Filomat 37:2 (2023), 457–466
https://doi.org/10.2298/FIL2302457Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New characterizations of weak group matrices

Mengmeng Zhou[a], Jianlong Chen[b], Néstor Thome[c]

[a]*College of Information Engineering, Nanjing Xiaozhuang University, Nanjing, Jiangsu 211171, China*
[b]*School of Mathematics, Southeast University, Nanjing, Jiangsu 210096, China*
[c]*Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia 46022, Spain*

Abstract. In this paper, we prove that a complex square matrix is weak group matrix if the m-th power of this matrix commutes with its weak group inverse, where m is an arbitrary positive integer. Firstly, some new characterizations of weak group matrices are investigated by means of core-EP decomposition. Secondly, we study new equivalent conditions of the weak group matrix by using commutator and rank equalities. Finally, the relationships between $\{m,k\}$-core EP matrices, k-EP matrices and weak group matrices are given.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. The symbol $\mathbb{Z}^+$ stands for the set of all positive integers. For $A\in\mathbb{C}^{m\times n}$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, and $\mathrm{rk}(A)$ stand for the range space, null space, conjugate transpose, and rank of A, respectively. The symbol I denotes the identity matrix of an appropriate size. We define $A^0=I$ and write $[A,B]=BA-AB$, for any matrices $A,B\in\mathbb{C}^{n\times n}$.

Let $A\in\mathbb{C}^{m\times n}$. The unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying the following equations:

$$AXA=A,\ \ XAX=X,\ \ (AX)^*=AX,\ \ (XA)^*=XA,$$

is called the Moore-Penrose inverse of A and denoted by $A^\dagger$ [17]. A matrix $X\in\mathbb{C}^{n\times m}$ is called a $\{2\}$-inverse of A if X satisfies the equality $XAX=X$. Recall that $A\in\mathbb{C}^{n\times n}$ is an EP matrix if $AA^\dagger=A^\dagger A$ [16]. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the smallest non-negative integer k such that $\mathrm{rk}(A^k)=\mathrm{rk}(A^{k+1})$. Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{ind}(A)=k$. The unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying the following equations:

$$A^{k+1}X=A^k,\ \ XAX=X,\ \ AX=XA,$$

2020 *Mathematics Subject Classification*. 15A09.
Keywords. Weak group inverse; Weak group matrix; $\{m,k\}$-core EP matrix.
Received: 13 July 2020; Accepted: 03 September 2020
Communicated by Dragana Cvetković Ilić
Corresponding author: Néstor Thome
This research is supported by the National Natural Science Foundation of China (No. 12101315,12101539,12171083), the China Scholarship Council (File No. 201906090122), Natural Science Foundation of Jiangsu Higher Education Institutions of China (21KJB110004), the Qing Lan Project of Jiangsu Province. The third author is partially supported by Ministerio de Economía y Competitividad of Spain (grant Red de Excelencia MTM2017-90682-REDT), by Project PGI 24/L108, Departamento de Matemática, Universidad Nacional del Sur (UNS), Argentina, by Universidad Nacional de Río Cuarto (Grant PPI 083/2020), and by Universidad Nacional de La Pampa, Facultad de Ingeniería (Grant Resol. Nro. 135/19).
Email addresses: mmz9209@163.com (Mengmeng Zhou), jlchen@seu.edu.cn (Jianlong Chen), njthome@mat.upv.es (Néstor Thome)

is called the Drazin inverse of A and denoted by A^D [4]. If $k = 1$, the Drazin inverse is reduced to the group inverse and denoted by $A^{\#}$.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. The DMP inverse and dual DMP inverse were introduced by Malik et al. [9]. The DMP inverse of A, denoted by $A^{D,\dagger}$, is defined to be the matrix $A^{D,\dagger} = A^DAA^\dagger$. The dual DMP inverse of A, denoted by $A^{\dagger,D}$, is defined as $A^{\dagger,D} = A^\dagger AA^D$. Manjunatha Prasad et al. [10] introduced the core-EP inverse. The unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X,\ \ \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

is called the core-EP inverse of $A \in \mathbb{C}^{n\times n}$ and is denoted by $A^{\oplus}$. Later, Gao et al. [7] presented the characterization of core-EP inverse by three equations. That is, the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies

$$XA^{k+1} = A^k,\ \ AX^2 = X,\ \ (AX)^* = AX,$$

is the core-EP inverse of A. They also obtained $A^{\oplus} = A^DA^k(A^k)^\dagger$. In particular, the DMP inverse and the core-EP inverse are reduced to the core inverse of A when $k = 1$, and it is denoted by $A^{\#\!\!\!\bigcirc}$ [1]. The CMP inverse of $A \in \mathbb{C}^{n\times n}$ [11], denoted by $A^{c,\dagger}$, is defined to be the matrix $A^{c,\dagger} = Q_AA^DP_A$, where $Q_A = A^\dagger A$ and $P_A = AA^\dagger$. In [11], the authors also introduced the core-EP matrix. The matrix $A \in \mathbb{C}^{n\times n}$ is called the core-EP matrix if $A^\dagger AA^DA = AA^DAA^\dagger$. Wang et al. [20] introduced the weak group inverse. If $A \in \mathbb{C}^{n\times n}$, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following equations:

$$AX^2 = X,\ \ AX = A^{\oplus}A,$$

is called the weak group inverse of A and is denoted by $A^{Ⓦ}$. In [20], the authors obtain $A^{Ⓦ} = (A^{\oplus})^2A$ and $A^{Ⓦ}A^{k+1} = A^k$, where k is the index of A. For related results, we refer the reader to [3, 12–15, 22].

Recently, Wang et al. [21] defined the weak group matrix. A matrix $A \in \mathbb{C}^{n\times n}$ is called the weak group matrix if $AA^{Ⓦ} = A^{Ⓦ}A$. On the other hand, Ferreyra et al. [5] presented some characterizations of k-commutative equalities for some outer generalized inverses, where k is the index of a given matrix. In [5], the symbols

$$\mathbb{C}_n^{k,\dagger} = \{A \in \mathbb{C}^{n\times n} : A^kA^\dagger = A^\dagger A^k\}$$

and

$$\mathbb{C}_n^{k,\oplus} = \{A \in \mathbb{C}^{n\times n} : A^kA^{\oplus} = A^{\oplus}A^k\}$$

denote the classes of all k-EP matrices [8] and k-core EP matrices, respectively.

Motivated by above discussion, we investigate equivalent conditions such that $A^mA^{Ⓦ} = A^{Ⓦ}A^m$ holds by core-EP decomposition, where $A \in \mathbb{C}^{n\times n}$ and m is an arbitrary positive integer. Furthermore, we also prove that the matrix A is weak group matrix if and only if $A^mA^{Ⓦ} = A^{Ⓦ}A^m$. Moreover, some new characterizations of weak group matrices are studied by commutator and rank equalities. Finally, we relate the weak group matrix to $\{m, k\}$-core EP matrix (or other special matrices, for example, k-EP matrices). We would like to highlight that the weak group inverse requires the computation of the core-EP inverse as its definition shows. In this paper, we provide some methods to characterize the weak group matrix without computing the core-EP inverse. Theses results can be calculated quickly by using packages like MATLAB, Mathematica, etc.

The paper is organized as follows. In Section 2, some preliminary results are given. In Section 3, we investigate equivalent conditions which ensure that $A^mA^{Ⓦ} = A^{Ⓦ}A^m$ holds by core-EP decomposition, where $A \in \mathbb{C}^{n\times n}$ and m is an arbitrary positive integer. In Section 4, some new characterizations of the weak group matrix are given by commutator and rank equalities. In Section 5, we study relations between weak group matrices and other special matrices.

2. Preliminaries

In this section, some necessary lemmas are given.

Lemma 2.1. [19] *(Core-EP decomposition) Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then A can be written as $A = A_1 + A_2$, where*

(i) $\operatorname{ind}(A_1) \leq 1$;

(ii) $A_2^k = 0$;

(iii) $A_1^* A_2 = A_2 A_1 = 0$.

Moreover, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*, \ A_2 = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*,$$

where $T \in \mathbb{C}^{r\times r}$ is non-singular, N is nilpotent and $\operatorname{rk}(A^k) = r$.

For $A \in \mathbb{C}^{n\times n}$ being as in Lemma 2.1, it is known [6, 20] that

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*, \ A^{\textcircled{w}} = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*, \ A^D = U\begin{pmatrix} T^{-1} & T^{-(k+1)}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*,$$

where $\widetilde{T}_k = \sum_{i=0}^{k-1} T^i S N^{k-1-i}$. From now on, we denote $\widetilde{T}_\ell = \sum_{i=0}^{\ell-1} T^i S N^{\ell-1-i}$ for an arbitrary $\ell \in \mathbb{Z}^+$.

Lemma 2.2. [21] *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *be written as in Lemma 2.1. Then the following conditions are equivalent:*

(i) $AA^{\textcircled{w}} = A^{\textcircled{w}}A$ *(i.e., A is a weak group matrix.);*

(ii) $SN = 0$.

Lemma 2.3. [5] *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then A is k-EP matrix if and only if $A^{c,\dagger} = A^{D,\dagger} = A^{\dagger,D} = A^D$.*

Lemma 2.4. [11] *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then A is k-EP matrix if and only if A is core-EP matrix.*

Lemma 2.5. [11] *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *be written as in Lemma 2.1. Then $A \in \mathbb{C}_n^{k,\dagger}$ if and only if the following conditions simultaneously hold:*

(i) $\mathcal{N}(N) \subseteq \mathcal{N}(S)$;

(ii) $\mathcal{N}(N^*) \subseteq \mathcal{N}(\widetilde{T}_k)$.

3. The weak group matrix is characterized by core-EP decomposition

In this section, we prove that $A^m A^{\textcircled{w}} = A^{\textcircled{w}} A^m$ if and only if $AA^{\textcircled{w}} = A^{\textcircled{w}}A$, where m is an arbitrary positive integer. Some new characterizations of the weak group matrix are investigated by core-EP decomposition.

Remark 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and m be the natural number. It is clear that $A^m A^{\textcircled{w}} = A^{\textcircled{w}} A^m$ when A is non-singular (that is, $A^{\textcircled{w}} = A^{-1}$) or $m = 0$.*

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *be written as in Lemma 2.1. Then the following conditions are equivalent:*

(i) *A is a weak group matrix;*

(ii) $A^m A^{\textcircled{w}} = A^{\textcircled{w}} A^m$, *for arbitrary $m \in \mathbb{Z}^+$;*

(iii) $\widetilde{T}_m N = 0$, *for arbitrary* $m \in \mathbb{Z}^+$;

(iv) $SN = 0$;

(v) $A^k A^{\text{Ⓦ}} = A^{\text{Ⓦ}} A^k$;

(vi) $\widetilde{T}_k = T^{k-1}S$.

Proof. Assume that $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ is written as in Lemma 2.1.

(ii) $\Leftrightarrow$ (iii) : Set an arbitrary $m \in \mathbb{Z}^+$. We proceed by induction.

(a): if $m = 1$. Then $\widetilde{T}_m N = SN$. It is clear that $SN = 0 \Leftrightarrow AA^{\text{Ⓦ}} = A^{\text{Ⓦ}}A$ by Lemma 2.2.

(b): Suppose that $A^\ell A^{\text{Ⓦ}} = A^{\text{Ⓦ}} A^\ell \Leftrightarrow \widetilde{T}_\ell N = 0$. We have

$$\begin{aligned} A^{\ell+1}A^{\text{Ⓦ}} &= U\begin{pmatrix} T^{\ell+1} & \sum_{i=0}^{\ell} T^i S N^{\ell-i} \\ 0 & N^{\ell+1}\end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0\end{pmatrix}U^* = U\begin{pmatrix} T^{\ell} & T^{\ell-1}S \\ 0 & 0\end{pmatrix}U^*, \\ A^{\text{Ⓦ}}A^{\ell+1} &= U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0\end{pmatrix}\begin{pmatrix} T^{\ell+1} & \sum_{i=0}^{\ell} T^i S N^{\ell-i} \\ 0 & N^{\ell+1}\end{pmatrix}U^* \\ &= U\begin{pmatrix} T^{\ell} & \sum_{i=0}^{\ell} T^{i-1} S N^{\ell-i} + T^{-2}SN^{\ell+1} \\ 0 & 0\end{pmatrix}U^*. \end{aligned}$$

So $A^{\ell+1}A^{\text{Ⓦ}} = A^{\text{Ⓦ}}A^{\ell+1} \Leftrightarrow T^{\ell-1}S = \sum_{i=0}^{\ell} T^{i-1}SN^{\ell-i} + T^{-2}SN^{\ell+1} \Leftrightarrow$

$$T^{\ell-1}S = T^{\ell-1}S + \sum_{i=0}^{\ell-1} T^{i-1}SN^{\ell-i} + T^{-2}SN^{\ell+1} \;\Leftrightarrow$$

$$T^{-2}\Big(\sum_{i=0}^{\ell-1} T^{i+1}SN^{\ell-i} + SN^{\ell+1}\Big) = 0 \;\Leftrightarrow\; \sum_{i=0}^{\ell-1} T^{i+1}SN^{\ell-i} + SN^{\ell+1} = 0.$$

Since $\sum_{i=0}^{\ell-1} T^{i+1}SN^{\ell-i} + SN^{\ell+1} = \sum_{i=1}^{\ell} T^{i}SN^{\ell+1-i} + SN^{\ell+1} = \sum_{i=0}^{\ell} T^{i}SN^{\ell+1-i}$,

$$A^{\ell+1}A^{\text{Ⓦ}} = A^{\text{Ⓦ}}A^{\ell+1} \;\Leftrightarrow\; \sum_{i=0}^{\ell} T^{i}SN^{\ell+1-i} = 0 \;\Leftrightarrow\; \widetilde{T}_{\ell+1}N = 0.$$

(iii) $\Leftrightarrow$ (iv) : Suppose that an arbitrary $m \in \mathbb{Z}^+$ and

$$\widetilde{T}_m N = SN^m + TSN^{m-1} + \cdots + T^{m-2}SN^2 + T^{m-1}SN = 0. \tag{1}$$

We study the following cases:

(a): If $m = 1$, then it is clear by Lemma 2.2.

(b): When $1 < m < k-1$. Multiplying by N^{k-2} on the right of the Eq. (1), we have $T^{m-1}SN^{k-1} = 0$, i.e., $SN^{k-1} = 0$. Multiplying by N^{k-3} on the right of the Eq. (1), we get $SN^{k-2} = 0$. In the same way, we get $SN = 0$. The converse is clear.

(c): When $m = k-1$, multiplying by N^{k-2} on the right of Eq. (1). Since $N^k = 0$, we get $T^{m-1}SN^{k-1} = 0$, i.e., $T^{m-1}SN^m = 0$. Since T is non-singular, $SN^m = 0$. Hence we have

$$TSN^{m-1} + T^2SN^{m-2} + \cdots + T^{m-2}SN^2 + T^{m-1}SN = 0. \tag{2}$$

Multiplying by N^{k-3} on the right of Eq. (2), we obtain $T^{m-1}SN^{k-2} = T^{m-1}SN^{m-1} = 0$, i.e., $SN^{m-1} = 0$. Similarly, we get $SN^{m-2} = 0, \cdots, SN = 0$. The converse is obvious.

(d): When $m > k-1$, i.e, $m \geq k$, $N^m = 0$. By Theorem 3.2, we obtain the Eq. (2). Multiplying by N^{m-2} on the right of the Eq. (2), we have

$$TSN^{2m-3} + T^2SN^{2m-4} + \cdots + T^{m-2}SN^m + T^{m-1}SN^{m-1} = 0. \tag{3}$$

Hence, $T^{m-1}SN^{m-1} = 0$. That is, $SN^{m-1} = 0$. In the same way, we obtain $SN^{m-2} = 0, \cdots, SN = 0$. The converse is evident.

(ii) $\Rightarrow$ (v) and (i) $\Rightarrow$ (ii) are clear.

(v) $\Rightarrow$ (vi) : We have

$$A^kA^{\circledW} = U\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{k-1} & T^{k-2}S \\ 0 & 0 \end{pmatrix}U^*,$$

$$A^{\circledW}A^k = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{k-1} & T^{-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*.$$

If $A^kA^{\circledW} = A^{\circledW}A^k$, then $T^{k-2}S = T^{-1}\widetilde{T}_k \Rightarrow \widetilde{T}_k = T^{k-1}S$.

(vi) $\Rightarrow$ (i) : We know that $A^D = U\begin{pmatrix} T^{-1} & T^{-(k+1)}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*$. If $\widetilde{T}_k = T^{k-1}S$, then $T^{-(k+1)}\widetilde{T}_k = T^{-(k+1)}T^{k-1}S = T^{-2}S$. So

$$A^D = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = A^{\circledW}.$$

Thus, $AA^{\circledW} = A^{\circledW}A$. That is, A is a weak group matrix. $\square$

4. The weak group matrix is characterized by commutator and rank equalities

In this section, new characterizations of the weak group matrix are studied by commutator and rank equalities. Some results in [21] are revised.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\text{ind}(A) = k$. Then the following conditions are equivalent:*

(i) *A is a weak group matrix;*

(ii) $[A^{\oplus}A,\ A^{\oplus}A^k] = 0$;

(iii) $[A^{\oplus}A,\ A^{\oplus}A^{k+j}] = 0$, *for arbitrary $j \in \mathbb{Z}^+$;*

(iv) $[A^{\oplus}A,\ A^{\oplus}A^{j+1}] = 0$, *for arbitrary $j \in \mathbb{Z}^+$.*

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ with $\text{ind}(A) = k$ is written as in Lemma 2.1. We have

$$A^{\oplus}A = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^* = U\begin{pmatrix} I & T^{-1}S \\ 0 & 0 \end{pmatrix}U^*.$$

(i) $\Leftrightarrow$ (ii): By direct computation,

$$A^{\oplus}A^k = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{k-1} & T^{-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*.$$

Since $A^{\oplus}A^kA^{\oplus}A - A^{\oplus}AA^{\oplus}A^k = A^{\oplus}A^k(A^{\oplus}A - I)$ and

$$A^{\oplus}A^k(A^{\oplus}A - I) = U\begin{pmatrix} T^{k-1} & T^{-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & T^{-1}S \\ 0 & -I \end{pmatrix}U^* = U\begin{pmatrix} 0 & T^{k-2}S - T^{-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*,$$

$[A^{\oplus}A,\ A^{\oplus}A^k] = 0 \Leftrightarrow (A^{\oplus}A^k)(A^{\oplus}A) - (A^{\oplus}A)(A^{\oplus}A^k) = 0 \Leftrightarrow \widetilde{T}_k = T^{k-1}S \Leftrightarrow A$ is a weak group matrix by Theorem 3.2.

(i) ⇔ (iii): Set an arbitrary $j \in \mathbb{Z}^+$. By direct computation, we have

$$A^{\oplus}A^{k+j} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^j & \widetilde{T}_j \\ 0 & N^j \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{k+j-1} & T^{j-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*.$$

Since $A^{\oplus}A^{k+j}A^{\oplus}A - A^{\oplus}AA^{\oplus}A^{k+j} = A^{\oplus}A^{k+j}(A^{\oplus}A - I)$ and

$$A^{\oplus}A^{k+j}(A^{\oplus}A - I) = U\begin{pmatrix} 0 & T^{k+j-2}S - T^{j-1}\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*,$$

we have $[A^{\oplus}A,\ A^{\oplus}A^{k+j}] = 0 \Leftrightarrow (A^{\oplus}A^{k+j})(A^{\oplus}A) - (A^{\oplus}A)(A^{\oplus}A^{k+j}) = 0 \Leftrightarrow T^{k+j-2}S = T^{j-1}\widetilde{T}_k \Leftrightarrow \widetilde{T}_k = T^{k-1}S \Leftrightarrow A$ is a weak group matrix by Theorem 3.2.

(i) ⇔ (iv): Set an arbitrary $j \in \mathbb{Z}^+$. Similar to reasoning as in the proof of (i) ⇔ (iii), we get that $[A^{\oplus}A,\ A^{\oplus}A^{j+1}] = 0 \Leftrightarrow T^{j-1}S = T^{-1}\widetilde{T}_{j+1} \Leftrightarrow \sum_{i=0}^{j} T^iSN^{j-i} - T^jS = 0 \Leftrightarrow \widetilde{T}_jN = 0 \Leftrightarrow A$ is a weak group matrix by using Theorem 3.2. □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then the following conditions are equivalent:*

(i) *A is a weak group matrix;*

(ii) $A^k(I - P_{A^k})A = 0$;

(iii) $A^{k+j}(I - P_{A^k})A = 0$, *for arbitrary $j \in \mathbb{Z}^+$;*

(iv) $(A^j)^*A^k(I - P_{A^k})A = 0$, *for arbitrary $j \in \mathbb{Z}^+$;*

(v) $A^*A^k(I - P_{A^k})A = 0$.

Proof. Assume that $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ is written as in Lemma 2.1. We know that $A^k = U\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*$ and $P_{A^k} = A^k(A^k)^\dagger = U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}U^*$.

(i) ⇔ (iv): Set an arbitrary $j \in \mathbb{Z}^+$. We have

$$(A^j)^*A^k = U\begin{pmatrix} (T^j)^* & 0 \\ (\widetilde{T}_j)^* & (N^j)^* \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} (T^j)^*T^k & (T^j)^*\widetilde{T}_k \\ (\widetilde{T}_j)^*T^k & (\widetilde{T}_j)^*\widetilde{T}_k \end{pmatrix}U^*.$$

So

$$\begin{aligned} (A^j)^*A^k(I - P_{A^k})A &= U\begin{pmatrix} (T^j)^*T^k & (T^j)^*\widetilde{T}_k \\ (\widetilde{T}_j)^*T^k & (\widetilde{T}_j)^*\widetilde{T}_k \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^* \\ &= U\begin{pmatrix} 0 & (T^j)^*\widetilde{T}_kN \\ 0 & (\widetilde{T}_j)^*\widetilde{T}_kN \end{pmatrix}U^*. \end{aligned}$$

Since T is non-singular, $(A^j)^*A^k(I - P_{A^k})A = 0$ if and only if $\widetilde{T}_kN = 0$. From Theorem 3.2, we have $\widetilde{T}_kN = 0 \Leftrightarrow A$ is a weak group matrix.

(i) ⇔ (iii): Set an arbitrary $j \in \mathbb{Z}^+$. We have

$$A^{k+j} = A^jA^k = U\begin{pmatrix} T^j & \widetilde{T}_j \\ 0 & N^j \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{k+j} & T^j\widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^*.$$

So,

$$\begin{aligned} A^{k+j}(I-P_{A^k})A &= U\begin{pmatrix} T^{k+j} & T^j\widetilde{T}_k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^* \\ &= U\begin{pmatrix} 0 & T^j\widetilde{T}_k N \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

Since T is non-singular, $A^{k+j}(I-P_{A^k})A = 0 \Leftrightarrow \widetilde{T}_k N = 0$. Now, reasoning as in the proof of (i) $\Leftrightarrow$ (iv), the result holds.

(i) $\Leftrightarrow$ (ii): Set $j = 0$ in the proof of (i) $\Leftrightarrow$ (iii). It is easy to check that $A^k(I-P_{A^k})A = 0 \Leftrightarrow \widetilde{T}_k N = 0$. Now, reasoning as in the proof of (i) $\Leftrightarrow$ (iv), the result is verified.

(iv) $\Leftrightarrow$ (v): It is obvious. $\square$

Remark 4.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. It is known that $\mathrm{rk}(A) = 0$ if and only if $A = 0$. So, we observe that A is a weak group matrix if and only if $\mathrm{rk}(A^k(I-P_{A^k})A) = 0$ in Theorem 4.2.*

Next, we improve the result in [21, Theorem 4.8].

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k \geq 2$. Then the following conditions are equivalent:*

(i) *A is a weak group matrix;*

(ii) $\mathrm{rk}((A^*A^k, (A^*)^kA^k)) = \mathrm{rk}(A^k)$;

(iii) $\mathrm{rk}((A^*A^k, (A^*)^{j+1}A^k)) = \mathrm{rk}(A^k)$, *for arbitrary $j \in \mathbb{Z}^+$.*

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k \geq 2$ is written as in Lemma 2.1.

(i) $\Leftrightarrow$ (iii) : Set an arbitrary $j \in \mathbb{Z}^+$. We have $A^* = U\begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}U^*$ and

$$(A^*)^{j+1} = U\begin{pmatrix} (T^{j+1})^* & 0 \\ (\sum\limits_{i=0}^{j} T^iSN^{j-i})^* & (N^{j+1})^* \end{pmatrix}U^*.$$

Denoting $M = (\sum\limits_{i=0}^{j-1} T^iSN^{j-i})^*T^k$ and $\widetilde{N} = (\sum\limits_{i=0}^{j-1} T^iSN^{j-i})^*\widetilde{T}_k$. We get

$$A^*A^k = U\begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^*T^k & T^*\widetilde{T}_k \\ S^*T^k & S^*\widetilde{T}_k \end{pmatrix}U^*$$

and

$$\begin{aligned} (A^*)^{j+1}A^k &= U\begin{pmatrix} (T^{j+1})^* & 0 \\ (\sum\limits_{i=0}^{j} T^iSN^{j-i})^* & (N^{j+1})^* \end{pmatrix}\begin{pmatrix} T^k & \widetilde{T}_k \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} (T^{j+1})^*T^k & (T^{j+1})^*\widetilde{T}_k \\ (\sum\limits_{i=0}^{j} T^iSN^{j-i})^*T^k & (\sum\limits_{i=0}^{j} T^iSN^{j-i})^*\widetilde{T}_k \end{pmatrix}U^*, \end{aligned}$$

Now,

$$\begin{aligned} &(A^*A^k, (A^*)^{j+1}A^k) = \\ &U\begin{pmatrix} T^*T^k & T^*\widetilde{T}_k & (T^{j+1})^*T^k & (T^{j+1})^*\widetilde{T}_k \\ S^*T^k & S^*\widetilde{T}_k & (T^jS)^*T^k + M & (T^jS)^*\widetilde{T}_k + \widetilde{N} \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} = \\ &U\begin{pmatrix} I & 0 \\ S^*(T^*)^{-1} & I \end{pmatrix}\begin{pmatrix} T^*T^k & T^*\widetilde{T}_k & (T^{j+1})^*T^k & (T^{j+1})^*\widetilde{T}_k \\ 0 & 0 & M & \widetilde{N} \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix}. \end{aligned}$$

Since T is non-singular, $\mathrm{rk}(T) = \mathrm{rk}(A^k) = \mathrm{rk}((A^*A^k, (A^*)^{j+1}A^k)) = \mathrm{rk}(T) + \mathrm{rk}((M, \widetilde{N})) \Leftrightarrow \sum_{i=0}^{j-1} T^iSN^{j-i} = 0 \Leftrightarrow \widetilde{T}_jN = 0$. By Theorem 3.2, the result is verified.

(iii) $\Rightarrow$ (ii) : Taking $j = k-1$ in (iii), it is clear.

(ii) $\Rightarrow$ (i) : Similar to the proof of (i) $\Leftrightarrow$ (iii), we obtain $\widetilde{T}_{k-1}N = 0$. So we have

$$\widetilde{T}_k = \sum_{i=0}^{k-1} T^iSN^{k-1-i} = T^{k-1}S + \sum_{i=0}^{k-2} T^iSN^{k-1-i} = T^{k-1}S.$$

By Theorem 3.2, A is a weak group matrix. $\square$

Remark 4.5. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = 1$. Then $A^{Ⓦ} = A^{\#}$. So, A is a weak group matrix. In this case, the rank equalities $\mathrm{rk}((A^*A, A^*A)) = \mathrm{rk}(A)$ and $\mathrm{rk}((A^*A, (A^*)^{j+1}A)) = \mathrm{rk}(A)$ are always right. Therefore, we require $k \geq 2$ in Theorem 4.4.*

5. Relations with other special matrices

In this section, we present definition and an equivalent characterization of the $\{m, k\}$-core EP matrix firstly. Then, relationships between the weak group matrix, the k-EP matrix and the $\{m, k\}$-core EP matrix are studied.

Definition 5.1. *Let $A \in \mathbb{C}^{n\times n}$. A is a $\{m, k\}$-core EP matrix if $A^mA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^m$ for $\mathrm{ind}(A) = k$ and $m \in \mathbb{Z}^+$.*

From above definition, we observe that a $\{k, k\}$-core EP matrix is a k-core EP matrix. Next, we present a characterization of $\{m, k\}$-core EP matrices.

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ be written as in Lemma 2.1. Then A is a $\{m, k\}$-core EP matrix if and only if $\widetilde{T}_m = 0$.*

Proof. Suppose that A has a core-EP decomposition be as in Lemma 2.1. Similar to the proof of Theorem 3.2, by Lemma 2.1, it is easy to verify that A is a $\{m, k\}$-core EP matrix $\Leftrightarrow \sum_{i=0}^{m-1} T^iSN^{m-1-i} = 0 \Leftrightarrow \widetilde{T}_m = 0$. $\square$

Combining Theorem 3.2 and Theorem 5.2, we have the following corollary.

Corollary 5.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If A is a $\{m, k\}$-core EP matrix for arbitrary $m \in \mathbb{Z}^+$, then A is a weak group matrix.*

In the following example, we explain that a matrix is weak group matrix and not a $\{m, k\}$-core EP matrix for arbitrary $m \in \mathbb{Z}^+$.

Example 5.4. *Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with $\mathrm{ind}(A) = 2$. We have*

$$A^{\textcircled{\dagger}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{Ⓦ} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $AA^{Ⓦ} = A^{Ⓦ}A$, but $A^mA^{\textcircled{\dagger}} \neq A^{\textcircled{\dagger}}A^m$. So, A is not a $\{m, k\}$-core EP matrix for arbitrary $m \in \mathbb{Z}^+$.

According to the relationship between the core-EP inverse and the weak group inverse, some auxiliary results are given.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{rk}(A) = r$, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^n$ of dimension $n-s$. Then A has a $\{2\}$-inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^n$, in which case X is unique and denoted by $A^{(2)}_{T,S}$ [2]. For more details see [2, 18].

Lemma 5.5. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then $A^{\circledW} = A^{(2)}_{\mathcal{R}(A^k),\ \mathcal{N}((A^k)^*A)}$.*

Proof. From [20, Remark 3.5], we know that $\mathcal{R}(A^{\circledW}) = \mathcal{R}(A^k)$. On the one hand, $A^{\circledW}AA^{\circledW} = A^{\circledW}$ and

$$\mathcal{N}(A^{\circledW}) \subseteq \mathcal{N}(AA^{\circledW}) = \mathcal{N}(A^{\oplus}A) = \mathcal{N}(A^DA^k(A^k)^{\dagger}A) \subseteq \mathcal{N}(A^{\circledW}).$$

On the other hand,

$$\begin{aligned} \mathcal{N}((A^k)^*A) &\subseteq \mathcal{N}(A^k(A^k)^{\dagger}A) \subseteq \mathcal{N}(A^DA^k(A^k)^{\dagger}A) \\ &\subseteq \mathcal{N}(AA^DA^k(A^k)^{\dagger}A) = \mathcal{N}(A^k(A^k)^{\dagger}A) \subseteq \mathcal{N}((A^k)^*A). \end{aligned}$$

So, $\mathcal{N}((A^k)^*A) = \mathcal{N}(A^DA^k(A^k)^{\dagger}A) = \mathcal{N}(A^{\circledW})$. $\square$

Lemma 5.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then $A^{\oplus} = A^{\circledW}P_{A^k}$.*

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Set $X = A^{\circledW}P_{A^k}$. Then we have

$$XA^{k+1} = A^{\circledW}P_{A^k}A^{k+1} = A^{\circledW}A^{k+1} = A^k,$$

and $AX = AA^{\circledW}P_{A^k} = A^{\oplus}AP_{A^k} = A^{\oplus}A^{k+1}(A^k)^{\dagger} = P_{A^k}$. So, AX is Hermitian. Since $\mathcal{R}(A^{\circledW}) = \mathcal{R}(A^k)$ by Lemma 5.5,

$$AX^2 = AA^{\circledW}P_{A^k}A^{\circledW}P_{A^k} = AA^{\circledW}A^{\circledW}P_{A^k} = A^{\circledW}P_{A^k} = X.$$

Hence, $A^{\oplus} = X = A^{\circledW}P_{A^k}$ by definition of the core-EP inverse. $\square$

Proposition 5.7. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ be written as in Lemma 2.1. Then the following conditions are equivalent:*

(i) $S = 0$;

(ii) $A^{\circledW} = A^{\oplus}$;

(iii) $\mathcal{N}((A^k)^*) \subseteq \mathcal{N}((A^k)^*A)$.

In this case, A is a $\{m,k\}$-core EP matrix, for an arbitrary $m \in \mathbb{Z}^+$.

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ be written as in Lemma 2.1.

(i) $\Leftrightarrow$ (ii) : We have

$$A^{\oplus} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*,\ \ A^{\circledW} = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*.$$

So, $S = 0$ if and only if $A^{\oplus} = A^{\circledW}$.

(ii) $\Leftrightarrow$ (iii) : By Lemma 5.5 and 5.6, we know that

$$A^{\circledW} = A^{\oplus} \ \Leftrightarrow\ A^{\circledW}(I - P_{A^k}) = 0 \ \Leftrightarrow\ \mathcal{N}((A^k)^*) \subseteq \mathcal{N}(A^{\circledW}) = \mathcal{N}((A^k)^*A).$$

In this case, since $S = 0$, $SN = 0$. By Lemma 2.2, we have $AA^{\circledW} = A^{\circledW}A$. From conditions (i) and (ii), we get $AA^{\oplus} = A^{\oplus}A$. So, for arbitrary $m \in \mathbb{Z}^+$, we have $A^mA^{\oplus} = A^{\oplus}A^m$. That is, A is a $\{m,k\}$-core EP matrix. $\square$

Now, the relationships between $\{m,k\}$-core EP matrices, k-EP matrices and weak group matrices are investigated.

Theorem 5.8. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ be written as in Lemma 2.1. If $S = SNN^{\dagger} = SN^{\dagger}N$, then the following conditions are equivalent:*

(i) *A is a weak group matrix;*

(ii) *A is a k-EP matrix;*

(iii) *A is a core-EP matrix;*

(iv) $A^{ⓦ} = A^{c,\dagger} = A^{D,\dagger} = A^{\dagger,D} = A^{D}$.

Proof. Since $S = SN^{\dagger}N$, $\mathcal{N}(N) \subseteq \mathcal{N}(S)$.

(i) $\Leftrightarrow$ (ii) : Since $S = SNN^{\dagger}$, by Theorem 3.2, we have

$$AA^{ⓦ} = A^{ⓦ}A \;\Leftrightarrow\; \widetilde{T}_k = T^{k-1}S \;\Leftrightarrow\; \widetilde{T}_k = T^{k-1}SNN^{\dagger} = \widetilde{T}_kNN^{\dagger} \;\Leftrightarrow\; \mathcal{N}(N^*) \subseteq \mathcal{N}(\widetilde{T}_k).$$

By Lemma 2.5, $A \in \mathbb{C}_n^{k,\dagger} \Leftrightarrow A$ is a weak group matrix

(ii) $\Leftrightarrow$ (iii) : It is clear by Lemma 2.4.

(ii) $\Leftrightarrow$ (iv) : By Lemma 2.3 and Theorem 3.2, we know that $A \in \mathbb{C}_n^{k,\dagger}$ if and only if $A^{c,\dagger} = A^{D,\dagger} = A^{\dagger,D} = A^{D}$, and A is a weak group matrix if and only if $A^{ⓦ} = A^{D}$ [21]. Since (i) and (ii) are equivalent, we get $A^{ⓦ} = A^{c,\dagger} = A^{D,\dagger} = A^{\dagger,D} = A^{D}$. The converse is evident. □

6. Acknowledgement

We would like to thank the Referees for their valuable comments and suggestions that help us to improve the reading of the paper.

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