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Representation of essentially semi regular linear relations and perturbations

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Abstract. In the case of linear operator the property $P(B,k)$ was introduced by M.A.Kaashoek. In this paper, we characterize the essentially semi regular linear relation in terms of the property $P(B,k)$. After that and as an application of this result we give some connection between essentially semi regular and semi regular linear relations. Further, we will give some supplementary conditions on essentially semi regular linear relation to be semi Fredholm. Then, we analyze the stability of the class of essentially semi regular linear relations under small perturbations and Riesz operators. Finally, we study some properties of the essentially semi regular spectrum of a linear relation and we establish a spectral mapping theorem.

1. Introduction and Preliminary

Throughout this paper, we shall denote by X and Y two Banach spaces. A linear relation $T : X \to Y$ is a mapping having a nonempty subspace $D(T)$ of X called the domain of T and taking values in the collection of nonempty subsets of Y such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all nonzero α, β scalars and $x, y \in D(T)$. We define $D(T) := \{x \in X : \; Tx \neq \emptyset\}$. The class of all linear relations from X to Y is denoted by $\mathcal{LR}(X,Y)$ and abbreviate $\mathcal{LR}(X,X)$ by $\mathcal{LR}(X)$. A linear relation $T \in \mathcal{LR}(X,Y)$ is uniquely determined by its graph $G(T)$, which is defined by $G(T) = \Big\{(x,y) \in X \times Y : x \in D(T), \; y \in Tx\Big\}$. T is said a closed linear relation if $G(T)$ is a closed subspace of $X \times Y$.

The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \Big\{(y,x)/ \; (x,y) \in G(T)\Big\}$. If X is a normed linear space, then X' will denote the norm dual of X, i.e. the space of all continuous linear functionals on X, with the norm $\|x'\| := \inf\{\lambda : | \, x'x \, |\leq \lambda\|x\|, \;$ for all $\; x \in X\}$. We shall adopt the following notation: If $M \subset X$ and $N \subset X'$, then

$$M^{\perp} := \{x' \in X' : \; x'x = 0 \text{ for all } x \in M\}$$

$$N^{\top} := \{x \in X : \; x'x = 0 \text{ for all } x' \in N\}$$

The adjoint or the conjugate T^* of $T \in \mathcal{LR}(X,Y)$ is defined by

$$G(T^*) := G(-T^{-1})^{\perp} \subset Y' \times X'.$$

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This means that $(y', x') \in G(T^*)$ if and only if $y'(y) - x'(x) = 0$ for all $(x, y) \in G(T)$.
Recall that if T is a linear relation in X, then T^* is a closed linear relation in X' such that

$$D(T^*) = \{y' \in X' : \ y'T \text{ is continuous and single valued}\}.$$

The range of a linear relation $T \in \mathcal{LR}(X, Y)$ is defined by $R(T) = T(D(T))$ and T is called surjective (or onto) if $R(T) = Y$. The null space (or the kernel) of T is defined by $N(T) = \{x \in D(T) : \ (x, 0) \in G(T)\} = T^{-1}(0)$ and T is called injective if $N(T) = \{0\}$. If T is injective and has closed range, then it is bounded below. The nullity and the defect of a linear relation T is defined respectively by $\alpha(T) = dimN(T)$ and $\beta(T) = dimX/R(T)$.
For $T, S \in \mathcal{LR}(X)$, the product ST is given by $G(ST) := \{(x, y) \in X \times X : \ (x, y) \in G(T), \ (y, z) \in G(S) \text{ for some} y \in X\}$. Hence T^n, $n \in \mathbb{N}$ is defined as usual with $T^0 = I$ where I is the identity operator in X and $T^n = TT^{n-1}$.
Let $\lambda \in \mathbb{K}$ and $T \in \mathcal{LR}(X)$, then λT is the linear relation defined by $G(\lambda T) := \{(x, \lambda y) : \ (x, y) \in G(T)\}$. The algebraic resolvent set of a closed linear relation T is defined by Sandovici [16] as follows

$$\rho(T) := \{\lambda \in \mathbb{K} : \ T - \lambda \text{ is injective and surjective}\}$$

where $G(T - \lambda) := \{(x, y - \lambda x) : \ (x, y) \in G(T)\}$ and the spectrum of T is the set $\sigma(T) = \mathbb{K} \setminus \rho(T)$. The minimum modulus of $T \in \mathcal{LR}(X, Y)$ is the quantity

$$\gamma(T) := \sup\{\lambda : \lambda d(x, N(T)) \leq \|Tx\|, \text{ for all } x \in D(T)\},$$

where $d(x, N(T)) := \inf\{\|x - y\| \text{ with } y \in N(T)\}$.
Recall that $T \in \mathcal{LR}(X)$ is upper semi Fredholm (resp. lower semi Fredholm), denoted by $T \in \phi_+$ (resp. $T \in \phi_-$), if T is closed everywhere defined with closed range and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). If T is both upper and lower semi Fredholm, then T is said to be Fredholm. If T is either upper or lower semi Fredholm, then T is said to be semi Fredholm and its index is defined by $ind(T) = \alpha(T) - \beta(T)$. Recall that the ascent of $T \in \mathcal{LR}(X)$ is defined by $asc(T) = min\{k \in \mathbb{N} \cup \{0\} : \ N(T^k) = N(T^{k+1})\}$.
The generalized kernel and the generalized range of $T \in \mathcal{LR}(X)$ are the subspaces of X defined by $N^\infty(T) = \bigcup_{n=1}^{\infty} N(T^n)$ and $R^\infty(T) = \bigcap_{n=1}^{\infty} R(T^n)$, respectively. $T \in \mathcal{LR}(X)$ is called semi regular if its range is closed and $N(T) \subset R^\infty(T)$ (or equivalently, $N^\infty(T) \subset R(T)$).
For a closed subspace M of X, $T_{|_M}$ denotes the restriction of T to M. Thus $T_{|_M} := T \cap (M \times X)$. We denote by T_M the linear relation defined by $T_M := T \cap (M \times M)$. For $T, \ S \in \mathcal{LR}(X)$ the linear relation $T \hat{+} S$ is defined by

$$G(T \hat{+} S) := \{x + u, y + v) : \ (x, y) \in G(T), \ (u, v) \in G(S)\},$$

moreover when $G(T) \cap \ G(S) = \{(0, 0)\}$ we write $T \oplus S$. T is said to be a Kato decomposition of finite type, abbreviated KDF, if there exists a pair of closed subspaces (M, N) of X such that $X = M \oplus N$, $dimN < \infty$, $T = T_M \oplus T_N$ with T_M is a semi regular linear relation and T_N is a bounded nilpotent operator.
T is said to be a Kato decomposition of finite type of degree d, abbreviated KDF(d), if there exists a pair of closed subspaces (M, N) of X such that $X = M \oplus N$, $dimN < \infty$, $T = T_M \oplus T_N$ with T_M is a semi regular linear relation and T_N is a bounded nilpotent operator of degree d.
For two subspaces M and N of X, we write $M \subset_e N$ if there exists a finite-dimensional subspace F of X such that $M \subset N + F$. Obviously $M \subset_e N$ if and only if $dim[M/(M \cap N)] < \infty$. Notice that we can assume that F is a subset of M. Similarly, we write $M =_e N$ if both $M \subset_e N$ and $N \subset_e M$.
This notation is used in the definition of essentially semi regular linear relation which is appeared in the first time in [5] then in [6] as follows: Let T be an everywhere defined closed linear relation. Then T is essentially semi regular linear relation if $R(T)$ is closed and $N^\infty(T) \subset_e R^\infty(T)$.
For two linear operators A and B from X to Y, M.A.Kaashoek introduced the property $P(B, k)$ in [10] as a generalization of the notion of semi Fredholm operators. In the present paper and in section 2, we associate to a linear relation T two operators A and B defined from $G(T)$ into X by $A(x, y) = y$ and $B(x, y) = x$. Then we characterize the essentially semi regular linear relation in terms of the property $P(B, k)$. More precisely, we prove that T is essentially semi regular if and only if A has the property $P(B, k)$. This brings in particular

the study of a linear relation to the study of bounded operators associated with this linear relation. This characterization helps us to give a representation of essentially semi regular linear relations made by semi regular and semi Fredholm linear relations.
In section 3, we analyze the stability of the class of essentially semi regular linear relations under small perturbations and Riesz operators.
Finally in section 4, we give some properties of the product and the power of essentially semi regular linear relations and the essentially semi regular linear relation spectrum. Then we establish a spectral mapping theorem.

2. Representation of an essentially semi regular linear relation

In this section we recall the definition of essentially semi regular linear relation and we introduce the notion of the property $P(B, k)$. After that, given a relation $T \in \mathcal{LR}(X)$, we define two operators A and B from $G(T)$ into X by $A(x, y) = y$ and $B(x, y) = x$. Our objective is to characterize an essentially semi regular linear relation in terms of the property $P(B, k)$. We shown that T is an essentially semi regular relation if and only if A has the property $P(B, k)$. As an application of this result we give an interesting connection between essentially semi regular and semi regular linear relations. Further, we give some supplementary conditions to essentially semi regular linear relations to be semi Fredholm.
We start this section by recalling the definition of an essentially semi regular linear relation.

Definition 2.1. *[5, Definition 3.1] Let T be an everywhere defined closed linear relation. T is called essentially semi regular linear relation if $R(T)$ is closed and $N^\infty(T) \subset_e R^\infty(T)$.*

The class of all essentially semi regular linear relations in X will be denoted by $ESR(X)$.

Now, we state the following theorem which is an amelioration of Theorem 3.1 in [5]. More precisely we have the associated same conclusions of Theorem 3.1 in [5] without the hypothesis $\rho(T) \neq \emptyset$.

Theorem 2.2. *Let T be an everywhere defined closed linear relation with closed range. The following properties are equivalent:*
(1) *T is essentially semi regular linear relation,*
(2) *T has a Kato decomposition of finite type,*
(3) $N^\infty(T) \subset_e R(T)$,
(4) $N(T) \subset_e R^\infty(T)$.

Proof: We consider the same proof of the Theorem 3.1 in [5]. Just we replace [4, Lemma 2.3 (ii)] by [2, Lemma 3.5], because in [4, Lemma 2.3 (ii)] we need the hypothesis $\rho(T) \neq \emptyset$ to deduce $(T^*)^n = (T^n)^*$ but in [2, Lemma 3.5] and since T is everywhere defined we have the same conclusion without $\rho(T) \neq \emptyset$. □

Remark 2.3. *We can see by the previous theorem that $T \in ESR(X)$ if and only if $R(T)$ is closed and $dim N(T)/N(T) \cap R^\infty(T) < \infty$.*

Definition 2.4. *Let $T \in ESR(X)$. We call order of T the integer k such that*

$$dim N(T)/N(T) \cap R^\infty(T) = k.$$

Corollary 2.5. *Let T be a closed linear relation everywhere defined in a Banach space X. If T is essentially semi regular, then $T(R^\infty(T)) = R^\infty(T)$.*

Proof: Assume that $T \in ESR(X)$. By Theorem 2.2, we have $X = M \oplus N$ such that $\dim N < \infty$, $T_N = T \cap (N \times N)$ is a bounded nilpotent operator and $T_M = T \cap (M \times M)$ is a semi regular linear relation with $T = T_M \oplus T_N$. We can see that $R^\infty(T) \subset M$ and by [1, Lemma 20], we obtain $T(R^\infty(T)) = T_M(R^\infty(T)) = R^\infty(T)$. □

Corollary 2.6. *Let T be a closed linear relation everywhere defined in a Banach space X with $\rho(T) \neq \emptyset$. If T is essentially semi regular, then T^* has a Kato decomposition of finite type.*

Proof: Since $T \in ESR(X)$, then by Theorem 2.2, T has a Kato decomposition of finite type. Let (M, N) be KDF associated to T. Hence by [4, Theorem 3.2] $(N^{\perp}, M^{\perp}) \in KDF(T^*)$. □

Corollary 2.7. *Let T be an everywhere defined closed linear relation in X with closed range. If there exists a subspace $X_0 \subset X$ such that $TX_0 = X_0$ and $\dim N(T)/(X_0 \cap N(T)) = k < \infty$, then T is essentially semi regular of order lower than k.*

Proof: Since $TX_0 = X_0$, then $X_0 \subset R^{\infty}(T)$. Thus $dim N(T)/(N(T) \cap R^{\infty}(T)) \leq dim\ N(T)/(X_0 \cap N(T)) = k < \infty$. Hence by Theorem 2.2, T is essentially semi regular linear relation of order lower than k. □

We will now introduce the notion we are interested in, which is due to M.A.Kaashoek[10]. For any two operators, we will define the notion of $D(A, B)$ and $R(A, B)$ as follows:
Let A and B be two linear operators with domains $D(A)$ and $D(B)$, linear subspaces of X satisfying $D(A) \subset D(B)$ and with ranges in Y.
We introduce the following sequences of subspaces $D_n(A, B)$ $(n = 0, 1, 2...)$ in X and $R_n(A, B)$ $(n = 0, 1, 2...)$ in Y :

$$D_0(A, B) = X,\ R_0(A, B) = Y,$$

for $n = 1, 2, ...$

$$R_n(A, B) = AD_{n-1}(A, B) \text{ and } D_n(A, B) = B^{-1}R_n(A, B).$$

In other words, for $n = 1, 2...$

$$R_n(A, B) = AB^{-1}R_{n-1}(A, B) \text{ and } D_n(A, B) = B^{-1}AD_{n-1}(A, B).$$

We denote $D(A, B) = \bigcap\limits_{n \geq 0} D_n(A, B)$ and $R(A, B) = \bigcap\limits_{n \geq 0} R_n(A, B)$.

Definition 2.8. *[10, Definition 2.1] Let A and B be two operators from X into Y. We say that A has the property $P(B, k)$ if there exists a non negative integer k such that*

$$dim N(A)/(N(A) \cap D(A, B)) = k.$$

Now, to express the main theorem of this section which gives a characterization of essentially semi regular linear relations of order k in terms of the property $P(B, k)$, we need to express the technical lemma.

Lemma 2.9. *Let J be a bijective linear relation in X and M and N be two closed subspaces of X. Then $(M + N)/N \sim J(M + N)/J(N)$.*

Proof: Let $\phi : (M + N)/N \to J(M + N)/J(N)$

$$\bar{x} \mapsto \phi(\bar{x}) = \{\tilde{y};\ y \in Jx\}.$$

It is clear that ϕ is a linear operator.
Let $\bar{x} \in N(\phi)$. So $\phi(\bar{x}) = \tilde{0}$. Thus for all $y \in Jx$, $\tilde{y} = \tilde{0}$. Hence $Jx \subset J(N)$. Then there exists $\alpha \in N$ such that $Jx = J\alpha$. So $x - \alpha \in N(J) = \{0\}$. Thus $x = \alpha$. Therefore $\bar{x} = \bar{\alpha} = \bar{0}$. Consequently $N(\phi) = \{\bar{0}\}$. Let $\tilde{z} \in J(M + N)/J(N)$. So there exists $y \in J(M + N)$ such that $\tilde{z} = \tilde{y}$. Thus there exists $x \in M + N$ such that $y \in Jx$. Then $\phi(\bar{x}) = \tilde{y} = \tilde{z}$. Therefore ϕ is bijective. □

Theorem 2.10. *Let T be an everywhere defined closed linear relation in X with closed range. Let A and B be the two linear operators defined from $G(T)$ into X by $A(x, y) = y$ and $B(x, y) = x$ respectively. Then T is essentially semi regular of order k if and only if A has the property $P(B, k)$.*

Proof: Observe that the linear relation J, defined on $D(T) = X$ by

$$J : X \to G(T)$$

$$x \mapsto \{(x, y)/y \in Tx\}$$

is bijective. We claim that $AJ = T$. Indeed, let $(x, y) \in G(T)$. So $(x, y) \in Jx$. Thus $A(x, y) = y \in AJ(x)$. Then

$(x, y) \in G(AJ)$. Now, let $(x, y) \in G(AJ)$. So $y \in AJx$. Hence there exists $(x, z) \in Jx$ such that $y \in A(x, z) = z \in Tx$. Then $(x, y) \in G(T)$.
We claim that $BJ = I$. Indeed, let $x \in X$. So $BJx = B(x, z) = x$, where $z \in Tx$.
We claim that, for all $n \in \mathbb{N}$, $D_n(A, B) = JR(T^n)$. The proof will be given by induction on $n \in \mathbb{N}$. For $n = 0$, $D_0(A, B) = D(A) = G(T) = J(X)$. Assume that the assertion is true for $n \geq 0$. We have $D_{n+1}(A, B) = B^{-1}AD_n(A, B) = B^{-1}AJ(R(T^n)) = B^{-1}T(R(T^n)) = B^{-1}(R(T^{n+1})) = J(R(T^{n+1}))$.
We claim that, for all $n \in \mathbb{N}$, $R_n(A, B) = R(T^n)$. We have $R_n(A, B) = AD_{n-1}(A, B) = AJR(T^{n-1}) = TR(T^{n-1}) = R(T^n)$.
We claim that $D(A, B) = JR^\infty(T)$. Indeed, we have $D(A, B) = \bigcap_{n\geq 0} D_n(A, B) = \bigcap_{n\geq 0} J(R(T^n))$. Now, we prove that $\bigcap_{n\geq 0} J(R(T^n)) = J(\bigcap_{n\geq 0} R(T^n))$. We have for all $n \in \mathbb{N}$, $\bigcap_{n\geq 0} R(T^n) \subset R(T^n)$. So,

$$J(\bigcap_{n\geq 0} R(T^n)) \subset J(R(T^n)) \quad \forall n \in \mathbb{N}.$$

Thus $J(\bigcap_{n\geq 0} R(T^n)) \subset \bigcap_{n\geq 0} J(R(T^n))$. For the reverse inclusion, let $z \in \bigcap_{n\geq 0} J(R(T^n))$. So for all $n \in \mathbb{N}$, $z \in J(R(T^n))$. Thus for all $n \in \mathbb{N}$, $J^{-1}z \subset R(T^n) \cap D(J) + J^{-1}(0)$. Since $D(J) = X$ and $N(J) = \{0\}$, then for all $n \in \mathbb{N}$, $J^{-1}z \in R(T^n)$. So $J^{-1}z \in \bigcap_{n\geq 0} R(T^n)$. Hence $z + J(0) \subset J(\bigcap_{n\geq 0} R(T^n))$. As $J(0) \subset J(\bigcap_{n\geq 0} R(T^n))$, then $z \in J(\bigcap_{n\geq 0} R(T^n))$. Therefore $\bigcap_{n\geq 0} J(R(T^n)) = J(\bigcap_{n\geq 0} R(T^n))$. Consequently,

$$D(A, B) = J(\bigcap_{n\geq 0} R(T^n)) = JR^\infty(T).$$

We claim that $N(A) + J(0) = J(N(T))$. Indeed, since $AJ = T$, so $J^{-1}A^{-1}(0) = T^{-1}(0)$. Thus $JJ^{-1}N(A) = JN(T)$. Hence $N(A) \cap R(J) + J(0) = J(N(T))$. Therefore $N(A) + J(0) = J(N(T))$.
Using Lemma 2.9, it follows that

$$\begin{aligned} N(T)/(N(T) \cap R^\infty(T)) &\sim (N(T) + R^\infty(T))/R^\infty(T) \\ &\sim (J(N(T) + R^\infty(T)))/J(R^\infty(T)) \\ &= (N(A) + J(0) + J(R^\infty(T)))/J(R^\infty(T)) \\ &= (N(A) + D(A, B))/D(A, B) \\ &\sim N(A)/(N(A) \cap D(A, B)). \end{aligned}$$

Consequently, $T \in ESR(X)$ if and only if A has the property $P(B, k)$. □

In [10, Lemma 2.4], Kaashoek gave a sufficient condition for an operator T to have the property $P(S, k)$. In the particular case where $D(T) = X$ and S is the identity operator, this gives a sufficient condition for T to be essentially semi regular of order k. In the following corollary and using Theorem 2.10, we generalize this result in the general setting of linear relations to give a sufficient condition for a linear relation T to be essentially semi regular.

Corollary 2.11. *Let T be an everywhere defined closed linear relation in X with closed range. If there exists a closed subspace $X_0 \subset X$ such that*
(i) $TX_0 \subset X_0$,
(ii) $N(T) \subset X_0$,
(iii) $dim X_0/TX_0 = l < \infty$,
then $T \in ESR(X)$ of order $k \leq l$.

Proof: Let $T_0 = T_{|X_0}$. Then T_0 is an everywhere defined closed linear relation in the Banach space X_0. We have $dim X_0/R(T_0) = l$, so there exists an l-dimensional subspace L such that $X_0 = L \oplus R(T_0)$. Let A and B be the linear operators defined on $G(T_0)$ by $A(x, y) = y$ and $B(x, y) = x$. By the proof of Theorem 2.10, we have $R_n(A, B) = R(T_0^n)$. Hence $X_0 = R_1(A, B) \oplus L$. By the same procedure of proof of [10, Lemma 2.4], we obtain that

$dimN(A)/(D(A,B) \cap N(A)) \leq l$. Thus A has the property $P(B,k')$ for some k'. Since X_0/TX_0 and X_0 are closed, then $R(T_0)$ is closed. It follows by Theorem 2.10 that $T_0 \in ESR(X)$ and $dimN(T_0)/(N(T_0) \cap R^\infty(T_0)) = k' \leq l$. As $N(T) \subset X_0$, then $dimN(T)/(N(T) \cap R^\infty(T) \leq dimN(T_0)/(N(T_0) \cap R^\infty(T_0)) = k' \leq l$. So the desired result. □

In the following corollary, we give a necessary condition of essentially semi regular linear relation. This result relates to giving a relationship between the order of essentially semi regular linear relation and its defect.

Corollary 2.12. *If T is essentially semi regular of order k, then $dimX/R(T) \geq k$.*

Proof: Let $l = dimX/R(T)$. If $l = +\infty$, then the inequality is trivial. Assume therefore that $dimX/R(T) = l < \infty$. Then by Corollary 2.11 with $X_0 = X$, we obtain that $dimN(T)/N(T) \cap R^\infty(T) = k \leq l$. □

As an application of Theorem 2.2 and Theorem 2.10 we give some supplementary conditions on essentially semi regular linear relations to be semi Fredholm and we give some connection between essentially semi regular and semi regular linear relations. This showed the interesting representation of an essentially semi regular linear relation. More precisely, in the following first theorem we prove that if $T \in ESR(X)$, then there exists a bounded linear operator C with $dimR(C) < \infty$ such that $T + C$ is semi regular. We note that this brings an extension of Theorem 3.2 and Theorem 3.3 in [10] to the case of linear relations in the particular case where B is the identity operator and $k = 0$.

Theorem 2.13. *Let T be an everywhere defined closed linear relation in X. If T is essentially semi regular of order k, then there exists a bounded linear operator C in X with $dimR(C) = k$ such that $T + C$ is semi regular.*

Proof: Since $T \in ESR(X)$ of order k, then $dimN(T)/(N(T) \cap R^\infty(T)) = k$. So there exists a k-dimensional subspace $N \subset N(T)$ such that $N \cap R^\infty(T) = \{0\}$ and $N \oplus R^\infty(T) = N(T) + R^\infty(T)$. The subspace $R^\infty(T)$ is closed in X. Then, as a consequence of the Hahn-Banach Theorem, there exists a closed subspace $X_0 \subset X$ such that $X = N \oplus X_0$ and $R^\infty(T) \subset X_0$. If $x_1, ..., x_k$ form a basis of N, we have for each $x \in X$ that

$$x = \sum_{i=1}^{k} \alpha_i x_i + x_0$$

with $x_0 \in X_0$. We may take $\|x\|_N = \sum_{i=1}^{i=k} |\alpha_i|$. As $dimN = k < \infty$, there exists $\beta > 0$ such that for all $x \in N$ we have

$$\frac{1}{\beta}\|x\|_N \leq \|x\| \leq \beta\|x\|_N.$$

We have by Corollary 2.12, $dimX/R(T) \geq k$. Let $\epsilon > 0$. Choose $y_1, ..., y_k \in X$ linearly independent modulo $R(T)$ and denoted by P the continuous projection from X to N. Define for each $x \in X$

$$Cx = \sum_{i=1}^{k} \alpha_i y_i.$$

As
$$\begin{aligned}\|Cx\| &= \|\sum_{i=1}^{i=k} \alpha_i y_i\| \\ &\leq \sum_{i=1}^{i=k} |\alpha_i| sup_{1\leq i\leq k}\|y_i\| \\ &\leq \beta sup_{1\leq i\leq k}\|y_i\|\|P\|\|x\|.\end{aligned}$$

Hence C is a bounded linear operator in X with $dimR(C) = k$. To prove that $T + C$ is semi regular, we claim in the first way that $R(T + C)$ is closed. Indeed, since $dimR(C) < \infty$, so $R(T + C) \subset_e R(T)$. Thus if we replaced

T by $T + C$ and C by $-C$, we obtain that $R(T) \subset_e R(T + C)$. Hence $R(T + C) =_e R(T)$. Then there exits a finite dimensional subspace H such that $R(T + C) + H = R(T) + H$ which is closed. So, it follows from [8, Lemma 3.2] that $R(T + C)$ is closed.
In the second way, we claim that $dimN(T + C)/(N(T + C) \cap R^\infty(T + C)) = 0$. For that we prove that $N(T + C) = N(T) \cap R^\infty(T)$. Since $R(T) \cap R(C) = \{0\}$, we have $x \in N(T + C)$ if and only if $x \in N(T) \cap N(C) = N(T) \cap X_0 = N(T) \cap R^\infty(T)$. Thus $dimN(T + C)/(N(T + C) \cap R^\infty(T)) = dimN(T + C)/((N(T) \cap R^\infty(T)) \cap R^\infty(T)) = dimN(T + C)/(N(T) \cap R^\infty(T)) = dimN(T + C)/N(T + C) = 0$.
As $(T + C)R^\infty(T) = TR^\infty(T) = R^\infty(T)$, then by Corollary 2.7, $dimN(T + C)/(N(T + C) \cap R^\infty(T + C)) = 0$. Therefore $T + C$ is semi regular. □

Now, in the following second theorem we prove under certain conditions that the sum of a semi regular linear relation and a finite rank operator is essentially semi regular linear relation. Note that, in the particular case of the operators, we find, as a consequence, the result given in [10, Theorem 3.3] with B considered as the identity operator.

Theorem 2.14. *Let T be an everywhere defined closed linear relation in X. If T is semi regular and C is a bounded linear operator in X with $dimR(C) = k$ and $C(R^\infty(T)) = \{0\}$, then $T + C$ is essentially semi regular.*

Proof: We will use Corollary 2.11 to prove that $T + C$ is essentially semi regular. Since T is semi regular, then $N(T) \subset R^\infty(T)$ which is closed. Let $X_0 = (T + C)^{-1}R^\infty(T)$ which is closed by [3, Proposition 2.6]. We have $N(T + C) \subset X_0$. By the same procedure of the previous theorem we prove that $R(T + C)$ is closed. Now, we claim that $(T + C)X_0 \subset X_0$. Indeed, $(T + C)X_0 = R^\infty(T) \cap R(T + C) + T(0) = R^\infty(T) \cap R(T + C)$. Now we prove that $R^\infty(T) \subset R(T + C)$. Let $x \in R^\infty(T) = TR^\infty(T)$. So there exists $y \in R^\infty(T)$ such that $x \in Ty$. As $C(R^\infty(T)) = \{0\}$, it follows that $x \in (T + C)y \subset R(T + C)$. Hence $R^\infty(T) \subset R(T + C)$ and $(T + C)X_0 = R^\infty(T)$. We claim that $R^\infty(T) \subset X_0$. Indeed, let $x_0 \in R^\infty(T)$. Thus $(T + C)x_0 = Tx_0 + Cx_0 = Tx_0 \subset R^\infty(T)$. Hence $x_0 + (T + C)^{-1}(0) \subset (T + C)^{-1}R^\infty(T)$. Then $x_0 \in (T + C)^{-1}R^\infty(T) = X_0$. Therefore $(T + C)X_0 = R^\infty(T) \subset X_0$.
We claim that $dimX_0/R^\infty(T) < \infty$. Indeed, let $x_0 \in X_0 = (T + C)^{-1}R^\infty(T)$. So $(T + C)x_0 \cap R^\infty(T) \neq \emptyset$. Thus there exists $y \in R^\infty(T)$ such that $y \in (T + C)x_0$. Since $y \in R^\infty(T) = TR^\infty(T)$, so there exists $y_0 \in R^\infty(T)$ such that $y \in Ty_0 = (T + C)y_0$. Hence $y \in (T + C)x_0 \cap (T + C)y_0$. Then $(T + C)x_0 = (T + C)y_0$. Therefore $x_0 - y_0 \in N(T + C)$. Thus there exists $\alpha \in N(T + C)$ such that $x_0 = y_0 + \alpha$. Since $y_0 \in R^\infty(T)$, so $\bar{y_0} = \bar{0}$. Hence $\bar{x_0} = \bar{\alpha}$. As $\alpha \in N(T + C)$, it follows that $T\alpha \cap R(C) \neq \emptyset$. Then $\alpha \in T^{-1}(R(C))$. Since $dimR(C) < \infty$, so there exists a finite dimensional subspace F such that $T^{-1}(R(C)) = F + T^{-1}(0)$. Hence there exist $\alpha_1 \in F$ and $\alpha_2 \in N(T)$ such that $\alpha = \alpha_1 + \alpha_2$. We have $\alpha_2 \in N(T) \subset R^\infty(T)$, so $\bar{\alpha_2} = \bar{0}$. Thus $\bar{\alpha} = \bar{\alpha_1} \in (F + R^\infty(T))/R^\infty(T) = \bar{F}$ which has finite dimensional. Therefore $X_0/R^\infty(T) \subset \bar{F}$. Consequently $dimX_0/R^\infty(T) = dimX_0/(T + C)X_0 < \infty$. Now, using Corollary 2.11, we get $T + C \in ESR(X)$. □

In [5] we say that a semi Fredholm linear relation is essentially semi regular. Now, we define two classes of linear relations which help us to study the opposite direction.

Definition 2.15. *We say that a relation $T \in \mathcal{LR}(X)$ is almost bounded below if there exists $\delta > 0$ such that for all $0 < |\lambda| < \delta$ we have $T - \lambda I$ is bounded below.*

Theorem 2.16. *Let $T \in ESR(X)$. If T is almost bounded below, then T is upper semi Fredholm.*

Proof: Since $T \in ESR(X)$, then by Theorem 2.2, $T = T_1 \oplus T_2$ with T_1 is a semi regular linear relation and T_2 is a nilpotent operator. Since T_1 is semi regular, then by [1, Theorem 23], there exists $\nu > 0$ such that $T_1 - \lambda$ is a semi regular for $|\lambda| < \nu$. Hence $R(T_1 - \lambda)$ is closed for $|\lambda| < \nu$. As T_2 is nilpotent, it follows that $T_2 - \lambda$ is invertible for $\lambda \neq 0$. We say by hypothesis that there exists $\delta > 0$ such that for all $0 < |\lambda| < \delta$, $T - \lambda I$ is bounded below. On the other hand, for $0 < |\lambda| < \delta$, $T - \lambda = T_1 - \lambda \oplus T_2 - \lambda$. By [15, (8.2)], we have $N(T - \lambda) = N(T_1 - \lambda) + N(T_2 - \lambda)$. Thus $N(T_1 - \lambda) = \{0\}$ for $0 < |\lambda| < \delta$. Hence $T_1 - \lambda$ is bounded below for $0 < |\lambda| < min(\delta, \nu)$. So T_1 is almost bounded below. Thus by [9, Theorem 2.1], we conclude that T_1 is bounded below. It follows that T_1 is upper semi Fredholm. By [15, Theorem 8.2], we have $\alpha(T) = \alpha(T_1) + \alpha(T_2)$. Consequently, T is upper semi Fredholm. □

Definition 2.17. *We say that a relation $T \in \mathcal{LR}(X)$ is almost onto if there exists $\delta > 0$ such that for all $0 < |\lambda| < \delta$ we have $T - \lambda I$ is onto.*

Theorem 2.18. *Let $T \in ESR(X)$. If T is almost onto, then T is lower semi Fredholm.*

Proof: Since $T \in ESR(X)$, then by Theorem 2.2, $T = T_1 \oplus T_2$ with T_1 is a semi regular linear relation and T_2 is a nilpotent operator. We say by hypothesis that there exists $\delta > 0$ such that for all $0 < |\lambda| < \delta$, $T - \lambda I$ is onto. On the other hand, for $0 < |\lambda| < \delta$, $T - \lambda = T_1 - \lambda \oplus T_2 - \lambda$. As T_2 is nilpotent, it follows that $T_2 - \lambda$ is invertible for $\lambda \neq 0$. By [15, (8.3)], we have $R(T-\lambda) = R(T_1-\lambda) \oplus R(T_2-\lambda)$. Thus $T_1 - \lambda$ is onto for $0 < |\lambda| < \delta$. Hence T_1 is almost onto. So by [9, Theorem 2.1], we conclude that T_1 is onto. It follows that T_1 is lower semi Fredholm. By [15, Theorem 8.2], we have $\beta(T) = \beta(T_1) + \beta(T_2)$. Consequently, T is lower semi Fredholm. □

3. Some perturbation results of essentially semi regular linear relations

We analyze in this section the stability of the class of essentially semi regular linear relations under small perturbations and Riesz perturbations. After that we will show when an essentially semi regular linear relation is semi Fredholm.

Lemma 3.1. *Let T be a closed linear relation everywhere defined and $d \in \mathbb{N}^*$. Suppose that T has a KDF(d). Then $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$. Furthermore, we have $T(D(T) \cap R^\infty(T)) = R^\infty(T)$.*

Proof: Since T has a KDF(d), then by [3, Proposition 2.5], for every nonnegative integer $n \geq d$, we have $N(T) \cap R(T^n) = N(T) \cap R(T^d)$. So $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$. It remains to show that $T(D(T) \cap R^\infty(T)) = R^\infty(T)$. For every linear relation, we have $T(D(T) \cap R^\infty(T)) \subset R^\infty(T)$. So it suffices to prove the opposite inclusion. For this, let $y \in R^\infty(T)$. Then for each $n \in \mathbb{N}$, there exists $x_n \in D(T) \cap R(T^n)$ such that $y \in Tx_n$. Thus $0 \in T(x_n - x_m)$, for all $n, m \in \mathbb{N}$. This implies that $x_n - x_m \in N(T)$. We get that, for all $n, m \in \mathbb{N}$,

$$x_{n+d} - x_d \in N(T) \cap R(T^d) = N(T) \cap R(T^{d+n}) \subset R(T^{d+n}).$$

It follows that $x_d \in R(T^{d+n})$, for all $n \geq 0$. Hence $x_d \in R^\infty(T)$ and since $y \in Tx_d$, we deduce that $y \in T(D(T) \cap R^\infty(T))$. □

Now, we analyze in the following theorem the stability of essentially semi regular linear relations under small perturbations which is a generalization of [12, Theorem 14 of Chapter III].

Theorem 3.2. *Let X be a Banach space, T be a closed linear relation everywhere defined with $\rho(T) \neq \emptyset$ and S be a bounded operator satisfying $TS = ST$. Suppose that $T \in ESR(X)$. Then, there exists $\epsilon > 0$ such that, if $\|S\| < \epsilon$, then $T + S \in ESR(X)$ and*

$$R^\infty(T+S) \cap \overline{N^\infty(T+S)} \subset R^\infty(T) \cap \overline{N^\infty(T)}. \tag{1}$$

Furthermore, if $asc(T) < \infty$, then $asc(T+S) < \infty$.

Proof: Set $M = R^\infty(T)$. Then M is a closed subspace of X invariant with respect to T and S. Let $T_1 = T_{|_M}$ and $S_1 = S_{|_M}$ be the corresponding restrictions. Denote further, $\tilde{T} : X/M \to X/M$ and $\tilde{S} : X/M \to X/M$ the operators induced by T and S, respectively. By Theorem [5, Theorem 3.2], T_1 is surjective and $\tilde{T}$ is upper semi Fredholm. If $\|S\| < \gamma(T_1)$, we have by [7, Corollary 1.4.3] $T_1 + S_1$ is surjective and if $\|S\| < lim_{n\to\infty}\gamma((\tilde{T}^n)^{1/n})$, we have by [17] $\tilde{T} + \tilde{S}$ is upper semi Fredholm. It remains to show that $\rho(T+S) \neq \emptyset$. Since $\rho(T) \neq \emptyset$, then there exists λ_0 such that $T - \lambda_0$ is injective and surjective. Hence by [7, Corollary 1.4.3], $T + S - \lambda_0$ is injective and surjective for all $\|S\| < \gamma(T - \lambda_0)$. If we take $\epsilon_1 = min\{\gamma(T_1),\ lim_{n\to\infty}\gamma((\tilde{T}^n)^{1/n}),\ \gamma(T-\lambda_0)\}$, Then by [5, Theorem 3.2], $T + S \in ESR(X)$ for all S bounded operator satisfying $TS = ST$ and $\|S\| < \epsilon_1$. To prove (1) we claim first that

$$R^\infty(T^*) \subset R^\infty(T^* + S^*). \tag{2}$$

Let $T^*_\infty : D(T^*) \cap R^\infty(T^*) \to R^\infty(T^*)$ be the restriction of T^* to $R^\infty(T^*)$ and S^*_∞ be the restriction of S^* to $R^\infty(T^*)$. Since $T \in ESR(X)$, then by Proposition [5, Proposition 4.1] $T^k \in ESR(X)$ for all $k \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$, T^k is closed and $R(T^k)$ is closed. By the closed range theorem $R(T^{k*})$ is also closed. Thus $R^\infty(T^*)$ is closed and so T^*_∞ is a closed operator. Since $T \in ESR(X)$ then T^* has a KDF. Thus by Lemma 3.1, $T^*(D(T^*) \cap R^\infty(T^*)) = R^\infty(T^*)$. Hence T^*_∞ is surjective and thus $\gamma(T^*_\infty) > 0$. Now, if we take $\epsilon = min\{\epsilon_1,\ \gamma(T^*_\infty)\}$, then $\|S^*_\infty\| < \|S^*\| = \|S\| < \gamma(T^*_\infty)$. Since T and S commute, we obtain that $T^*_\infty + S^*_\infty : D(T^*) \cap R^\infty(T^*) \longrightarrow R^\infty(T^*)$ and so by [7, Corollary 1.4.3], $T^*_\infty + S^*_\infty$ is surjective for all $\|S^*_\infty\| < \gamma(T^*_\infty)$. Thus for all $\|S^*_\infty\| < \gamma(T^*_\infty)$, $(T^*_\infty + S^*_\infty)(D(T^*) \cap R^\infty(T^*)) = R^\infty(T^*)$. Let $E = D(T^*) \cap R^\infty(T^*)$. So $((T^*_\infty + S^*_\infty)(E)) \cap E = E$. Thus $(T^*_\infty + S^*_\infty)[((T^*_\infty + S^*_\infty)(E)) \cap E] = (T^*_\infty + S^*_\infty)(E) = R^\infty(T^*)$. Hence $R^\infty(T^*) \subset ((T^*_\infty + S^*_\infty)^2(E)) \cap R^\infty(T^*) \subset R((T^*_\infty + S^*_\infty)^2)$. Repeat the same procedure, we obtain that for all $\|S^*_\infty\| < \gamma(T^*_\infty)$, $R^\infty(T^*) \subset R^\infty(T^* + S^*)$.
Now, we will prove that

$$\overline{N^\infty(T+S)} \subset \overline{N^\infty(T)}. \tag{3}$$

In fact, since $\rho(T) \neq \emptyset$, then by [4, Lemma 2.3], we have for all $k \in \mathbb{N}$, $T^{k*} = T^{*k}$. It follows that

$$\overline{N^\infty(T)} = (N^\infty(T))^{\perp\top} = (\bigcap_{k=0}^{\infty} N(T^k)^\perp)^\top = (\bigcap_{k=0}^{\infty} R(T^{k*}))^\top = (\bigcap_{k=0}^{\infty} R(T^{*k}))^\top = R^\infty(T^*)^\top.$$

The above equality remains true for $T + S$ instead of T. Hence, we get by (2) that,

$$\overline{N^\infty(T+S)} = R^\infty(T^* + S^*)^\top \subset R^\infty(T^*)^\top = \overline{N^\infty(T)}.$$

We claim now, that

$$R^\infty(T+S) \cap \overline{N^\infty(T+S)} \subset R^\infty(T). \tag{4}$$

Since $T + S \in ESR(X)$, then by [6, Lemma 5.5], it is sufficient to show that $R^\infty(T+S) \cap N((T+S)^k) \subset R^\infty(T)$, for all $k \in \mathbb{N}$. We will do this by induction on k. For $k = 0$, the statement is obvious. Let $k \geq 1$ and assume that the inclusion holds for $k-1$. Let $x_0 \in R^\infty(T+S) \cap N((T+S)^k)$. Since $(T+S)(R^\infty(T+S)) = R^\infty(T+S)$, then we can find an infinite sequence $x_0, x_1, \ldots \in R^\infty(T+S)$ such that $x_{j-1} \in (T+S)x_j$ for $j = 1, 2, \ldots$. We claim that for all $j = 1, 2, \ldots$, $x_j \in \overline{N^\infty(T)}$. Indeed, we have $x_0 \in N((T+S)^k) \subset \overline{N^\infty(T+S)} \subset \overline{N^\infty(T)}$. Since $x_0 \in (T+S)x_1$ and $x_0 \in N((T+S)^k)$, then $0 \in (T+S)^k(T+S)x_1 = (T+S)^{k+1}x_1$. Hence $x_1 \in N((T+S)^{k+1}) \subset \overline{N^\infty(T+S)} \subset \overline{N^\infty(T)}$. By induction, we get that $x_j \in N((T+S)^{k+j}) \subset \overline{N^\infty(T+S)} \subset \overline{N^\infty(T)}$ for all $j = 1, 2, \ldots$.
On the other hand, since $T \in ESR(X)$, then $\dim \overline{N^\infty(T)}/(R^\infty(T) \cap \overline{N^\infty(T)}) = m < \infty$. Thus, $x_0, x_1, \ldots, x_m$ are linearly dependent, i.e. there exists a non trivial linear combination $x = \sum_{i=0}^{m} \mu_i x_i \in R^\infty(T)$. Let l be such that $\mu_l \neq 0$ and $\mu_j = 0$ for $j = l+1, \ldots, m$. We obtain

$$\begin{aligned}
(T+S)^l x =&(T+S)^l(\sum_{i=0}^{l} \mu_i x_i)\\
=&\sum_{i=0}^{l} \mu_i (T+S)^l x_i\\
=&\sum_{i=0}^{l-1} \mu_i (T+S)^l x_i + \mu_l (T+S)^l x_l.
\end{aligned}$$

However, $x_0 \in (T+S)x_1 \subset (T+S)^2 x_2 \subset \ldots \subset (T+S)^l x_l$. This implies that $(T+S)^l x_l = x_0 + (T+S)^l(0)$. So

$$(T+S)^l x = \mu_l x_0 + \sum_{i=0}^{l-1} \mu_i (T+S)^l x_i.$$

Since $(T+S)^l(R^\infty(T)) \subset \sum_{j=0}^{l} C_l^j T^j S^{l-j}(R^\infty(T)) \subset R^\infty(T)$ and $x \in R^\infty(T)$), then $(T+S)^l x \in R^\infty(T)$. Thus, to prove that $x_0 \in R^\infty(T)$ it remains only to show that $\sum_{i=0}^{l-1} \mu_i (T+S)^l x_i \subset R^\infty(T)$. To do this, we first claim that

$$\sum_{i=0}^{l-1} \mu_i (T+S)^l x_i \subset N((T+S)^{k-1}) + (T+S)^l(0).$$

Indeed, since $x_0 \in N((T+S)^k)$, then $0 \in (T+S)^k x_0$. It follows that $0 \in (T+S)^{l-(j+1)}(T+S)^k x_0$ for all $j = 0, 1, 2, \dots, l-1$. Thus,

$$0 \in (T+S)^{k-1}(T+S)^{l-j} x_0 \subset (T+S)^{k-1}(T+S)^l x_j,$$

which implies that $(T+S)^l x_j \subset N((T+S)^{k-1}) + (T+S)^l(0)$ and this gives the desired inclusion. Now by the use of the induction assumption and the fact that $(T+S)^l(0) \subset R^\infty(T+S)$, we get

$$\begin{aligned}\sum_{i=0}^{l-1} \mu_i (T+S)^l x_i &\subset R^\infty(T+S) \cap (N((T+S)^{k-1}) + (T+S)^l(0)) \\ &\subset (R^\infty(T+S) \cap N((T+S)^{k-1})) + (T+S)^l(0) \\ &\subset R^\infty(T) + T^l(0) = R^\infty(T).\end{aligned}$$

Therefore $x_0 \in R^\infty(T)$ and so $R^\infty(T+S) \cap \overline{N^\infty(T+S)} \subset R^\infty(T)$.
To finish the proof of (1), we have by (3) that

$$R^\infty(T+S) \cap \overline{N^\infty(T+S)} \subset \overline{N^\infty(T+S)} \subset \overline{N^\infty(T)}.$$

Consequently,

$$R^\infty(T+S) \cap \overline{N^\infty(T+S)} \subset R^\infty(T) \cap \overline{N^\infty(T)}.$$

Now, we assume that $asc(T) < \infty$. Since $\rho(T) \neq \emptyset$, then by [16, Lemma 6.1] $R_c(T) = \{0\}$. Since $asc(T) < \infty$, then by [15, Lemma 5.5 (ii)] and [6, Lemma 5.5], $R^\infty(T) \cap \overline{N^\infty(T)} = \{0\}$. This last equality together with (1) ensure that $R^\infty(T+S) \cap \overline{N^\infty(T+S)} = \{0\}$. By [2, Lemma 2.4], we have $R_c(T+S) = \{0\}$. Since $N(T+S) \cap R^\infty(T+S) = \{0\}$, then by Lemma 3.1 there exists $p \in \mathbb{N}$ such that $N(T+S) \cap R(T+S)^p = N(T+S) \cap R^\infty(T+S) = \{0\}$. Then by [15, Lemma 5.5 (i)] , $asc(T+S) < \infty$. □

In the following theorem we study the stability of essentially semi regular linear relations under perturbation with Riesz operators.
Recall first that a bounded operator R in a Banach space X is called Riesz operator if $R - \lambda I$ is Fredholm for all nonzero scalar λ.

Theorem 3.3. *Let X be a Banach space, T be a closed linear relation everywhere defined with $\rho(T) \neq \emptyset$ and R be a Riesz operator satisfying $TR = RT$, $\rho(T+R) \neq \emptyset$ and $(T+R)(R^\infty(T)) = R^\infty(T)$. Suppose that $T \in ESR(X)$, then $T + R \in ESR(X)$.*

Proof: Set $M = R^\infty(T)$. Since $T \in ESR(X)$, then by [5, Theorem 3.2], $T_1 = T_{|_M}$ is surjective and $\tilde{T} : X/M \to X/M$ is upper semi Fredholm. Since $TR = RT$, then $R(M) \subset M$. So we can define the operators $R_1 = R_{|_M}$ and $\tilde{R} : X/M \to X/M$. Clearly, $T_1 R_1 = R_1 T_1$ and $\tilde{T}\tilde{R} = \tilde{R}\tilde{T}$. Now, we have by [11, Lemma 15] that $\tilde{R}$ is a Riesz operator. So, from [13, Proof of Theorem 1], we deduce that $\tilde{T} + \tilde{R}$ is upper semi Fredholm. Since $T_1 + R_1$ is surjective and $\rho(T+R) \neq \emptyset$, then $T + R \in ESR(X)$. □

4. Some properties of essentially semi regular spectrum

The goal of this section is to establish the spectral mapping theorem of essentially semi regular linear relation and we give some properties of the essentially semi regular spectrum of a linear relation. We start by giving some properties of the product and the power of essentially semi regular linear relations.

Proposition 4.1. *Let T, S be two closed linear relations everywhere defined in a Banach space X satisfying $TS = ST$. If TS is essentially semi regular, then T and S are essentially semi regular.*

Proof: Since $ST \in ESR(X)$, then by Theorem 2.2 $N(ST) \subset_e R^{\infty}(ST)$. So

$$N(T) \subset N(ST) \subset_e R^{\infty}(ST) \subset R^{\infty}(T).$$

It remains only to prove that $R(T)$ is closed. By assumption we know that there exists a finite-dimensional subspace F such that $N(ST) \subset R(ST) + F$. We claim that $R(T) + F$ is closed. Indeed, let $u \in \overline{R(T) + F}$. So,

$$Su \subset S(\overline{R(T) + F}) \subset \overline{S(R(T) + F)} = \overline{R(ST) + S(F)}.$$

Since $dimF < \infty$, then there exists a finite dimensional subspace G such that

$$R(ST) + S(F) = R(ST) + G + S(0) = R(ST) + G.$$

Since $dimG < \infty$ and $R(ST)$ is closed, then $R(ST)+G$ is closed. Thus $Su \subset R(ST)+G = R(ST)+S(F) = S(R(T)+F)$. Let $y \in Su$. So $Su = y + S(0)$ and there exists $v \in R(T) + F$ such that $y \in Sv$. Thus $Sv = y + S(0)$. Hence $S(v - u) = S(0)$. So $v - u \in N(S) \subset N(ST) \subset R(ST) + F \subset R(T) + F$. Then $u \in R(T) + F$. Therefore $R(T) + F$ is closed. Thus $R(T)$ is closed. Consequently T is essentially semi regular. By a similar proof we show that S is essentially semi regular. □

Proposition 4.2. *Let T, S be two closed linear relations everywhere defined in a Banach space X with $TS = ST$ and $0 \in \rho(S)$. If T is essentially semi regular, then TS is essentially semi regular.*

Proof: We claim that $R(TS)$ is closed. Indeed, since $0 \in \rho(S)$ then $S(X) = X$. Hence $R(TS) = TS(X) = T(X) = R(T)$ which is closed. We claim that $N^{\infty}(T) \subset_e R^{\infty}(T)$. Indeed, since $0 \in \rho(S)$, then $S^{-1}(0) = \{0\}$. Thus,

$$N(ST) = (ST)^{-1}(0) = T^{-1}S^{-1}(0) = N(T).$$

Since $TS = ST$, then we can see that for all $n \in \mathbb{N}$, $N((TS)^n) = N(T^n)$. Hence $N^{\infty}(TS) = N^{\infty}(T)$. In the same way, we can see that $R^{\infty}(TS) = R^{\infty}(T)$. Therefore $N^{\infty}(TS) = N^{\infty}(T) \subset_e R^{\infty}(T) = R^{\infty}(TS)$. Consequently, TS is essentially semi regular. □

For a closed linear relation T let us define the Fredholm spectrum, the regular spectrum and the essentially semi regular spectrum respectively,
$\sigma_{\phi}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}$,
$\sigma_{reg}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not semi regular}\}$,
$\sigma_{esr}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not essentially semi regular}\}$.

In the following theorem we will establish the relationship between the semi regular spectrum and the essentially semi regular spectrum of a linear relation.

Theorem 4.3. *Let X be a Banach space and let T be a closed linear relation everywhere defined with $\rho(T) \neq \emptyset$. Then*

(i) *$\sigma_{reg}(T) \setminus \sigma_{esr}(T)$ is at most countable.*

(ii) *$\sigma_{reg}(T) \setminus (\sigma_{\phi}(T) \cap \sigma_{reg}(T))$ is at most countable.*

Proof: **(i)** Let $\lambda_0 \in \sigma_{reg}(T) \setminus \sigma_{esr}(T)$. Then $\lambda_0 - T$ is essentially semi regular and not semi regular. So by [5, Theorem 5.3], there exists $r > 0$ such that if $0 <| \lambda - \lambda_0 |< r$ we have $\lambda - T$ is semi regular. Thus there exists $r > 0$ such that $B(\lambda_0, r) \cap \sigma_{reg}(T) = \{\lambda_0\}$. Therefore, λ_0 is an isolated point of $\sigma_{reg}(T)$. So $\sigma_{reg}(T) \setminus \sigma_{es}(T) \subset A := \{\lambda \in \mathbb{C} : \lambda \text{ is an isolated point of } \sigma_{reg}(T)\}$. A classical result of topology ensures that if A is a subset of $\mathbb{C}$ such that all its points are isolated and as $\mathbb{K}$ is a separable metric space, then A is countable. This leads to $\sigma_{reg}(T) \setminus \sigma_{esr}(T)$ is at most countable.
(ii) We say that $\sigma_{esr}(T) \subset (\sigma_{\phi}(T) \cap \sigma_{reg}(T))$. So $\sigma_{reg}(T) \setminus (\sigma_{\phi}(T) \cap \sigma_{reg}(T)) \subset \sigma_{reg}(T) \setminus \sigma_{esr}(T)$. Using **(i)**, we get the desired result. □

Our next aim is to establish the spectral mapping theorem for essentially semi regular linear relations. To this end we need to recall the following definition and theorem.

Definition 4.4. [14, (1.1)] *Let T be a linear relation in a linear space E. Let n and m_i, $1 \le i \le n$ be some positive integers and let $\lambda_i \in \mathbb{K}$, $1 \le i \le n$ be some distinct constants. The polynomial $P(T)$ in T is the linear relation*

$$P(T) := \prod_{i=1}^{n} (T - \lambda_i)^{m_i}.$$

Theorem 4.5. *[5, Theorem 4.1] Let T be an everywhere defined closed linear relation in a Banach space X with $\rho(T) \neq \emptyset$. Then*

$$P(T) \in ESR(X) \text{ if and only if } T - \lambda_i \in ESR(X),\ \ 1 \le i \le n.$$

We are in the position to give the spectral mapping theorem for linear relations.

Theorem 4.6. *Let X be a Banach space and T be a closed linear relation everywhere defined in X with $\rho(T) \neq \emptyset$. Then for any complex polynomial P, we have*

$$\sigma_{esr}(P(T)) = P(\sigma_{esr}(T)).$$

Proof: Fix $\lambda \in \mathbb{C}$ and let $P(\lambda) - \alpha = \prod_{i=1}^{k} (\lambda - \beta_i)^{m_i}$, for $k \in \mathbb{N}$. Then

$$P(T) - \alpha I = \prod_{i=1}^{k} (T - \beta_i I)^{m_i}.$$

Let $\alpha \in \sigma_{esr}(P(T))$. Then $P(T) - \alpha I$ is not essentially semi regular. According to Theorem 4.5, there exists $i \le 1 \le n$, such that $(T - \beta_i I)$ is not essentially semi regular. Therefore $\beta_i \in \sigma_{esr}(T)$. However $\alpha = P(\beta_i)$, so $\alpha \in P(\sigma_{esr}(T))$.
For the reverse inclusion, let $\alpha \in P(\sigma_{esr}(T))$. So there exists $\lambda \in \sigma_{esr}(T)$, such that $\alpha = P(\lambda)$. Since $P(\lambda) - \alpha = \prod_{i=1}^{k} (\lambda - \beta_i)^{m_i}$, then there exists $i \le 1 \le k$, such that $\lambda = \beta_i$. It follows that $T - \beta_i I$ is not essentially semi regular. Theorem 4.5 leads to $P(T) - \alpha I$ is not essentially semi regular, which means that $\alpha \in \sigma_{esr}(P(T))$. □

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