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Reconstruction of Szász-Mirakyan operators preserving exponential type functions

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Abstract. In this paper, we construct a modification of Szász-Mirakyan operators with a new technique that preserved the exponential functions i.e. $exp(\mu t)$ and $exp(2\mu t)$, for a fixed real parameter $\mu > 0$. We study the asymptotic behaviour and weighted approximation of these operators. Comparisons about one approximate better between the recent operators and the classical Szász-Mirakyan operators have also been presented. In the end, we compare the convergence of these operators and modified Baskakov operators to certain functions by illustrative graphics using the Mathematica algorithms.

1. Introduction

In 1912, Bernstein [7] defined the Bernstein operators in order to give constructive proof of the Weierstrass approximation theorem. The uniform convergence of these operators is ensured by Korovkin theorem [19]. For the convergence, the moments of the linear positive operators play a very important role and helps to reproduce and to get better approximation. Bernstein polynomials fix constant as well as linear functions. To obtain better error estimation, King [18] has taken initiative to modify the well-known Bernstein operators that preserve constant and quadratic functions. After that many authors [3, 10, 14, 15, 17] etc. followed his approach on several positive linear operators. It is an effective technique of preservation of moments to get better results from modified operators than the classical ones. In 1950, Szász [25] introduced the generalization of Bernstein polynomials to infinite intervals called the classical Szász-Mirakyan operators which are defined by

$$\mathcal{S}_n(f;x) = e^{-nx}\sum_{l=0}^{\infty}\frac{(nx)^l}{l!}f\left(\frac{l}{n}\right), \ \ x \geq 0, \ \ n \in \mathbb{N}$$

for all functions $f : [0, \infty) \to \mathbb{R}$ for which the infinite series at the right hand side is absolutely convergent. These operators are positive linear operators and preserve constant as well as linear functions.

In order to obtain the uniform convergence of positive linear operators to continuous functions, Korovkin [19] discovered a important concept by giving a very simple criterion for the functions $1, x$ and x^2. The foremost applications are concerned with constructive approximation theory which uses it as a valuable

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tool. Also, to determine the convergence behaviour, direct results of approximation e.g. degree of approximation, rate of convergence and asymptotic behaviour have been studied for many well-known positive linear operators. The various qualitative and quantitative results of Voronovskaja type theorem provided asymptotic behaviour, the uniform convergence and the approximation order of the approximated function for these operators. Such type of results have attracted the attention of many great mathematicians in last century which can be seen in literature cf. [8, 22, 24] etc.

A lot of generalizations have been investigated on Szász-Mirakyan operators. In 2017, Srivastava et al. [23] introduced $q-$Szász-Mirakjan-Kantorovich type positive linear operators of one and two variables that are generated by Dunkl's generalization of the exponential function and presented their approximation properties. Very recently, Gupta et al. [12] provided the estimates for the difference of the operators (especially Szász-Mirakyan and Baskakov operators) which are associated with different fundamental functions.

Nowadays, several authors have modified the linear positive operators that reproduce the exponential functions instead of the usual polynomial type ones which gives more better results. Initially, Aldaz and Render [4] have defined linear positive operators which preserve the exponential functions. We mention some of the important papers in this direction as [6, 9, 11, 13, 16, 20].

Recently, Acar et al. [1] and [5] modified the Szász-Mirakyan operators that preserve the exponential function $e^{2\mu x}, \mu > 0$. Acar et al. [2] also changed the same operators which fix the exponential functions $e^{\mu x}$, $e^{2\mu x}$, $\mu > 0$ by taking two sequences $\alpha_n(x)$ and $\beta_n(x)$. Here, we present the generalization of the classical Szász operators in a different manner that also regenerate the exponential functions $e^{\mu t}$ and $e^{2\mu t}$, where $\mu > 0$ is a fixed real parameter. We modify the Szász operators by using a single sequence $\alpha_n(x)$, which is defined as follows:

$$M_{\mu,n}(f;x) = e^{(\mu x - n\alpha_n(x))} \sum_{l=0}^{\infty} e^{-\mu l/n} \frac{(n\alpha_n(x))^l}{l!} f\left(\frac{l}{n}\right), \tag{1.1}$$

where $\alpha_n(x) = \dfrac{\mu x}{n(e^{\mu/n} - 1)}$.

For $f \in C[0,\infty)$ *i.e* the space of all continuous functions on $[0,\infty)$, $M_{\mu,n}(f;x)$ converges to f as n tends to ∞ uniformly on $[0,\infty)$. Our operators have close relation with the classical Szász-Mirakyan operators that is given by

$$M_{\mu,n}(f;x) = e^{\mu x} \mathcal{S}_n\left(\frac{f}{e^{\mu}}; \alpha_n(x)\right).$$

Clearly $M_{\mu,n}(f;x)$ are positive and linear operators. It is known that the operators $\mathcal{S}_n(f;x)$ preserve constant as well as linear functions but our operators preserve $e^{\mu t}$ and $e^{2\mu t}$ i.e.

$$M_{\mu,n}(e^{\mu t};x) = e^{\mu x}, \qquad M_{\mu,n}(e^{2\mu t};x) = e^{2\mu x}. \tag{1.2}$$

The aim of this paper is to study the Voronovskaja type theorems and weighted approximation results for the modified Szász operators. In order to obtain these convergence results, we obtain some auxiliary results. Also, we give a comparison of these operators with classical Szász operators by using asymptotic results and with modified Baskakov operators through graphics.

2. Auxiliary results

For $\mu > 0$, consider the exponential function $exp_\mu(t) = e^{\mu t}$, $exp_\mu^2(t) = e^{2\mu t}$ and its inverse function is denoted by $\log_\mu$ which is the logarithmic function with base e^μ.

Lemma 2.1. *For each $n \in \mathbb{N}$ and $x \in [0,\infty)$, the following identities hold:*

(i) $M_{\mu,n}(1;x) = e^{(1-e^{-\mu/n})\mu x}$

(ii) $M_{\mu,n}(exp_\mu^3(t);x) = e^{\left(1+\frac{e^{2\mu/n}-1}{e^{\mu/n}-1}\right)\mu x}$

(iii) $M_{\mu,n}(exp_\mu^4(t);x) = e^{\left(1+\frac{e^{3\mu/n}-1}{e^{\mu/n}-1}\right)\mu x}$.

For each $x \in (0,\infty)$, we shall consider the functions e_x and $exp_{\mu,x}$ defined for $t \in [0,\infty)$ by

$$e_x(t) = t - x, \qquad exp_{\mu,x}(t) = e^{\mu t} - e^{\mu x}.$$

For $x \in [0,\infty)$ and $t \in [0,\infty)$, by using equation (1.2) and Lemma 2.1, we get

Lemma 2.2. *For each $n \in \mathbb{N}$ and $x \in [0,\infty)$, the following identities hold:*

(i) $M_{\mu,n}(exp_{\mu,x}(t);x) = e^{\mu x}\left(1 - e^{(1-e^{-\mu/n})\mu x}\right)$

(ii) $M_{\mu,n}(exp_{\mu,x}^2(t);x) = e^{2\mu x}\left(e^{(1-e^{-\mu/n})\mu x} - 1\right)$

(iii) $M_{\mu,n}(exp_{\mu,x}^4(t);x) = e^{\left(1+\frac{e^{3u/n}-1}{e^{\mu/n}-1}\right)\mu x} + e^{(5-e^{-\mu/n})\mu x} + 2e^{4\mu x} - 4e^{\left(2+\frac{e^{2\mu/n}-1}{e^{\mu/n}-1}\right)\mu x}$.

Remark 2.3. $\lim\limits_{n\to\infty} n(M_{\mu,n}(1;x) - 1) = \lim\limits_{n\to\infty} n\left(e^{(1-e^{-\mu/n})\mu x} - 1\right) = \mu^2 x.$

Theorem 1. *[21] Let $\mathcal{S}_n(f;x)$ be a classical Szász-Mirakyan operators and f is a continuous and bounded on $[0,\infty)$ such that f', f'' are continuous and bounded on $[0,\infty)$, then the following uniform convergence holds on any compact interval $[0,a], a > 0$:*

$$\lim_{n\to\infty} 2n\left(\mathcal{S}_n(f;x) - f(x)\right) = xf''(x).$$

3. Main results

Let $C^*[0,\infty)$ denote the space of all real valued and continuous functions on $[0,\infty)$ in which $\lim\limits_{x\to\infty} f(x)$ exists and finite endowed with the norm $\|f\| = \sup\limits_{x\in[0,\infty)} |f(x)|$.

Theorem 2. *If $f \in C^*[0,\infty)$ has a second derivative at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} 2n\left(M_{\mu,n}(f;x) - f(x)\right) = \left(2\mu^2 f(x) - 3\mu f'(x) + f''(x)\right)x. \tag{3.1}$$

Proof. By Taylor's theorem, we have

$$f(t) = (f\circ\log_\mu)e^{\mu t} = (f\circ\log_\mu)(e^{\mu x}) + (f\circ\log_\mu)'e^{\mu x}exp_{\mu,x}(t) + \frac{(fo\log_\mu)''}{2!}e^{\mu x}exp_{\mu,x}^2(t) + h_x(t)exp_{\mu,x}^2(t), \tag{3.2}$$

where $h_x(t) = h(t-x)$ and h is continuous function which vanishes at 0.
Applying $M_{\mu,n}(.;x)$ to both the sides, we get

$$\begin{aligned} M_{\mu,n}(f;x) &= (f\circ\log_\mu)(e^{\mu x})M_{\mu,n}(1;x) + (f\circ\log_\mu)'e^{\mu x}M_{\mu,n}(exp_{\mu,x}(t);x) \\ &\quad + \frac{(f\circ\log_\mu)''}{2!}e^{\mu x}M_{\mu,n}(exp_{\mu,x}^2(t);x) + M_{\mu,n}(h_x(t)exp_{\mu,x}^2(t);x). \end{aligned}$$

Since $(f\circ\log_\mu)e^{\mu x} = f(x)$, $(f\circ\log_\mu)'e^{\mu x} = f'(\log_\mu e^{\mu x})\left(\frac{d}{dx}\left(\log_\mu e^{\mu x}\right)\right) = f'(x)e^{-\mu x}\mu^{-1}$ and $(f\circ\log_\mu)''e^{\mu x} = e^{-2\mu x}\left(f''(x)\mu^{-2} - f'(x)\mu^{-1}\right)$, then by using Lemma 2.2, we have

$$\begin{aligned} M_{\mu,n}(f;x) - f(x) &= f(x)M_{\mu,n}(1;x) - f(x) + (f\circ\log_\mu)'e^{\mu x}M_{\mu,n}(exp_{\mu,x}(t);x) \\ &\quad + \frac{(f\circ\log_\mu)''}{2!}e^{\mu x}M_{\mu,n}(exp_{\mu,x}^2(t);x) + M_{\mu,n}(h_x(t)exp_{\mu,x}^2(t);x) \\ n(M\mu,n(f;x) - f(x)) &= \frac{\left(2\mu^2 f(x) - 3\mu f'(x) + f''(x)\right)}{2\mu^2}n(M_{\mu,n}(1;x) - 1) + nM_{\mu,n}(h_x(t)exp_{\mu,x}^2(t);x). \end{aligned} \tag{3.3}$$

Taking limit $n \to \infty$ to both sides of the equation (3.3) and then by using Remark 2.3, we get

$$\lim_{n\to\infty} n(M_{\mu,n}(f;x) - f(x)) = \frac{\left(2\mu^2 f(x) - 3\mu f'(x) + f''(x)\right)x}{2} + \lim_{n\to\infty} nM_{\mu,n}(h_x(t)exp^2_{\mu,x}(t);x).$$

To get the desired result, it is sufficient to prove that $\lim_{n\to\infty} nM_{\mu,n}(h_x(t)exp^2_{\mu,x}(t);x) = 0$. From Cauchy-Schwarz inequality we have

$$n|M_{\mu,n}(h_x(t)exp^2_{\mu,x}(t);x)| \leq \sqrt{M_{\mu,n}(h_x^2(t);x)}\sqrt{n^2 M_{\mu,n}(exp^4_{\mu,x};x)}.$$

It is an easily found that $\lim_{n\to\infty}(h_x^2(t);x) = h_x^2(x) = 0$ and $\lim_{n\to\infty} n^2 M_{\mu,n}(exp^4_{\mu,x};x) = 3\mu^4 x^2$.
Thus the proof is completed. □

We get differential operators in the right hand side of equation (3.1). It can be expressed as follows:

$$x(f''(x) - 3\mu f'(x) + 2\mu^2 f(x)) = \frac{1}{\omega_2(x)}\left(\frac{1}{\omega_1(x)}\left(\frac{f(x)}{\omega_0(x)}\right)'\right)',$$

where $\omega_0(x) = \omega_1(x) = e^{\mu x}$, $\omega_2(x) = \dfrac{1}{xe^{2\mu x}}$.

Theorem 3. *Let $f \in C^*[0,\infty)$ then for each $x \in (a,b) \subset (0,1)$, $2n(M_{\mu,n}(f;x) - f(x)) = o(1)$ if and only if f is a solution of differential equation $f'' - 3\mu f' + 2\mu^2 f = 0$ in (a,b).*

Theorem 4. *Let $f \in C^*[0,\infty)$ and let $M \geq 0$ then for each $x \in (a,b) \subset (0,1)$*

$$2n|M_{\mu,n}(f;x) - f(x)| \leq M + o(1)$$

if and only if $|f'' - 3\mu f' + 2\mu^2 f| \leq M$ in (a,b).

4. Weighted Approximation

For $x \in (0,\infty)$ assume that $\phi(x) = 1 + e^{2\mu x}$, $\mu > 2$ and consider the following weighted spaces:

$$C_\phi(0,\infty) = \{f \in C(0,\infty) : |f(x)| \leq M_f \phi(x), x \geq 0, \text{ for some } M_f > 0\}$$

$$C^l_\phi(0,\infty) = \{f \in C_\phi(0,\infty) : \lim_{x\to\infty}\frac{f(x)}{\phi(x)} = k_f \text{ exists and finite}\},$$

where k_f is a constant depending on f. $C_\phi(0,\infty)$ and $C^k_\phi(0,\infty)$ are normed spaces with the norm $\|f\|_\phi = \sup\limits_{x\in(0,\infty)} \dfrac{|f(x)|}{\phi(x)}$.

Theorem 5. *For any $f \in C_\phi(0,\infty)$, the inequality $\|M_{\mu,n}(f)\|_\phi \leq \|f\|_\phi$ holds.*

Theorem 6. *For each $f \in C^k_\phi(0,\infty)$, $\lim_{n\to\infty}\|M_{\mu,n}(f) - f\|_\phi = 0$.*

Proof. To prove the above result, it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\|M_{\mu,n}(e^{k\mu t};x) - l_{\mu,k}\|_\phi = 0, \text{ where } l_{\mu,k}(x) = e^{k\mu x}, \ k = 0,1,2.$$

Since $M_{\mu,n}(e^{\mu t};x) = e^{\mu x}$ and $M_{\mu,n}(e^{2\mu t};x) = e^{2\mu x}$, the conditions are true for $k = 1,2$, we need to prove only for $k = 0$.
From Remark 2.3, we get

$$M_{\mu,n}(1;x) - 1 = \frac{1}{2n^2}\left(\mu^4 x^2 - \mu^3 x + 2\mu^2 nx\right) + O\left(\frac{1}{n^3}\right)$$

$$\|M_{\mu,n}(1;x) - 1\|_\phi = \frac{1}{2n^2}\left(\mu^4 - \mu^3 + 2\mu^2 n\right) + O\left(\frac{1}{n^3}\right).$$

Thus, the condition is true for $k = 0$. This completes the proof of theorem. □

5. Comparison with Szász and modified Baskakov operators

Theorem 7. *Let $f \in C^2[0,\infty) := \{f \in C[0,\infty) : \; f', f'' \in C[0,\infty)\}$. If there exists $n_0 \in \mathbb{N}$ such that*

$$f(t) \leq M_{\mu,n}(f;x) \leq \mathcal{S}_n(f;x), \;\; \forall \; n \geq n_0, \;\; t \in (0,\infty), \tag{5.1}$$

then

$$f''(t) \geq 3\mu f'(t) - 2\mu^2 f(t) \geq 0, \;\; t \in (0,\infty). \tag{5.2}$$

In particular, $f''(x) \geq 0$. Conversely, if (5.2) holds with strict inequalities at a point $x \in (0,\infty)$, then (5.1) also holds for strict inequalities.

Proof. From (5.1), we have

$$0 \leq 2n(M_{\mu,n}(f;x) - f(t)) \leq 2n(\mathcal{S}_n(f;x) - f(t)), \; n \geq n_0, \; t \in (0,\infty).$$

By using Theorems 1 and 2, we get

$$0 \leq f''(t) - 3\mu f'(t) + 2\mu^2 f(t) \leq f''(t),$$

from which (5.2) follows.
Conversely, if (5.2) holds with strict inequalities for a given $x \in (0,\infty)$, then we have

$$0 < f''(t) - 3\mu f'(t) + 2\mu^2 f(t) < f''(t),$$

and using Theorems 1 and 2, we obtain the required result. □

Now, we compare the modified Szász operators $M_{\mu,n}(f;x)$ and the modified Baskakov operators that also preserve the exponential functions i.e. $exp(\mu t)$ and $exp(2\mu t)$, $\mu > 0$, defined in [20], to a certain continuous function through graphics using Mathematica software in the following examples:

Example 5.1. *For $\mu = 1, 2$, $n = 50, 100, 200$, the rate of convergence of the modified Szász operators $M_{\mu,n}(f;x)$ (blue) and the modified Baskakov operators $K_{\mu,n}(f;x)$ (say), (Green) to the $f(x) = x^3 + 1$ (Brown) is illustrated in the following figures:*

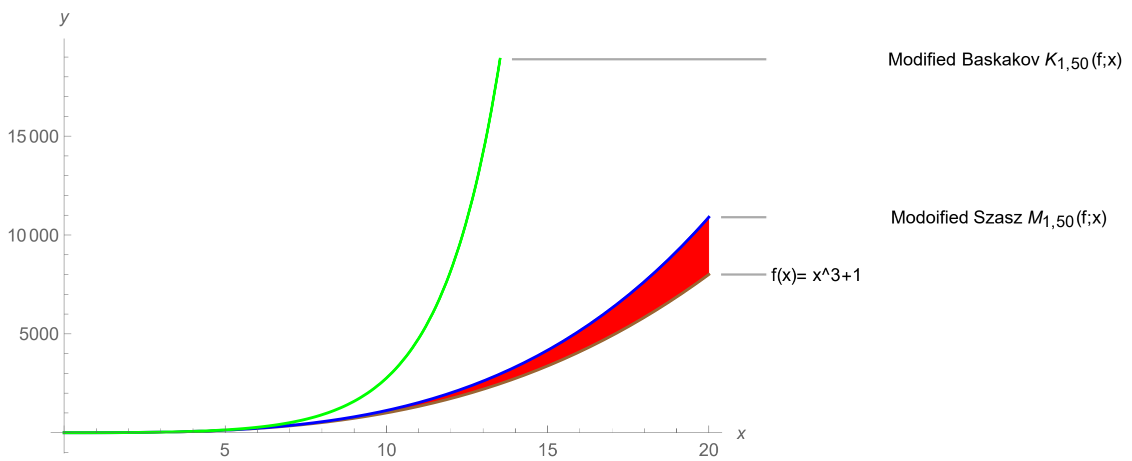

Figure 1. The convergence of $M_{1,50}(f;x)$ *and* $K_{1,50}(f;x)$ *to* $f(x) = x^3 + 1$

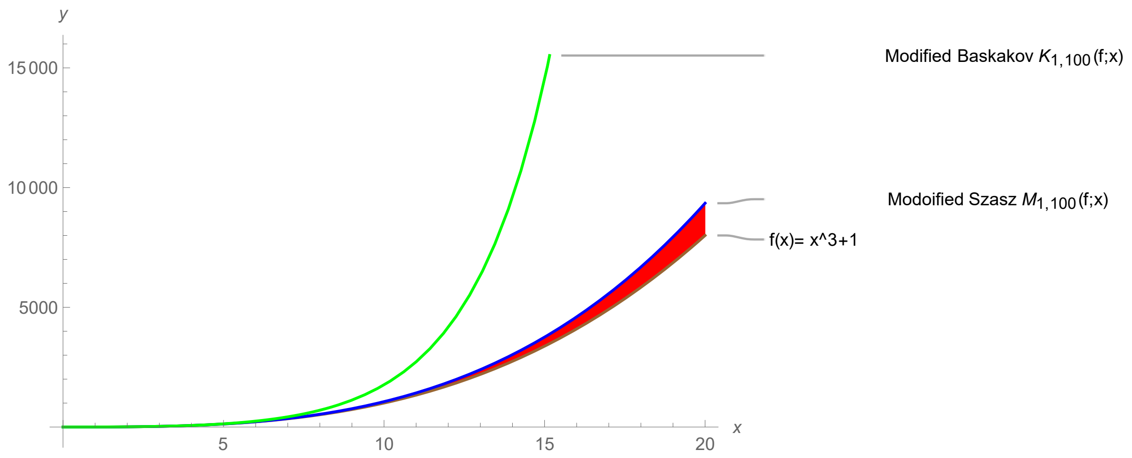

Figure 2. The convergence of $M_{1,100}(f;x)$ *and* $K_{1,100}(f;x)$ *to* $f(x) = x^3 + 1$

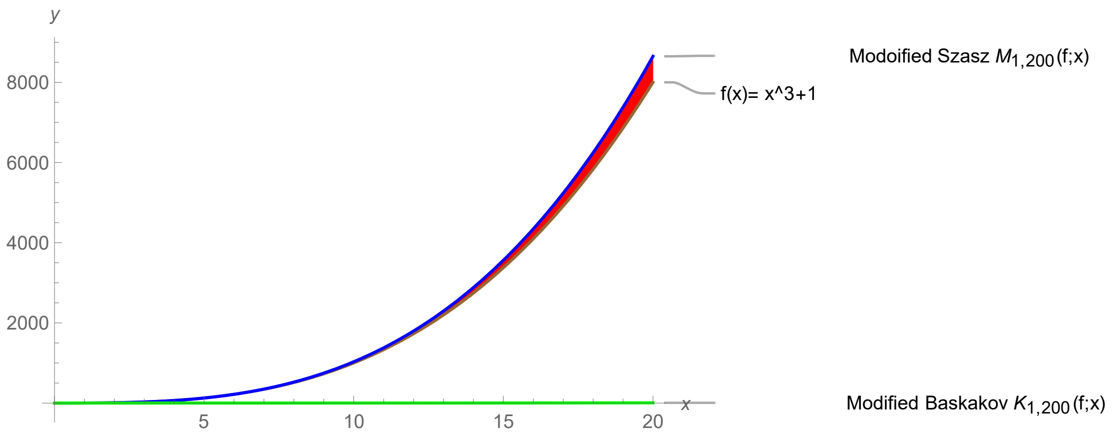

Figure 3. The convergence of $M_{1,200}(f;x)$ *and* $K_{1,200}(f;x)$ *to* $f(x) = x^3 + 1$

From figures 1-3, we observe that the error between function $f(x)$ *and the modified Szász operators (red shaded area) is less than the error between function and the modified Baskakov operators for the values of* $\mu = 1$ *and* $n = 50, 100, 200$. *So, as* n *increases the approximation of* $f(x) = x^3 + 1$ *by the operators* $M_{1,n}(f;x)$ *becomes better than the approximation given by the operators* $K_{1,n}(f;x)$.

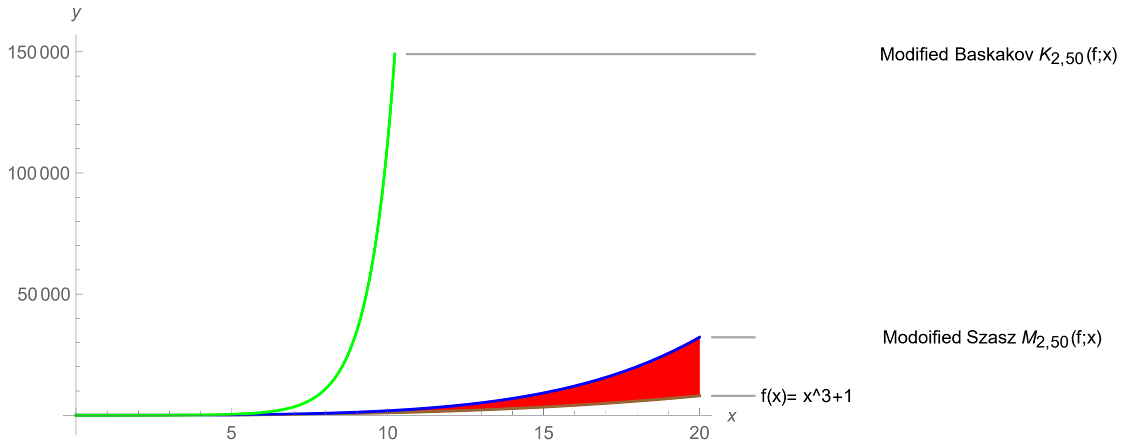

Figure 4. The convergence of $M_{2,50}(f;x)$ and $K_{2,50}(f;x)$ to $f(x) = x^3 + 1$

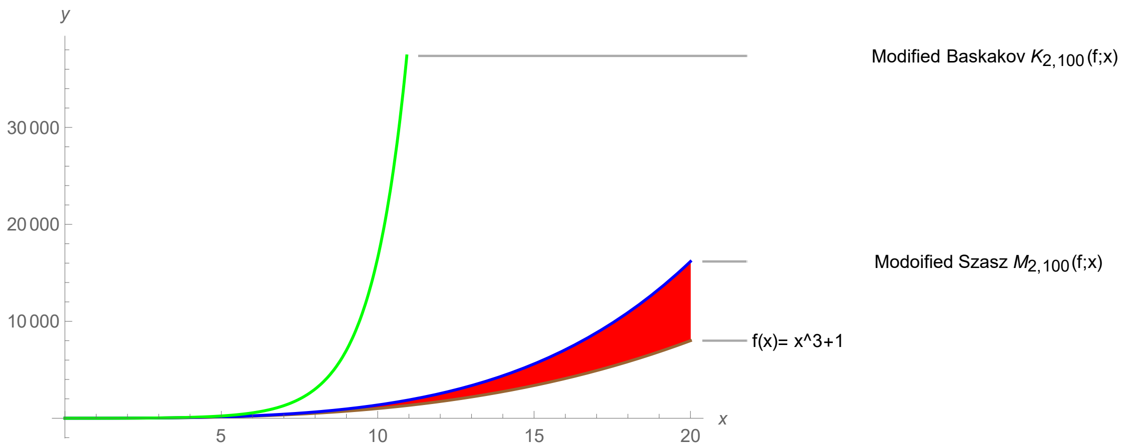

Figure 5. The convergence of $M_{2,100}(f;x)$ and $K_{2,100}(f;x)$ to $f(x) = x^3 + 1$

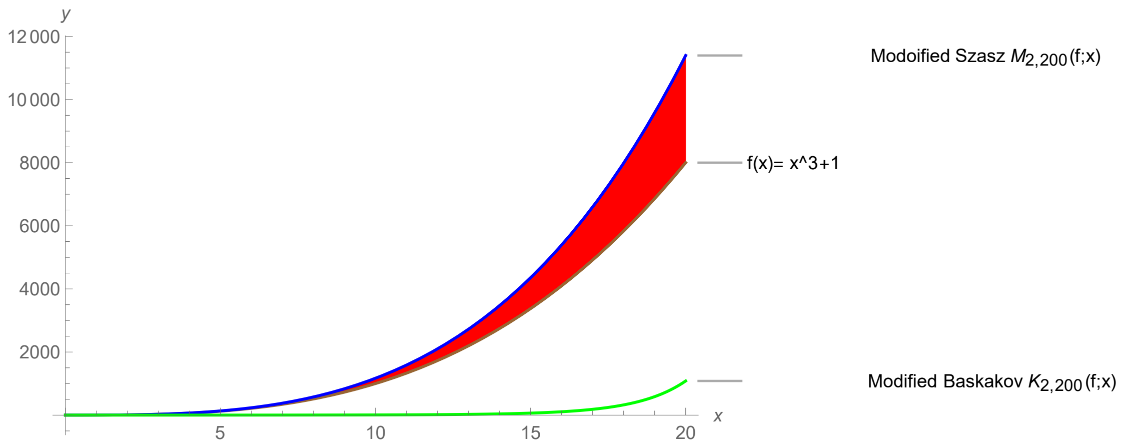

Figure 6. The convergence of $M_{2,200}(f;x)$ and $K_{2,200}(f;x)$ to $f(x) = x^3 + 1$

In figures 4-6, for $\mu = 2$, the convergence of the operators $M_{2,n}(f;x)$ to $f(x)$ is slow as n increases but it is still better than the convergence given by the operators $K_{2,n}(f;x)$.

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