{q}' \times \vec{q}}{|\vec{q}'|}. \tag{7}$$

The plane of ruled surface ϕ which goes through its rulling u_1. It makes a right angle with the vector $\vec{a}$, which is defined as asymptotic plane α. The tangent plane γ goes through the rulling u_1 and makes a right angle with the asymptotic plane, which is defined as central plane. The point, where $\vec{m}$ makes a right angle with $\vec{a}$ is defined as the striction point (or central point) β on the rulling u_1(Fig. 1). The set of central points of all rullings is defined as striction curve of the ruled surface. The straight lines go through point β. They make a right angle with to the planes α and γ are defined as central tangent and central normal, respectively.

Taking the orthogonality of the vectors $\vec{q}$, $\vec{q}'$ and correspondence (7), the unit vector $\vec{h}$ of the central normal is represented as:

$$\vec{h} = \frac{\vec{q}'}{|\vec{q}'|}. \tag{8}$$

Substituting the parameter v of central point β into (6) we have $\vec{h} \times \vec{m} = 0$. Therefore, the following equation is presented as:

$$\vec{q}' \times [(\vec{k}' + v\vec{q}') \times \vec{q}] = \langle \vec{q}', \vec{k}' \rangle (\vec{k}' + v\vec{q}') + v\langle \vec{q}', \vec{q}' \rangle \vec{q} = 0. \tag{9}$$

From (9), we acquire

$$v = -\frac{\langle \vec{q}', \vec{k}' \rangle}{\langle \vec{q}', \vec{q}' \rangle}. \tag{10}$$

According to the arc length parameter u, the parametrization of the striction curve of the ruled surface is defined as:

$$\vec{c}(u) = \vec{k}(u) - \frac{\langle \vec{q}'(u), \vec{k}'(u)\rangle}{\langle \vec{q}'(u), \vec{q}'(u)\rangle}\vec{q}(u). \tag{11}$$

If the striction curve $\vec{c}(u)$ coincides the base curve $\vec{k}(u)$, the ruled surface is called a developable ruled surface, see [12].

The orthonormal system $\{\beta, \vec{q}, \vec{h}, \vec{a}\}$ is defined as Frenet frame of the ruled surface ϕ along the striction curve. For the Frenet formula of the ruled surface ϕ with respect to the arc length u of striction curve, we have

$$\begin{pmatrix} \vec{q}' \\ \vec{h}' \\ \vec{a}' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{pmatrix},$$

where $\vec{q}, \vec{h}$ and $\vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively. Here k_1 and k_2 denote curvature and torsion of ϕ in turn, see [16].

The orthonormal system $\{\vec{q}, \vec{h}, \vec{a}\}$ is obtained as Frenet frame of slant ruled surface generated by striction curve of natural lift curve for the rest of paper. Here $\vec{q}, \vec{h}$ and $\vec{a}$ are unit vectors of rulling, central normal and central tangent of slant ruled surface generated by striction curve of natural lift curve, respectively.

3. Unit dual sphere and ruled surfaces

In this section, some basic definitions and theorems about the dual vectors are represented. Furthermore, the correspondence between $T\bar{M}$ and DS^2 is given.

The set of dual numbers is defined as

$$ID = \{X = x + \varepsilon x^*; (x, x^*) \in IR \times IR, \;\; \varepsilon^2 = 0\}.$$

The combination of $\vec{x}$ and $\vec{x}^*$ is called dual vectors in IR^3. These vectors are real part and dual part of $\vec{X}$, respectively. If $\vec{x}$ and $\vec{x}^*$ are vectors in IR^3, then $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is defined as dual vector. Assume that $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$ are dual vectors. The addition, inner product and vector product are represented, respectively, as follows:
The addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*)$$

and their inner product is

$$\langle \vec{X}, \vec{Y}\rangle = \langle \vec{x}, \vec{y}\rangle + \varepsilon(\langle \vec{x}^*, \vec{y}\rangle + \langle \vec{x}, \vec{y}^*\rangle).$$

Also the vector product is given as

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}).$$

The norm of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is defined as

$$|\vec{X}| = \sqrt{\langle \vec{x}, \vec{x}\rangle} + \varepsilon\frac{\langle \vec{x}, \vec{x}^*\rangle}{\sqrt{\langle \vec{x}, \vec{x}\rangle}}. \tag{12}$$

The norm of $\vec{X}$ exists only for $\vec{x} \neq 0$. If the norm of X is equal to 1, the dual vector is called unit dual vector. The unit dual sphere which consists of the all unit dual vectors is defined as

$$DS^2 = \{\vec{X} = \vec{x} + \varepsilon\vec{x}^* \in ID^3 : |\vec{X}| = 1\}. \tag{13}$$

For detailed information for dual vectors, see [1]. The correspondence between the unit dual sphere and the tangent bundle of unit 2-sphere of the natural lift curve is given via (2) and (13):

$$\begin{array}{ccc} T\bar{M} & \longrightarrow & DS^2, \\ \bar{\Gamma} = (\bar{q}, \bar{\vartheta}) & \longmapsto & \vec{\bar{\Gamma}} = \vec{\bar{q}} + \varepsilon \vec{\bar{\vartheta}}. \end{array}$$

Here $\bar{q}$ and $\bar{\vartheta}$ are taken as $q^{'}$ and $\vartheta^{'}$, respectively.

Theorem 3.1. *(E. Study mapping) There exists one-to-one correspondence between the oriented lines in IR^3 and the points of DS^2, see [1].*

Theorem 3.2. *Assume that $\vec{\bar{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u)$ is a natural lift curve on DS^2 with parameter u. In IR^3, the ruled surface obtained by the natural lift curve $\bar{\Gamma}(u)$ can be represented as*

$$\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u), \tag{14}$$

where

$$\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) \tag{15}$$

is the base curve of $\bar{\phi}$.

Consequently, the isomorphism among $T\bar{M}$, DS^2 and IR^3 can be given as:

$$\begin{array}{ccc} T\bar{M} & \longrightarrow & DS^2 \longrightarrow IR^3, \\ \bar{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}(u)) & \longmapsto & \vec{\bar{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u) \longmapsto \bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u). \end{array}$$

Here $\bar{\phi}(u,v)$ is the ruled surface in IR^3 corresponding to the dual curve $\bar{\Gamma}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u) \in DS^2$ (or to the natural lift curve $\bar{\Gamma}(u) \in T\bar{M}$), see [9].

4. Tangent bundle of unit 2-sphere and slant ruled surfaces

In this section, the properties and the conditions for being a slant ruled surface generated by the striction curve of a natural lift curve are considered.

Definition 4.1. *A ruled surface $\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u)$ in IR^3 is called a $\vec{\bar{q}}-$(resp., $\vec{\bar{h}}-, \vec{\bar{a}}-$) slant ruled surface if the following three conditions are satisfied:*
(i) *The base curve $\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u)$ of the ruled surface $\bar{\phi}(u,v)$ must be taken as*

$$\bar{\beta}(u) = (\vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}^*(u)) - \frac{\langle (\vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}^*(u))', \vec{\bar{q}}'(u) \rangle}{\langle \vec{\bar{q}}'(u), \vec{\bar{q}}'(u) \rangle} \vec{\bar{q}}(u), \quad \vec{\bar{\vartheta}}^* \subseteq \vec{\bar{\vartheta}}.$$

(ii) *The equations $\langle \vec{\bar{q}}(u), \bar{\vartheta}^*(u) \rangle = 0$ and $|\vec{\bar{q}}(u)| = 1$ must be satisfied.*
(iii) $\vec{\bar{q}} -$ *(resp., $\vec{\bar{h}}-, \vec{\bar{a}}-$) must make a constant angle with a fixed non-zero direction.*

Let $\hat{\Gamma}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}^*(u)$ be a striction curve of natural lift curve on DS^2 with parameter u. In IR^3, the slant ruled surface generated by $\hat{\Gamma}(u)$ is represented as

$$\hat{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}^*(u) + v\vec{\bar{q}}(u), \; u \in I, \; \vec{\bar{\vartheta}}^* \subseteq \vec{\bar{\vartheta}}, \tag{16}$$

where

$$\bar{\beta}(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}^*(u) \tag{17}$$

is the base curve of $\hat{\phi}$.
Consequently, we can write the isomorphism

$$\begin{array}{rcl} T\bar{M} & \longrightarrow & DS^2 \longrightarrow IR^3, \\ \hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u)) & \longmapsto & \vec{\hat{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}^*(u) \longmapsto \hat{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}^*(u) + v\vec{\bar{q}}(u), \end{array}$$

where $\hat{\phi}(u,v)$ is the slant ruled surface in IR^3 corresponding to the striction curve $\vec{\hat{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}^*(u) \in DS^2$ (or to the smooth curve $\hat{\Gamma}(u) \in T\bar{M}$).
The following sections are about the characterizations for $\vec{\bar{q}}-, \vec{\bar{h}}-, \vec{\bar{a}}-$ slant ruled surfaces, respectively.

4.1. Tangent bundle of unit 2-sphere and $\vec{\bar{q}}-$ slant ruled surfaces

In this section, the definition and characterizations of $\vec{\bar{q}}-$ slant ruled surfaces in IR^3 are introduced.

Definition 4.2. *Let $\hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u)) \in T\bar{M}$ be striction curve of natural lift curve. Therefore, the ruled surface $\hat{\phi}(u,v)$ corresponding to $\hat{\Gamma}(u)$ in IR^3 is*

$$\hat{\phi}(u,v) = \bar{\beta}(u) + v\vec{\bar{q}}(u), \tag{18}$$

where $\bar{\beta}(u)$ is the striction curve of $\hat{\phi}(u,v)$. u denotes the arc length parameter of $\bar{\beta}(u)$. Let $\{\vec{\bar{q}}, \vec{\bar{h}}, \vec{\bar{a}}, \bar{k}_1, \bar{k}_2\}$ be Frenet operators of $\hat{\phi}$. The following equation exists

$$\langle \vec{\bar{q}}, \vec{u} \rangle = \cos\theta = constant; \;\; \theta \neq \frac{\pi}{2}$$

if the rulling makes constant angle θ with a fixed non-zero direction $\vec{u}$ in the space. $\hat{\phi}$ is called a $\vec{\bar{q}}-$ slant ruled surface.

Theorem 4.3. *The following equation*

$$\tan\theta = \frac{\bar{k}_1}{\bar{k}_2} \tag{19}$$

is constant if and only if $\hat{\phi}$ is a $\vec{\bar{q}}-$ slant ruled surface. Here θ denotes the angle between the rulling $\vec{\bar{q}}$ and a fixed direction.

Proof. Let $\hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u)) \in T\bar{M}$ be striction curve of natural lift curve and $\hat{\phi}(u,v)$ be the ruled surface corresponding to the curve $\hat{\Gamma}(u)$ in IR^3. Thus, $\hat{\phi}$ provides

$$\langle \vec{\bar{q}}, \vec{u} \rangle = \cos\theta = constant. \tag{20}$$

Differentiating (20) with respect to u, we get $\langle \vec{\bar{h}}, \vec{u} \rangle = 0$. Thus, u lies on the plane spanned by the vectors $\vec{\bar{q}}$ and $\vec{\bar{a}}$, i.e.,

$$\vec{u} = (\cos\theta)\vec{\bar{q}} + (\sin\theta)\vec{\bar{a}}. \tag{21}$$

Differentiating (21) with respect to u we obtain

$$0 = (\cos\theta\bar{k}_1 - \sin\theta\bar{k}_2)\vec{\bar{h}}. \tag{22}$$

Therefore, we obtain that $\tan\theta = \frac{\bar{k}_1}{\bar{k}_2}$ is constant.

Conversely, (19) is satisfied for the ruled surface $\hat{\phi}$. We define the following equation:

$$\vec{u} = (\cos\theta)\vec{\bar{q}} + (\sin\theta)\vec{\bar{a}}. \tag{23}$$

Differentiating (23) and using (19), we obtain $\vec{u}' = 0$. Therefore, $\vec{u}$ is a constant vector and $\langle \vec{\bar{q}}, \vec{u} \rangle = \cos\theta$ is constant. Thus, $\hat{\phi}$ is a $\vec{\bar{q}}-$ slant ruled surface in IR^3. $\square$

Theorem 4.4. *$det(\vec{q}', \vec{q}'', \vec{q}''') = 0$ if and only if $\hat{\phi}$ is a $\vec{q}-$ slant ruled surface.*

Proof. Let $\hat{\phi}$ be the ruled surface in IR^3. From Frenet formulas, we write

$$\begin{aligned} \vec{q}' &= \bar{k}_1\vec{h}, \\ \vec{q}'' &= -\bar{k}_1^2\vec{q} + \bar{k}_1'\vec{h} + \bar{k}_1\bar{k}_2\vec{a}, \\ \vec{q}''' &= (-3\bar{k}_1\bar{k}_1')\vec{q} + (\bar{k}_1'' - \bar{k}_1^3 - \bar{k}_1\bar{k}_2^2)\vec{h} + (2\bar{k}_1'\bar{k}_2 + \bar{k}_2'\bar{k}_1)\vec{a}. \end{aligned}$$

Thus, we obtain

$$det(\vec{q}', \vec{q}'', \vec{q}''') = \bar{k}_1^3\bar{k}_2^2(\frac{\bar{k}_1}{\bar{k}_2})'. \tag{24}$$

Assume that $\hat{\phi}$ is ruled surface. From Theorem 7, $\frac{\bar{k}_1}{\bar{k}_2}$ is constant. Therefore, $det(\vec{q}', \vec{q}'', \vec{q}''')$ is equal to zero.

Conversely, if $det(\vec{q}', \vec{q}'', \vec{q}''') = 0$, since the curvatures are non-zero from (24), it is obtained that $\frac{\bar{k}_1}{\bar{k}_2}$ is constant. Thus, $\hat{\phi}$ is $\vec{q}-$ slant ruled surface in IR^3. □

Theorem 4.5. *$det(\vec{a}', \vec{a}'', \vec{a}''') = 0$ if and only if $\hat{\phi}$ is a $\vec{q}-$ slant ruled surface.*

Proof. Let $\hat{\phi}$ be the ruled surface in IR^3. From Frenet formulas, we write

$$\begin{aligned} \vec{a}' &= -\bar{k}_2\vec{h}, \\ \vec{a}'' &= \bar{k}_1\bar{k}_2\vec{q} - \bar{k}_2'\vec{h} - \bar{k}_2^2\vec{a}, \\ \vec{a}''' &= (\bar{k}_1'\bar{k}_2 + 2\bar{k}_1\bar{k}_2')\vec{q} + (-\bar{k}_2'' + \bar{k}_2^3 + \bar{k}_1^2\bar{k}_2)\vec{h} - 3\bar{k}_2\bar{k}_2'\vec{a}. \end{aligned}$$

Thus, we obtain

$$det(\vec{a}', \vec{a}'', \vec{a}''') = \bar{k}_2^5(\frac{\bar{k}_1}{\bar{k}_2})'. \tag{25}$$

Assume that $\hat{\phi}$ is ruled surface generated by striction curve of natural lift curve. From Theorem 4.3., we have $\frac{\bar{k}_1}{\bar{k}_2} = constant$. Therefore, $det(\vec{a}', \vec{a}'', \vec{a}''')$ is equal to zero.

Conversely, if $det(\vec{a}', \vec{a}'', \vec{a}''') = 0$, since the curvature $\bar{k}_2$ is non-zero, we obtain that $\frac{\bar{k}_1}{\bar{k}_2}$ is constant. Thus, $\hat{\phi}$ is $\vec{q}-$ slant ruled surface in IR^3. □

Theorem 4.6. *$\hat{\phi}$ is a $\vec{q}-$ slant ruled surface if and only if*

$$\vec{q}''' = m\vec{q}' + 3\bar{k}_1\vec{h}', \tag{26}$$

where m is obtained as $\frac{\bar{k}_1''}{\bar{k}_1} - (\bar{k}_1^2 + \bar{k}_2^2)$, exists.

Proof. Let $\hat{\phi}$ be $\vec{q}-$ slant ruled surface. From Frenet formulas, we get

$$\begin{aligned} \vec{q}'' &= -\bar{k}_1^2\vec{q} + \bar{k}_1'\vec{h} + \bar{k}_1\bar{k}_2\vec{a}, \\ \vec{q}''' &= (-3\bar{k}_1\bar{k}_1')\vec{q} + (\bar{k}_1'' - \bar{k}_1^3 - \bar{k}_1\bar{k}_2^2)\vec{h} + (2\bar{k}_1'\bar{k}_2 + \bar{k}_2'\bar{k}_1)\vec{a}. \end{aligned}$$

As we take derivative of $\frac{\bar{k}_1}{\bar{k}_2}$, we obtain

$$\bar{k}_1\bar{k}_2' = \bar{k}_2\bar{k}_1'. \tag{27}$$

Substituting (27) into $\vec{\bar{q}}'''$, we get

$$\vec{\bar{q}}''' = (\frac{\bar{k}_1''}{\bar{k}_1} - \bar{k}_1^2 - \bar{k}_2^2)\vec{\bar{q}} - (3\bar{k}_1\bar{k}_1')\vec{\bar{q}} + (3\bar{k}_2\bar{k}_1')\vec{\bar{a}}. \tag{28}$$

Using the Frenet formulas, (26) is obtained from (28).

Conversely, let (26) provide. Differentiating $\vec{\bar{h}} = \frac{\vec{\bar{q}}'}{\bar{k}_1}$, we have

$$\vec{\bar{h}}' = -(\frac{\bar{k}_1'}{\bar{k}_1^2})\vec{\bar{q}}' + (\frac{1}{\bar{k}_1})\vec{\bar{q}}'', \tag{29}$$

$$\vec{\bar{h}}'' = -(\frac{\bar{k}_1'}{\bar{k}_1^2})'\vec{\bar{q}}' - 2(\frac{\bar{k}_1'}{\bar{k}_1^2})\vec{\bar{q}}'' + (\frac{1}{\bar{k}_1})\vec{\bar{q}}'''. \tag{30}$$

Substutituting (26) in (30) it can be obtained

$$\vec{\bar{h}}'' = -2(\frac{\bar{k}_1'}{\bar{k}_1^2})\vec{\bar{q}}'' - [(\frac{\bar{k}_1'}{\bar{k}_1^2})' + \frac{m}{\bar{k}_1}]\vec{\bar{q}}' + 3(\frac{\bar{k}_1'}{\bar{k}_1})\vec{\bar{h}}'. \tag{31}$$

Hence, we can write

$$\vec{\bar{h}}'' = -[(\frac{\bar{k}_1}{\bar{k}_1^2})' + \frac{m}{\bar{k}_1}]\vec{\bar{q}}' - \bar{k}_1'\vec{\bar{q}} - 2(\frac{\bar{k}_1'}{\bar{k}_1})^2\vec{\bar{h}} + (\frac{\bar{k}_2\bar{k}_1'}{\bar{k}_1})\vec{\bar{a}}. \tag{32}$$

On the other hand, from Frenet formulas it is obtained

$$\vec{\bar{h}}'' = -\bar{k}_1\vec{\bar{q}}' - \bar{k}_1'\vec{\bar{q}} - \bar{k}_2^2\vec{\bar{h}} + \bar{k}_2'\vec{\bar{a}}. \tag{33}$$

Substituting (33) in (32) we have

$$\frac{\bar{k}_2'}{\bar{k}_2} = \frac{\bar{k}_1'}{\bar{k}_1}. \tag{34}$$

Integrating (34), we acquire that $\frac{\bar{k}_1}{\bar{k}_2}$ is constant and from Theorem 4.3. So, $\hat{\phi}$ is a $\vec{\bar{q}}-$ slant ruled surface. □

4.2. *Tangent bundle of unit 2-sphere and $\vec{\bar{h}}-$ slant ruled surfaces*

In this section, the definition and characterizations of $\vec{\bar{h}}-$ slant ruled surfaces in IR^3 are introduced.

Definition 4.7. *Let $\hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u)) \in T\bar{M}$ be striction curve of natural lift curve. Therefore, the ruled surface $\hat{\phi}(u, v)$ corresponding to $\hat{\Gamma}(u)$ in IR^3 is*

$$\hat{\phi}(u, v) = \vec{\beta}(u) + v\vec{\bar{q}}(u), \tag{35}$$

where $\bar{\beta}(u)$ is the striction curve of $\hat{\phi}$. u denotes the arc length parameter of $\bar{\beta}(u)$. Let $\{\vec{\bar{q}}, \vec{\bar{h}}, \vec{\bar{a}}, \bar{k}_1, \bar{k}_2\}$ be Frenet operators of $\hat{\phi}$. The following equation exists

$$\langle\vec{\bar{h}}, \vec{u}\rangle = \cos\varphi = constant; \quad \varphi \neq \frac{\pi}{2}$$

if the central normal vector makes constant angle φ with a fixed non-zero direction $\vec{u}$ in the space. $\hat{\phi}$ is called a $\vec{\bar{h}}-$ slant ruled surface.

Theorem 4.8. *The following equation*

$$\frac{\bar{k}_1^2}{(\bar{k}_1^2+\bar{k}_2^2)^{\frac{3}{2}}}(\frac{\bar{k}_1}{\bar{k}_2})' \tag{36}$$

is constant if and only $\hat{\phi}$ is a $\vec{\bar{h}}-$ slant ruled surface.

Proof. Let $\hat{\phi}$ be a $\vec{\bar{h}}-$ slant ruled surface in IR^3. Assume that $\vec{u}$ is a fixed constant vector such that $\langle\vec{\bar{h}},\vec{u}\rangle = \cos\varphi = c = constant$. Here φ is the constant angle between $\vec{\bar{h}}$ and $\vec{u}$. Therefore, we have

$$\vec{u} = b_1(u)\vec{\bar{q}}(u) + c\vec{\bar{h}}(u) + b_2(u)\vec{\bar{a}}(u), \tag{37}$$

where $b_1 = b_1(u)$ and $b_2 = b_2(u)$ are smooth functions of arc length parameter u for the vector $\vec{u}$. Since $\vec{u}$ is constant, derivative of (37) gives

$$\begin{aligned} b_1' - c\bar{k}_1 &= 0, \\ b_1\bar{k}_1 - b_2\bar{k}_2 &= 0, \\ b_2' + c\bar{k}_2 &= 0. \end{aligned}$$

For $b_1\bar{k}_1 - b_2\bar{k}_2 = 0$, we have

$$b_1 = b_2\frac{\bar{k}_2}{\bar{k}_1}. \tag{38}$$

Moreover, it is obtained that

$$\langle\vec{u},\vec{u}\rangle = b_1^2 + c^2 + b_2^2 = constant. \tag{39}$$

Substituting (38) in (39) gives

$$b_2^2(1+(\frac{\bar{k}_2}{\bar{k}_1})^2) = n^2 = constant. \tag{40}$$

If $n = 0$, then $b_2 = 0$ and we have $b_1 = 0$, $c = 0$. It means that $\vec{u} = \vec{0}$ which is a contradiction. Thus, $n \neq 0$. Then from (40) it is acquired that

$$b_2 = \pm\frac{n}{\sqrt{1+(\frac{\bar{k}_2}{\bar{k}_1})^2}} \tag{41}$$

Considering $b_2' + c\bar{k}_2 = 0$, from (41) we have

$$\frac{d}{du}[\pm\frac{n}{\sqrt{1+(\frac{\bar{k}_2}{\bar{k}_1})^2}}] = -c\bar{k}_2. \tag{42}$$

This can be written as

$$\frac{\bar{k}_1^2}{(\bar{k}_1^2+\bar{k}_2^2)^{\frac{3}{2}}} = \frac{c}{n} = l = constant, \tag{43}$$

which is desired.

Conversely, let the function in (36) be constant. That is,

$$\frac{\bar{k}_1^2}{(\bar{k}_1^2+\bar{k}_2^2)^{\frac{3}{2}}} = \frac{c}{n} = l = constant. \tag{44}$$

We define

$$\vec{u} = \frac{\bar{k}_2}{\sqrt{\bar{k}_1^2 + \bar{k}_2^2}}\vec{\bar{q}} + l\vec{\bar{h}} + \frac{\bar{k}_1}{\sqrt{\bar{k}_1^2 + \bar{k}_2^2}}\vec{\bar{a}}. \tag{45}$$

Differentiating (45) with respect to u and using (36) we have $\vec{u}' = 0$. That is, $\vec{u}$ is a constant vector. On the other hand, $\langle \vec{\bar{h}}, \vec{u} \rangle$ is constant. Therefore, $\hat{\phi}$ is a $\vec{\bar{h}}-$ slant ruled surface in IR^3. □

Theorem 4.9. *Let $\hat{\phi}$ be a regular ruled surface in IR^3 with first curvature $\bar{k}_1 = 1$. Moreover, we have*

$$\bar{k}_2(u) = \pm\frac{u}{\sqrt{\tan^2\varphi - u^2}} \tag{46}$$

for a fixed constant the unit vector $\vec{u}$.

Proof. Let $\hat{\phi}$ be a regular ruled surface in IR^3 with first curvature $\bar{k}_1 = 1$. Therefore, we have

$$\langle \vec{\bar{h}}, \vec{u} \rangle = \cos\varphi = constant \tag{47}$$

for a fixed constant unit vector $\vec{u}$. Differentiating (47) with respect to u gives

$$\langle -\vec{\bar{q}} + k_2\vec{\bar{a}}, \vec{u} \rangle = 0, \tag{48}$$

and from (48) we have

$$\langle \vec{\bar{q}}, \vec{u} \rangle = \bar{k}_2\langle \vec{\bar{a}}, \vec{u} \rangle. \tag{49}$$

If we put $\langle \vec{\bar{a}}, \vec{u} \rangle = x$, we can write

$$\vec{u} = \bar{k}_2 x\vec{\bar{q}} + \cos\varphi\vec{\bar{h}} + x\vec{\bar{a}}. \tag{50}$$

Since $\vec{u}$ is unit, from (50) we have

$$x = \pm\frac{\sin\varphi}{\sqrt{1 + \bar{k}_2^2}}. \tag{51}$$

Then the vector $\vec{u}$ is given as follows:

$$\vec{u} = \pm\frac{\bar{k}_2\sin\varphi}{\sqrt{1 + \bar{k}_2^2}} + \cos\varphi\vec{\bar{h}} \pm \frac{\sin\varphi}{\sqrt{1 + \bar{k}_2^2}}\vec{\bar{a}}. \tag{52}$$

Differentiating (48) with respect to u, it follows

$$\langle -(1 + \bar{k}_2^2)\vec{\bar{h}} + \bar{k}_2'\vec{\bar{a}}, \vec{u} \rangle = 0. \tag{53}$$

Writing x and substituting (47) in (53), we have

$$x = \frac{(1 + \bar{k}_2^2)\cos\varphi}{\bar{k}_2'}. \tag{54}$$

From (51) and (54), we obtain the following differential equation:

$$\pm\tan\varphi\frac{\bar{k}_2'}{(1 + \bar{k}_2^2)^{\frac{3}{2}}} + 1 = 0. \tag{55}$$

Integrating (55) we get

$$\pm\tan\varphi\frac{\bar{k}_2}{\sqrt{1+\bar{k}_2^2}}+u+c=0, \tag{56}$$

where c is integration constant. The integration constant can be subsumed thanks to a parameter change $u\longrightarrow u-c$. Then (56) can be written as

$$\pm\tan\varphi\frac{\bar{k}_2}{\sqrt{1+\bar{k}_2^2}}=-u, \tag{57}$$

which gives us $\bar{k}_2(u)=\pm\frac{u}{\sqrt{\tan^2\varphi-u^2}}$.

Conversely, assume that $\bar{k}_2(u)=\pm\frac{u}{\sqrt{\tan^2\varphi-u^2}}$ holds and let us put

$$x=\pm\frac{\sin\varphi}{\sqrt{1+\bar{k}_2^2}}=\pm\frac{\sin\varphi}{\sqrt{1+\frac{u^2}{\tan^2\varphi-u^2}}}=\pm\cos\varphi\sqrt{\tan^2\varphi-u^2}. \tag{58}$$

Here we are assuming that when $\bar{k}_2$ has the positive(negative) sign, then x gets the negative (positive) sign and φ is constant. Therefore, $k_2x=-s\cos\varphi$. Let consider the vector $\vec{u}$ given as

$$\vec{u}=\cos\varphi(u\vec{\bar{q}}+\vec{\bar{h}}\pm(\tan^2\varphi-u^2)\vec{\bar{a}}). \tag{59}$$

We will prove that $\vec{u}$ is constant and makes a constant angle φ with $\vec{\bar{h}}$. Differentiating (59) and using Frenet formulas we have $\vec{u}'=0$, i.e., the direction of $\vec{u}$ is constant and $\langle\vec{\bar{h}},\vec{u}\rangle=\cos\varphi=constant$. Then $\hat{\phi}$ is a $\vec{\bar{h}}-$ slant ruled surface in IR^3. □

On the other hand, if the striction curve $\bar{\beta}$ is a geodesic on $\hat{\phi}$, then the principal normal vector $\vec{n}$ of $\bar{\beta}$ and the central normal vector $\vec{\bar{h}}$ of $\hat{\phi}$ coincide. Then we have the following corollary.

Corollary 4.10. *Let the striction line $\bar{\beta}$ be a geodesic on $\hat{\phi}$. Then $\hat{\phi}$ is a $\vec{\bar{h}}-$ slant ruled surface if and only if the striction line is a slant helix in IR^3.*

If the ruled surface $\hat{\phi}$ is developable, the Frenet frame $\{\vec{t},\vec{n},\vec{b}\}$ of the striction line $\bar{\beta}$ coincides with the frame $\{\vec{\bar{q}},\vec{\bar{h}},\vec{\bar{a}}\}$ and we can give the following corollary.

Corollary 4.11. *Let $\hat{\phi}$ be a developable surface. Then $\hat{\phi}$ is a $\vec{\bar{h}}-$ slant ruled surface if and only if the striction line is a slant helix in IR^3.*

4.3. Tangent bundle of unit 2-sphere and $\vec{\bar{a}}-$ slant ruled surfaces

In this section, the definition and characterization of $\vec{\bar{a}}-$ slant ruled surfaces in IR^3 are introduced.

Definition 4.12. *Let $\hat{\Gamma}(u)=(\bar{q}(u),\bar{\vartheta}^*(u))\in T\bar{M}$ be striction curve . Therefore, the ruled surface $\hat{\phi}(u,v)$ corresponding to $\hat{\Gamma}(u)$ in IR^3 is*

$$\hat{\phi}(u,v)=\vec{\bar{\beta}}(u)+v\vec{\bar{q}}(u), \tag{60}$$

where $\bar{\beta}(u)$ is the striction curve of $\hat{\phi}$. u denotes the arc length parameter of $\bar{\beta}(u)$. Let $\{\vec{\bar{q}},\vec{\bar{h}},\vec{\bar{a}},\bar{k}_1,\bar{k}_2\}$ be Frenet operators of $\hat{\phi}$. The following equation exists

$$\langle\vec{\bar{a}},\vec{u}\rangle=\cos\mu=constant;\quad \mu\neq\frac{\pi}{2}$$

if the central tangent vector makes constant angle μ with a fixed non-zero direction $\vec{u}$ in the space. $\hat{\phi}$ is called a $\vec{\bar{a}}-$ slant ruled surface.

From (21) it is clear that a ruled surface $\hat{\phi}$ is a $\vec{a}-$ slant ruled surface if and only if it is a $\vec{q}-$ slant ruled surfaces. So, all the theorems for $\vec{q}-$ slant ruled surfaces also characterize the $\vec{a}-$ slant ruled surfaces.

Example 4.13. *Let us consider the slant ruled surface $\hat{\phi}(u,v)$ in IR^3.*

$$\hat{\phi}(u,v) = (\cos(u+1), v\cos u, v\sin u), \tag{61}$$

where the base curve and rulling of $\hat{\phi}(u,v)$ are $\bar{\beta}(u) = (\cos(u+1),0,0)$ and $\vec{q}(u) = (0,\cos u,\sin u)$, respectively. The striction curve of $\hat{\phi}(u,v)$ is given as

$$\bar{\beta}(u) = (\cos(u+1),0,0), \tag{62}$$

where arc parameter of the striction curve is $\vec{u} = \arccos(-s-\cos 1)-1$.
Since the striction curve $\bar{\beta}(u) = \vec{q}(u) \times \vec{\vartheta}^*(u)$, we obtain

$$\vec{\vartheta}^*(u) = (0, -\frac{1}{2}\tan u(\cos(2u+1)+\cos 1), \frac{1}{2}(\cos(2u+1)+\cos 1)). \tag{63}$$

Since $\langle \vec{q}(u), \vec{\vartheta}^*(u)\rangle = 0$ and $|\vec{q}(u)| = 1$, the curve $\hat{\Gamma} = (\vec{q}, \bar{\vartheta}^*)$ is in $T\bar{M}$. Therefore, the striction curve and Frenet vectors of $\hat{\phi}(u,v)$ are

$$\begin{aligned}
\bar{\beta}(s) &= (-s-\cos 1,0,0),\\
\vec{q}(s) &= (0,\cos(\arccos(-s-\cos 1)-1),\sin(\arccos(-s-\cos 1)-1)),\\
\vec{h}(s) &= (0,-\sin(\arccos(-s-\cos 1)-1),\cos(\arccos(-s-\cos 1)-1)),\\
\vec{a}(s) &= (1,0,0).
\end{aligned}$$

The derivatives of Frenet vectors are

$$\begin{aligned}
\frac{d\vec{q}}{ds} &= \frac{1}{\sqrt{1-(s+\cos 1)^2}}(0,-\sin(\arccos(-s-\cos 1)-1),\cos(\arccos(-s-\cos 1)-1)),\\
\frac{d\vec{h}}{ds} &= \frac{1}{\sqrt{1-(s+\cos 1)^2}}(0,-\cos(\arccos(-s-\cos 1)-1),-\sin(\arccos(-s-\cos 1)-1)),\\
\frac{d\vec{a}}{ds} &= (0,0,0),
\end{aligned}$$

where $\bar{k}_1 = \frac{1}{\sqrt{1-(s+\cos 1)^2}}$ and $\bar{k}_2 = 0$ are curvature and torsion of $\hat{\phi}$, respectively.

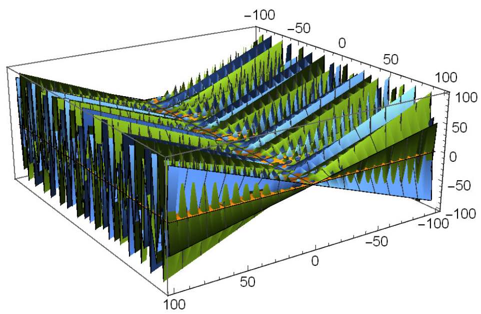

Figure 2: Slant ruled surface $\hat{\phi}(u,v)$ generated by striction curve $\hat{\Gamma}(u)$

5. Conclusion

E. Study mapping plays a significiant role in modeling motions in IR^3. It is a novel attempt to provide a corresponcence between the striction curves of natural lift curves on $T\bar{M}$ and the slant ruled surfaces in IR^3. Employing this mapping, it is possible to model motions by considering $T\bar{M}$ instead of DS^2. Therefore, the slant ruled surfaces generated by striction curves of natural lift curves on $T\bar{M}$ can also be used to model motions in IR^3.

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