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(ω, c)-almost periodic type functions and applications

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Abstract. In this paper, we introduce several various classes of (ω, c)-almost periodic type functions and their Stepanov generalizations. We also consider the corresponding classes of (ω, c)-almost periodic type functions depending on two variables and related composition principles. We provide several illustrative examples and applications to the abstract Volterra integro-differential equations in Banach spaces.

1. Introduction and preliminaries

The theory of almost periodic functions is an active area of research of many mathematicians. The notion of almost periodicity was studied by H. Bohr around 1925 and later generalized by many others. The interested reader may consult the monographs by Besicovitch [5], Diagana [8], Fink [9], Guérékata [11], Kostić [15], Levitan, Zhikov [20] and Zaidman [26] for the basic introduction to the theory of almost periodic functions. Almost periodic functions and almost automorphic functions play a significant role in the qualitative theory of differential equations, physics, mathematical biology, control theory and technical sciences.

As is well known, the class of Bloch periodic functions extends the classes of periodic functions and anti-periodic functions. The Bloch periodic functions are incredible important in the quantum mechanics and solid state physics. For more details about this class of functions, we refer the reader to [7], [18] and references cited therein.

The class of (ω, c)-periodic functions, which extends the class of Bloch periodic functions, has recently been introduced and investigated by Alvarez, Gómez and Pinto [2] and Alvarez, Castillo, Pinto [3]. The (ω, c)-periodic functions is of major relevance in the qualitative analysis of solutions to the Mathieu linear differential equation

$$y''(t) + [a - 2q \cos 2t]y(t) = 0,$$

arising in modeling of seasonally forced population dynamics. The linear delayed equations can have (ω, c)-periodic solutions, as well (see e.g., [2, Example 2.5]). Furthermore, the authors of [3] have proved the

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existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0.$$

This equation describes the survival of red blood cells in the blood of an animal (see, e.g., Wazewska–Czyzewska and Lasota [25]).

Let $(E, \|\cdot\|)$ be a complex Banach space. In a more abstract context, the authors of [2] have analyzed the existence and uniqueness of mild (ω, c)-periodic solutions to the abstract semilinear integro-differential equation

$$D^{\alpha}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \quad t \in \mathbb{R},$$

where $D^{\alpha}u(t)$ denotes the Weyl-Liouville fractional derivative of order $\alpha > 0$, $a \in L^1_{loc}([0,\infty))$ is a scalar-valued kernel, the function $f(\cdot,\cdot)$ enjoys some properties and A generates an α-resolvent operator family on E. Further on, Alvarez, Castillo and Pinto have analyzed in [3] the existence and uniqueness of mild (ω, c)-pseudo periodic solutions to the abstract semilinear differential equation of first order

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A generates a strongly continuous semigroup of operators. Concerning the applications to time varying impulsive differential equations, mention should be made of the article [24] by Wang, Ren and Zhou; cf. also the article [1] by Agaoglou, Fečkan, Panagiotidou, the article [23] by Mophou, Guérékata and the article [21] by Li, Wang and Fečkan for some other applications.

The main purpose of this paper is to continue the above mentioned researches by introducing and investigating the various spaces of (ω, c)-almost periodic type functions and their Stepanov generalizations. As mentioned in the abstract, we also consider the corresponding spaces of (ω, c)-almost periodic type functions depending on two variables and related composition principles, providing several applications to the abstract Volterra integro-differential equations.

The organization and main ideas of paper can be briefly described as follows. After recalling the basic definitions and results about almost periodic functions, almost automorphic functions and uniformly recurrent functions (cf. Subsection 1.1, where the only novelties are Proposition 1.2 and a simple computation of the Bohr spectrum of an (ω, c)-almost periodic function with $|c| = 1$), we introduce the spaces of (ω, c)-uniformly recurrent functions, (ω, c)-almost periodic functions and (compactly) (ω, c)-almost automorphic functions. In Definition 2.1, we use a simple trick from [2]; a continuous function $f : I \to E$ is said to be (ω, c)-almost periodic if and only if the function

$$f_{\omega,c}(t) := c^{-(t/\omega)} f(t), \quad t \in I, \tag{1}$$

is almost periodic (in contrast to [2]-[3], we consider the case in which $I = \mathbb{R}$ or $I = [0,\infty)$). The main aim of Definition 2.6 is to introduce the classes of asymptotically (ω, c)-uniformly recurrent functions, asymptotically (ω, c)-almost periodic functions and asymptotically (compactly) (ω, c)-almost automorphic functions; in Definition 2.9, we extend the notion from Definition 2.1 and Definition 2.6 by introducing the corresponding Stepanov classes of (ω, c)-almost periodic functions. In the remainder of Section 2, we state and prove several results about the introduced classes of functions.

Our main contributions are given in Section 3, where we introduce and analyze (ω, c)-uniformly recurrent functions of type 1 (type 2) and (ω, c)-almost periodic functions of type 1 (type 2). The main result of paper is Theorem 3.2, in which we completely profile the introduced classes of (ω, c)-uniformly recurrent functions and (ω, c)-almost periodic functions in the case that $I = \mathbb{R}$ and $|c| \neq 1$ (if $|c| = 1$, then the concept of (ω, c)-almost periodicity of type 1 (type 2) is equivalent with the concept of almost periodicity): any of the spaces $UR_{\omega,c,i}(I : E)$ and $AP_{\omega,c,i}(I : E)$ for $i = 1, 2$ equals to the set $M_{\omega,c}(I : E)$ in the set-theoretical sense, which consists of all continuous functions $f : I \to E$ such that $c^{-\cdot/\omega} f(\cdot)$ is periodic. Furthermore, in the case that $i = 1$, then the same statement continues to hold in the case that $I = [0,\infty)$. After that, we analyze the remaining case $I = [0,\infty)$ and $|c| < 1$ in more detail. We show that the classes $AP_{\omega,c,1}([0,\infty) : E)$ and $UR_{\omega,c,1}([0,\infty) : E)$ are rather non-attractive (see Corollary 3.8 and Proposition 3.11): the class $AP_{\omega,c,1}([0,\infty) : E)$ consists of those continuous functions $f : [0,\infty) \to E$ for which the function $f_{\omega,c}(\cdot)$ is bounded and continuous, while the class $UR_{\omega,c,1}([0,\infty) : E)$ coincides with the class $C_0([0,\infty) : E)$. The space $AP_{\omega,c,2}([0,\infty) : E)$ is much more interested for further analyses bacause it actually consists of those continuous functions $f : [0,\infty) \to E$ for which the function $f_{\omega,c} : [0,\infty) \to E$ is bounded and recurrent in the sense of [20, Definition 2, p. 80] with $I = [0,\infty)$; see Proposition 3.12. The space $UR_{\omega,c,2}([0,\infty) : E)$ is also interested for further analyses becuase for any uniformly recurrent function $f : [0,\infty) \to E$, the function $f_{\omega,c}(\cdot)$ belongs to this space (in fact, with the notation explained below, we have $UR_{\omega,c}([0,\infty) : E) \subseteq UR_{\omega,c,2}([0,\infty) : E) \subseteq UR_{\omega,c,1}([0,\infty) : E)$). If a function $f(\cdot)$ is (ω,c)-almost periodic of type 1 or 2, then it vanishes at plus infinity so that the case $|c| < 1$ is important because its analysis leads to some new spaces of ergodic components that are contained in the usually considered class $C_0([0,\infty) : E)$; for simplicity, we will not analyze here the (weighted) pseudo-ergodic components and Weyl (Besicovitch) ergodic components which generalize the ergodic components from the space $C_0([0,\infty) : E)$ in the opposite direction. The Stepanov classes of (ω,c)-uniformly recurrent functions and (ω,c)-almost periodic functions are introduced in Definition 3.14. Subsection 3.1 examines the composition principles for (ω,c)-almost periodic type functions. In Section 4, we analyze the invariance of (ω,c)-almost periodicity under the actions of convolution products and provide a few relevant applications of our abstract theoretical results to the abstract Volterra integro-differential equations and inclusions in the Banach spaces. The final section of paper is deserved for offering several conclusions and useful remarks about the results obtained.

We use the standard notation throughout the paper. Unless stated otherwise, we will always assume that $f : I \to E$ is a continuous function. By $C(I : E)$, $C_b(I : E)$ and $C_0(I : E)$ we denote the vector spaces consisting of all continuous functions $f : I \to E$, all bounded continuous functions $f : I \to E$ and all bounded continuous functions $f : I \to E$ satisfying that $\lim_{|t|\to+\infty} \|f(t)\| = 0$. As is well known, $C_b(I : E)$ and $C_0(I : E)$ are Banach spaces equipped with the sup-norm, denoted by $\|\cdot\|_\infty$. If X is also a complex Banach space, then $L(E,X)$ stands for the space of all continuous linear mappings from E into X; $L(E) \equiv L(E,E)$. The principal branches are always used for taking the powers of complex numbers.

1.1. *Almost periodic type functions and generalizations*

Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is *almost periodic* if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in $[0,\infty)$, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I : E)$. This space contains the space $P(I : E)$ consisting of all continuous periodic functions $f : I \to E$.

Let $f \in AP(I : E)$. Then the Bohr-Fourier coefficient

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds$$

exists for all $r \in \mathbb{R}$; furthermore, if $P_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in \mathbb{R}$, and $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ is at most countable. The function $f : I \to E$ is said to be *asymptotically almost periodic* if and only if there exist an almost periodic function $h : I \to E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$. This is equivalent to saying that, for every $\epsilon > 0$, we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I of length l contains, at least, one number τ such that $\|f(t+\tau) - f(t)\| \leq \epsilon$ provided $|t|,\ |t+\tau| \geq M$.

Let $f : I \to E$ be continuous. Following Haraux and Souplet [13], we say that the function $f(\cdot)$ is *uniformly recurrent* if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left\| f(t+\alpha_n) - f(t) \right\| = 0. \tag{2}$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general. We say that the function $f(\cdot)$ is *asymptotically uniformly recurrent* if and only if there exist a uniformly recurrent function $h : I \to E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$. Let us recall that, if $(f_n(\cdot))$ is a sequence of uniformly recurrent functions and $(f_n(\cdot))$ converges uniformly to a function $f : I \to E$, then the function $f(\cdot)$ is uniformly recurrent ([16]). But, it is not clear whether we can deduce the corresponding statement for asymptotically uniformly recurrent functions because uniformly recurrent functions do not form a vector space, so that the proof given for asymptotically almost periodic functions does not work in this extended framework (see e.g., [26, Theorem 1, pp. 37-38] as well as the proof of [16, Theorem 2.16(i)] and the point [7.] below).

Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. A continuous function $f : I \to E$ is said to be (ω, c)*-periodic* if and only if $f(t + \omega) = cf(t)$ for all $t \in I$; see [2]-[3] for more details. The number ω is said to be a c-period of $f(\cdot)$. The space of all (ω, c)-periodic functions $f : I \to E$ will be denoted with $P_{\omega,c}(I : E)$. Let we note that, by putting $c = 1$, we obtain the space of ω-periodic functions $f : I \to E$; by putting $c = -1$, we obtain the space of ω-antiperiodic functions $f : I \to E$; by putting $c = e^{ik\omega}$, we obtain the space of Bloch (ω, k)-periodic functions.

The following facts about the (ω, c)-periodic functions should be stated at the very beginning (see also [2]-[3]):

(i) If $f \in P_{\omega,c}([0,\infty) : E)$, $f(\cdot)$ is not identically equal to zero and $|c| > 1$, then $\limsup_{t\to+\infty} \|f(t)\| = +\infty$; if $f \in P_{\omega,c}(\mathbb{R} : E)$ and $|c| > 1$, then $\lim_{t\to-\infty} f(t) = 0$ and, if $f(\cdot)$ is not identically equal to zero, then $\limsup_{t\to+\infty} \|f(t)\| = +\infty$.

(ii) If $f \in P_{\omega,c}(I : E)$ and $f(x) \neq 0$ for all $x \in I$, then the function $(1/f)(\cdot)$ belongs to the class $P_{\omega,1/c}(I : E)$.

(iii) If $f \in P_{\omega,c}(I : E)$ and $|c| = 1$, then the function is almost periodic. To see this, observe that there exists a real number $k \in \mathbb{R}$ such that $f(x + \omega) = e^{ik\omega} f(x)$, $x \in I$, so that the function $f(\cdot)$ is Bloch (ω, k)-periodic. After that, the conclusion follows from [7, Remark 2.2]. In this case, we can simply compute the Bohr spectrum by using the computation:

$$\begin{aligned} P_r(f) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\,ds = \lim_{n\to+\infty} \frac{1}{n\omega} \int_0^{n\omega} e^{-irs} f(s)\,ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_{j\omega}^{(j+1)\omega} e^{-irs} f(s)\,ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_0^{\omega} e^{-ir(s+j\omega)} c^j f(s)\,ds \\ &= \frac{1}{\omega} \int_0^{\omega} e^{-irs} f(s)\,ds \times \lim_{n\to+\infty} \frac{\sum_{j=0}^{n-1} \big(ce^{-ir\omega}\big)^j}{n}. \end{aligned}$$

Therefore, if $c = e^{ir\omega}$, then $P_r = 1$; otherwise, we have $P_r = 0$ because:

$$\left| \sum_{j=0}^{n-1} \big(ce^{-ir\omega}\big)^j \right| = \left| \frac{c^n e^{-irn\omega} - 1}{ce^{-ir\omega} - 1} \right| \leq \frac{2}{ce^{-ir\omega} - 1}, \quad n \in \mathbb{N}.$$

Furthermore, arguing as in the above-mentioned remark, we may deduce that for each $k \in \mathbb{R}$ the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \left\| f(\cdot + \alpha_n) - e^{ik\alpha_n} f(\cdot) \right\|_\infty = 0$$

is equivalent to saying that the function $F(\cdot) := e^{-ik\cdot} f(\cdot)$ is uniformly recurrent. Due to the argumentation given in the proof of [2, Proposition 2.2], with $I = \mathbb{R}$, we have that the function $f(\cdot)$ is (ω, c)-periodic if and only if the function $c^{-\frac{\cdot}{\omega}} f(\cdot)$ belongs to the space $P_\omega(I : E)$.

Let $1 \leq p < \infty$. We continue by recalling that a function $f \in L^p_{loc}(I : E)$ is said to be *Stepanov p-bounded* if and only if

$$\|f\|_{S^p} := \sup_{t \in I} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L^p_S(I : E)$ consisted of all Stepanov p-bounded functions is a Banach space. A function $f \in L^p_S(I : E)$ is said to be *Stepanov p-almost periodic* if and only if the function $\hat{f} : I \to L^p([0,1] : E)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, \ s \in [0,1], \tag{3}$$

is almost periodic. Furthermore, we say that a function $f \in L^p_S(I : E)$ is *asymptotically Stepanov p-almost periodic* if and only if there exist a Stepanov p-almost periodic function $g \in L^p_S(I : E)$ and a function $q \in L^p_S(I : E)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : E))$. It is well known that, if $1 \leq p \leq q < \infty$ and $f(\cdot)$ is (asymptotically) Stepanov q-almost periodic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost periodic.

We need the following definition from [16].

Definition 1.1. *Let* $1 \leq p < \infty$.

(i) A function $f \in L^p_{loc}(I : E)$ *is said to be Stepanov p-uniformly recurrent if and only if the function* $\hat{f} : I \to L^p([0,1] : E)$, *defined by* (3), *is uniformly recurrent.*

(ii) A function $f \in L^p_{loc}(I : E)$ *is said to be asymptotically Stepanov p-uniformly recurrent if and only if there exist a Stepanov p-uniformly recurrent function* $h(\cdot)$ *and a function* $q \in L^p_S(I : E)$ *such that* $f(t) = h(t) + q(t)$, $t \in I$ *and* $\hat{q} \in C_0(I : L^p([0,1] : E))$.

Let $f : \mathbb{R} \to E$ be continuous. Then it is said that $f(\cdot)$ is *almost automorphic* if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to E$ such that

$$\lim_{n \to \infty} f\Big(t + a_n\Big) = g(t) \ \text{ and } \ \lim_{n \to \infty} g\Big(t - a_n\Big) = f(t), \tag{4}$$

pointwise for $t \in \mathbb{R}$. If the convergence of limits appearing in (4) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is *compactly almost automorphic*. It is worth noting that an almost automorphic function $f(\cdot)$ is compactly almost automorphic if and only if it is uniformly continuous as well as that an almost automorphic function is always bounded. The function $f : I \to E$ is said to be *asymptotically (compactly) almost automorphic* if and only if there exist a (compactly) almost automorphic function $h : \mathbb{R} \to E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$. The space consisting of all (compactly) almost automorphic functions will be denoted by $(AA_{\mathbf{c}}(\mathbb{R} : E))$ $AA(\mathbb{R} : E)$, while the space consisting of all asymptotically almost automorphic functions will be denoted by $AAA(I : E)$.

For the sake of completeness, we will include the proof of following simple proposition:

Proposition 1.2. *(i) Suppose that* $f \in AA(\mathbb{R} : \mathbb{C})$ *and* $g \in AA(\mathbb{R} : E)$. *Then* $fg \in AA(\mathbb{R} : E)$.

(ii) Suppose that $f \in AA_{\mathbf{c}}(\mathbb{C} : \mathbb{R})$ *and* $g \in AA_{\mathbf{c}}(\mathbb{R} : E)$. *Then* $fg \in AA_{\mathbf{c}}(\mathbb{R} : E)$.

Proof. Suppose that (b_n) is a given real sequence. Then there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to E$ such that (4) holds pointwise for $t \in \mathbb{R}$, with the function $g(\cdot)$ replaced therein with the function $h_1(\cdot)$. Further on, there exist a subsequence (a_{n_k}) of (a_n) and a map $h_2 : \mathbb{R} \to \mathbb{C}$ such that $\lim_{k\to\infty} f(t+a_{n_k}) = h_2(t)$ and $\lim_{k\to\infty} h_2(t-a_{n_k}) = f(t)$, pointwise for $t \in \mathbb{R}$. This simply implies that $\lim_{k\to\infty} f(t+a_{n_k})g(t+a_{n_k}) = h_1(t)h_2(t)$ and $\lim_{k\to\infty} h_1(t-a_{n_k})h_2(t-a_{n_k}) = f(t)g(t)$, pointwise for $t \in \mathbb{R}$, finishing the proof of (i). The proof of (ii) follows from (i) and the fact that the pointwise product of two bounded uniformly continuous functions is a uniformly continuous function, provided that one of them is scalar-valued. □

Let $1 \leq p < \infty$. A function $f \in L^p_{loc}(\mathbb{R} : E)$ is called *Stepanov p-almost automorphic* (see e.g., Guérékata and Pankov [12]) if and only if for every real sequence (a_n), there exist a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(\mathbb{R} : E)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \Big\| f\big(a_{n_k} + s\big) - g(s) \Big\|^p ds = 0 \text{ and } \lim_{k\to\infty} \int_t^{t+1} \Big\| g\big(s - a_{n_k}\big) - f(s) \Big\|^p ds = 0$$

for each $t \in \mathbb{R}$; a function $f \in L^p_{loc}(I : E)$ is called *asymptotically Stepanov p-almost automorphic* if and only if there exist an S^p-almost automorphic function $g(\cdot)$ and a function $q \in L^p_S(I : E)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : E))$. Any Stepanov p-almost automorphic function $f(\cdot)$ has to be Stepanov p-bounded. Furthermore, if $1 \leq p \leq q < \infty$ and a function $f(\cdot)$ is (asymptotically) Stepanov q-almost automorphic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost automorphic. Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic).

Let us finally recall ([17]) that a continuous function $F : I \times X \to E$ is called *uniformly recurrent* if and only if for every $\epsilon > 0$ and every compact $K \subseteq X$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| F(t + \alpha_n, x) - F(t, x) \Big\| = 0, \quad x \in K.$$

A function $F : I \times X \to E$ is called *Stepanov p-uniformly recurrent* if and only if the function $\hat{F} : I \times X \to L^p([0,1] : E)$ is uniformly recurrent, where $\hat{F} : I \times X \to L^p([0,1] : E)$ is defined by $\hat{F}(t,x) := F(t+\cdot, x)$, $t \in I$, $x \in X$.

2. (ω, c)-Almost periodic type functions and their Stepanov generalizations

In this section, we will consider three different approaches for introducing the spaces of (ω, c)-almost periodic type functions and their Stepanov generalizations. The first approach is the simplest one (in the case of consideration of (ω, c)-almost automorphic functions and their Stepanov generalizations, we will always tactily assume that $I = \mathbb{R}$):

Definition 2.1. *Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. Then it is said that a continuous function $f : I \to E$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(ω, c)-almost automorphic/compactly (ω, c)-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by (1), is uniformly recurrent (almost periodic/almost automorphic/compactly almost automorphic). By $UR_{\omega,c}(I : E)$, $AP_{\omega,c}(I : E)$, $AA_{\omega,c}(I : E)$ and $AA_{\omega,c;c}(I : E)$ we denote the space of all (ω, c)-uniformly recurrent functions, the space of all (ω, c)-almost periodic functions, the space of all (ω, c)-almost automorphic functions and the space of all compactly (ω, c)-almost automorphic functions, respectively.*

It is clear that the space $P_{\omega,c}(I : E)$ is contained in any of the above introduced spaces. With $c = 1$ and $\omega > 0$ arbitrary, the class of (ω, c)-almost periodic functions reduces to the class of almost periodic functions; in this case, the class of (ω, c)-uniformly recurrent functions has recently been analyzed in [16], where the author also examined the class of $\odot_g$-almost periodic functions. The class of $(\omega, c, \odot_g)$-almost periodic functions can be also introduced and analyzed but we will skip all related details concerning this class of functions for simplicity.

For positive real numbers c_1, $\omega_1 > 0$ and c_2, $\omega_2 > 0$, we have the identity

$$c_1^{-\frac{t}{\omega_1}} c_2^{-\frac{t}{\omega_2}} = \Big(c_1^{\frac{\omega}{\omega_1}} c_2^{\frac{\omega}{\omega_2}} \Big)^{\frac{-t}{\omega}}, \quad t \in I.$$

With the help of [15, Theorem 2.1.1(ii)], Proposition 1.2 and this equality, we can simply deduce the following

Proposition 2.2. *Suppose that $\omega > 0$, c_1, $\omega_1 > 0$, c_2, $\omega_2 > 0$, $f(\cdot)$ is (ω_1, c_1)-almost periodic ((ω_1, c_1)-almost automorphic/compactly (ω_1, c_1)-almost automorphic), $g(\cdot)$ is (ω_2, c_2)-almost periodic ((ω_2, c_2)-almost automorphic/compactly (ω_2, c_2)-almost automorphic) and the function $f(\cdot)$ or the function $g(\cdot)$ is scalar-valued. Set $c := c_1^{\frac{\omega}{\omega_1}} c_2^{\frac{\omega}{\omega_2}}$. Then the function $fg(\cdot)$ is (ω, c)-almost periodic ((ω, c)-almost automorphic/compactly (ω, c)-almost automorphic).*

Since the sum of two uniformly recurrent functions need not be uniformly recurrent, $UR_{\omega,c}(I : E)$ is not a vector space together with the usual operations of addition of functions and pointwise multiplication of functions with scalars ([16]). But, $AP_{\omega,c}(I : E)$, $AA_{\omega,c}(I : E)$ and $AA_{\omega,c;\mathbf{c}}(I : E)$ are vector spaces together with the above operations.

Remark 2.3. *If the function $f_{\omega,c}(\cdot)$ is bounded and $|c| < 1$, then $\lim_{t\to+\infty} f(t) = 0$; moreover, if $I = \mathbb{R}$, the function $f_{\omega,c}(\cdot)$ is bounded and $|c| > 1$, then $\lim_{t\to-\infty} f(t) = 0$.*

Remark 2.4. *In the equation* (1)*, one can consider an arbitrary function $c(\cdot)$ in place of function $c^{-(\cdot/\omega)}$ but the things then become much more complicated. For example, following the examination from the previous remark, it seems reasonable to use the function $c^{-(|\cdot|/\omega)}$ in place of function $c^{-(\cdot/\omega)}$. We will not follow this approach for simplicity and we will consider here only the asymptotically (ω, c)-almost periodic type functions defined on the non-negative real axis.*

Since the spaces of uniformly recurrent, almost periodic and (compactly) almost automorphic functions are closed under reflections at zero ([8], [16]), the following proposition simply follows:

Proposition 2.5. *Suppose that $I = \mathbb{R}$ and $f : \mathbb{R} \to E$. Then $f(\cdot)$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(ω, c)-almost automorphic/compactly (ω, c)-almost automorphic) if and only if the function $\check{f}(\cdot) \equiv f(-\cdot)$ is $(\omega, 1/c)$-uniformly recurrent ($(\omega, 1/c)$-almost periodic/$(\omega, 1/c)$-almost automorphic/compactly $(\omega, 1/c)$-almost automorphic).*

It is clear that any (ω, c)-almost periodic function is (ω, c)-uniformly recurrent and compactly (ω, c)-almost automorphic, as well as that any compactly (ω, c)-almost automorphic function is (ω, c)-almost automorphic. Even in the case that $c = 1$ and $\omega > 0$ is arbitrary, there exists a compactly almost automorphic function which is not uniformly recurrent and therefore not almost periodic; the first example of a bounded uniformly continuous, uniformly recurrent function that is not asymptotically almost periodic (automorphic) has been constructed by in [13, Theorem 2.2] (see also [16] for more details).

Definition 2.6. *Let $c \in \mathbb{C}$, $|c| \geq 1$ and $\omega > 0$. Then it is said that a continuous function $f : [0, \infty) \to E$ is asymptotically (ω, c)-uniformly recurrent (asymptotically (ω, c)-almost periodic, asymptotically (compactly) (ω, c)-almost automorphic) if and only if there exist an (ω, c)-uniformly recurrent ((ω, c)-almost periodic, (compactly) (ω, c)-almost automorphic) function $h : [0, \infty) \to E$ and a function $q \in C_0([0, \infty) : E)$ such that $f(t) = h(t) + q(t)$ for all $t \geq 0$.*

The following facts concerning the introduced classes of functions should be stated:

1. Suppose that $|c| = 1$ and $\omega > 0$. Then we can use [15, Theorem 2.1.1(ii)] and Proposition 1.2 in order to see that the function $f : I \to E$ is (ω, c)-almost periodic ((compactly) (ω, c)-almost automorphic) if and only if $f(\cdot)$ is almost periodic ((compactly) almost automorphic). In the case that $I = [0, \infty)$, the same assertion holds for the asymptotically (ω, c)-almost periodic functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.

2. Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to E$ is (ω, c)-uniformly recurrent or (ω, c)-almost automorphic. If $f(\cdot)$ is not identically equal to zero, then the supremum formula (see [16, Proposition 2.2] and [15, Lemma 3.9.9]) and the fact that the only uniformly recurrent function $h : I \to E$ such that $\lim_{t\to+\infty} h(t) = 0$ is the zero-function ([16, point (ix)]), together imply that the function $f(t)$ is unbounded as $t \to +\infty$. Similarly, if $|c| < 1$, $\omega > 0$ and $f : \mathbb{R} \to E$ is (ω, c)-uniformly recurrent [(ω, c)-almost automorphic], then the function $\check{f}(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent [$(\omega, 1/c)$-almost automorphic] and we easily get from the above that $f(t)$ is unbounded as $t \to -\infty$. In the case that $I = [0, \infty)$, similar assertions hold for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions; in particular, a constant non-zero function cannot be asymptotically (ω, c)-uniformly recurrent or asymptotically (ω, c)-almost automorphic.

3. Suppose $c \in \mathbb{C}\backslash\{0\}$, $\omega > 0$ and $f : [0,\infty) \to E$ is (ω, c)-almost periodic. Then it is well known that there exists a unique almost periodic function $F_{\omega,c} : \mathbb{R} \to E$ such that $F_{\omega,c}(t) = f_{\omega,c}(t)$, $t \geq 0$. Define $F(t) := c^{t/\omega} F_{\omega,c}(t)$, $t \in \mathbb{R}$. Then it simply follows that the function $F(\cdot)$ is a unique (ω, c)-almost periodic function which extends the function $f(\cdot)$ to the whole real line.

4. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent ((compactly) (ω, c)-almost automorphic) function $f(\cdot)$, we have that the function $\|f(\cdot)\|$ is $(\omega, |c|)$-uniformly recurrent ((compactly) $(\omega, |c|)$-almost automorphic). In the case that $I = [0,\infty)$, then the same assertion holds for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.

5. The spaces $UR_{\omega,c}(I : E)$, $AP_{\omega,c}(I : E)$, $AA_{\omega,c}(I : E)$ and $AA_{\omega,c;\mathbf{c}}(I : E)$ are invariant under pointwise multiplications with scalars. In the case that $I = [0,\infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.

6. The spaces $UR_{\omega,c}(I : E)$, $AP_{\omega,c}(I : E)$, $AA_{\omega,c}(I : E)$ and $AA_{\omega,c;\mathbf{c}}(I : E)$ are translation invariant. In the case that $I = [0,\infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.

7. If $I = [0,\infty)$, $|c| \geq 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c}(I : E)$ $(AP_{\omega,c}(I : E)/AA_{\omega,c}(I : E)/AA_{\omega,c;\mathbf{c}}(I : E))$ converges uniformly to a function $f : I \to E$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c}(I : E)$ $(AP_{\omega,c}(I : E)/AA_{\omega,c}(I : E)/AA_{\omega,c;\mathbf{c}}(I : E))$. In the case that $I = [0,\infty)$, then the same assertion holds for the asymptotically (ω, c)-almost periodic type function spaces (as already mentioned in the introductory part, the only exception is the space of asymptotically uniformly recurrent functions).

For completeness, we will include the most relevant details of the proofs of the following two propositions:

Proposition 2.7. *Suppose $E = \mathbb{C}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, $f : I \to \mathbb{C}$ and $\inf_{x \in I} |f(x)| > m > 0$. Then the following holds:*

(i) If $|c| = 1$ and $f(\cdot)$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(ω, c)-almost automorphic/compactly (ω, c)-almost automorphic), then the function $(1/f)(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent ($(\omega, 1/c)$-almost periodic/ $(\omega, 1/c)$-almost automorphic/compactly $(\omega, 1/c)$-almost automorphic).

(ii) If $|c| \leq 1$, $I = [0,\infty)$ and $f(\cdot)$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic), then the function $(1/f)(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent ($(\omega, 1/c)$-almost periodic).

Proof. The proof of (i) essentially follows from the simple argumentation and the conclusions obtained in the point [1.], while the proof of (ii) can be deduced as follows. Suppose that the function $f(\cdot)$ is (ω, c)-almost periodic, i.e., the function $f_{\omega,c}(\cdot)$ is almost periodic. This implies that for each number $\epsilon > 0$ there exists a finite number $l > 0$ such that any subinterval I' of I contains at least one point τ such that

$$\left| c^{-\frac{t+\tau}{\omega}} f(t + \tau) - c^{-\frac{t}{\omega}} f(t) \right| \leq \epsilon, \quad t \geq 0.$$

This implies

$$\left| f(t + \tau) - c^{-\frac{\tau}{\omega}} f(t) \right| \leq \epsilon \left| c^{\frac{t+\tau}{\omega}} \right|, \quad t \geq 0.$$

Then the final conclusion is a consequence of the following simple calculation:

$$\begin{aligned} \left| \frac{c^{\frac{t+\tau}{\omega}}}{f(t + \tau)} - \frac{c^{\frac{t}{\omega}}}{f(t)} \right| &= \left| c^{\frac{t}{\omega}} \right| \cdot \left| \frac{f(t + \tau) - c^{-\frac{\tau}{\omega}} f(t)}{f(t + \tau) \cdot f(t)} \right| \\ &\leq \frac{\epsilon}{m^2} \left| c^{\frac{2t+\tau}{\omega}} \right| \leq \frac{\epsilon}{m^2}, \quad t \geq 0. \end{aligned}$$

The proof for (ω, c)-uniform recurrence is similar and therefore omitted. □

Proposition 2.8. *Suppose that $I = \mathbb{R}$, $f : \mathbb{R} \to E$ satisfies that the function $f_{\omega,c}(\cdot)$ is a bounded uniformly recurrent (almost periodic, (compactly) almost automorphic) and $c^{-\frac{\cdot}{\omega}}\psi(\cdot) \in L^1(\mathbb{R})$. Then the function $c^{-\frac{\cdot}{\omega}}(\psi * f)(\cdot)$ is bounded uniformly continuous and the function $(\psi * f)(\cdot)$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(compactly) (ω, c)-almost automorphic).*

Proof. For every $x \in \mathbb{R}$, the convolution $(\psi * f)(x)$ is well defined and we have

$$c^{-\frac{x}{\omega}}(\psi * f)(x) = \int_{-\infty}^{\infty}\Big[c^{-\frac{x-y}{\omega}}\psi(x-y)\Big] \cdot \Big[c^{-\frac{y}{\omega}}f(y)\Big]\,dy, \quad x \in \mathbb{R}.$$

Then the corresponding statement follows from the fact that the space of all almost periodic ((compactly) almost automorphic) functions and the space of all bounded uniformly recurrent functions are convolution invariant (see [8] and [15]-[16]). □

The following definitons are logical analogues of Definition 2.1 and Definition 2.6 for Stepanov classes:

Definition 2.9. *Let $p \in [1, \infty)$, $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. Then it is said that a function $f \in L^p_{loc}(I : E)$ is Stepanov (p, ω, c)-uniformly recurrent (Stepanov (p, ω, c)-almost periodic/Stepanov (p, ω, c)-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by* (1), *is Stepanov p-uniformly recurrent (Stepanov p-almost periodic/Stepanov p-almost automorphic). By $S^pUR_{\omega,c}(I : E)$, $S^pAP_{\omega,c}(I : E)$ and $S^pAA_{\omega,c}(I : E)$ we denote the space of all Stepanov (p, ω, c)-uniformly recurrent functions, the space of all Stepanov (p, ω, c)-almost periodic functions and the space of all Stepanov (p, ω, c)-almost automorphic functions, respectively.*

Definition 2.10. *Let $p \in [1, \infty)$, $c \in \mathbb{C}$, $|c| \geq 1$ and $\omega > 0$. Then it is said that a function $f \in L^p_{loc}([0, \infty) : E)$ is asymptotically Stepanov (p, ω, c)-uniformly recurrent (asymptotically Stepanov (p, ω, c)-almost periodic, asymptotically Stepanov (p, ω, c)-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by* (1), *is asymptotically Stepanov p-uniformly recurrent (asymptotically Stepanov p-almost periodic, asymptotically Stepanov p-almost automorphic). By $AS^pUR_{\omega,c}(I : E)$, $AS^pAP_{\omega,c}(I : E)$ and $AS^pAA_{\omega,c}(I : E)$ we denote the space of all asymptotically Stepanov (p, ω, c)-uniformly recurrent functions, the space of all asymptotically Stepanov (p, ω, c)-almost periodic functions and the space of all asymptotically Stepanov (p, ω, c)-almost automorphic functions, respectively.*

The conclusion established in the points [1.-2., 4.-7.] can be simply reformulated for the Stepanov classes. For example, in the case of point [2.], we may conclude the following: Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to E$ is Stepanov (p, ω, c)-uniformly recurrent or Stepanov (p, ω, c)-almost automorphic. If $f(\cdot)$ is not almost everywhere equal to zero, then the function $f(\cdot)$ is not Stepanov p-bounded; moreover, in the case of consideration of Stepanov (p, ω, c)-almost automorphicity, the function $\hat{f}(\cdot)$ is unbounded as $t \to +\infty$ so that a constant non-zero function cannot be Stepanov (p, ω, c)-uniformly recurrent or Stepanov (p, ω, c)-almost automorphic.

Basically, any established result for almost periodic type functions and their Stepanov generalizations can be straightforwardly reformulated for (ω, c)-almost periodic type functions and their Stepanov generalizations (in the sequel, we will try not to consider such statements). For example, using [16, Theorem 2.16(iii)] we can immediately deduce the following:

Proposition 2.11. *Let $p \in [1, \infty)$. If $f : [0, \infty) \to E$ satisfies that the function $f_{\omega,c}(\cdot)$ is uniformly continuous and asymptotically Stepanov p-uniformly recurrent, then the function $f(\cdot)$ is asymptotically (ω, c)-uniformly recurrent.*

Let us only note that the uniform continuity of function $f_{\omega,c}(\cdot)$ is ensured provided that $|c| \geq 1$ and $f(\cdot)$ is a bounded uniformly continuous function. This follows from the fact that, for every two non-negative real numbers $t_1,\ t_2 \geq 0$ such that $t_1 < t_2$, the Darboux inequality yields

$$\begin{aligned}\Big\|c^{-\frac{t_1}{\omega}}f\big(t_1\big) - c^{-\frac{t_2}{\omega}}f\big(t_2\big)\Big\| &\leq \Big\|c^{-\frac{t_1}{\omega}}\Big[f\big(t_1\big) - f\big(t_2\big)\Big]\Big\| + \Big\|\Big[c^{-\frac{t_1}{\omega}} - c^{-\frac{t_2}{\omega}}\Big]f\big(t_2\big)\Big\| \\ &\leq \Big\|f\big(t_1\big) - f\big(t_2\big)\Big\| + \frac{1}{\omega}(\ln|c| + \pi) \cdot \big|t_1 - t_2\big| \cdot \|f\|_\infty.\end{aligned}$$

Now we would like to endow the introduced spaces of (asymptotically) (ω, c)-almost periodic type functions with certain norms. We start with the notion introduced in Definition 2.1 and Definition 2.6. Define

$$\|f\|_{\omega,c} := \sup_{t\in I}\left\|c^{-\frac{t}{\omega}} f(t)\right\|.$$

Proposition 2.12. *The spaces $AP_{\omega,c}(I : E)$, $AA_{\omega,c}(I : E)$, $AA_{\omega,c;\mathbf{c}}(I : E)$, $AAP_{\omega,c}([0,\infty) : E)$, $AAA_{\omega,c}([0,\infty) : E)$ and $AAA_{\omega,c;\mathbf{c}}([0,\infty) : E)$, equipped with the norm $\|\cdot\|_{\omega,c}$, are Banach spaces.*

Proof. Denote by $\mathcal{X}$ any of the above spaces. Suppose that $(f_n)_n$ is a Cauchy sequence in $\mathcal{X}$. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,\ n \geq N$, we have $\|f_n - f_m\|_{\omega,c} < \varepsilon$. So, there exist $u_m,\ u_n \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ (with the meaning clear) such that $f_m(t) = c^{\frac{t}{\omega}} u_m(t)$ and $f_n(t) = c^{\frac{t}{\omega}} u_n(t)$ for all $t \in I$. For $m,\ n \geq N$, we have

$$\begin{aligned}\left\|u_m - u_n\right\|_\infty &= \sup_{t\in I}\left\|u_m(t) - u_n(t)\right\| \\ &= \sup_{t\in I}\left\|c^{-\frac{t}{\omega}} f_m(t) - c^{-\frac{t}{\omega}} f_n(t)\right\| \\ &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}}\left[f_m(t) - f_n(t)\right]\right\| \\ &= \left\|f_n - f_m\right\|_{\omega,c} < \varepsilon.\end{aligned}$$

Hence, $(u_n)_n$ is a Cauchy sequence in $c^{-\frac{\cdot}{\omega}}\mathcal{X}$, which is a complete space. Then, there exists $u \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ such that $\lim_{n\to+\infty} u_n = u$. Define $f(t) := c^{\frac{t}{\omega}} u(t)$, $t \in I$. Thus,

$$\begin{aligned}\left\|f_n - f\right\|_{\omega,c} &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}}\left[f_n(t) - f(t)\right]\right\| \\ &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u_n(t) - |c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u(t)\right\| \\ &= \sup_{t\in I}\left\|u_n(t) - u(t)\right\| \to 0,\end{aligned}$$

when $n \to \infty$. Hence, $\mathcal{X}$ is a Banach space. □

For any $c \in \mathbb{C} \setminus \{0\}$ and $p \in [1,\infty)$, we denote by $L^p_{S,c}(I : E)$ the space of all functions $f \in L^p_{loc}(I : E)$ such that

$$\|f\|_{p,\omega,c} := \sup_{t\in I}\left(\int_t^{t+1} |c|^{-\frac{s}{\omega}} f(s)\, ds\right)^{1/p}.$$

Then $(L^p_{S,c}(I : E), \|\cdot\|_{p,\omega,c})$ is a Banach space. Arguing as above, we may conclude that $S^pAP_{\omega,c}(I : E)$ ($S^pAA_{\omega,c}(I : E)$/$AS^pAP_{\omega,c}(I : E)$, $AS^pAA_{\omega,c}(I : E)$) is a closed subspace of $L^p_{S,c}(I : E)$ and therefore the Banach space itself.

3. (ω, c)-Uniform recurrence of type i and (ω, c)-almost periodicity of type i ($i = 1, 2$)

Suppose temporarily that $f \in P_{\omega,c}(I : E)$ and $n \in \mathbb{N}$. Then we have $f(t + n\omega) = c^n f(t)$, $t \in I$. Setting $\alpha_n = n\omega$, we get that for each $\epsilon > 0$ and $t \in I$ we have

$$\left\|f(t + \alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\right\| \leq \epsilon \quad \text{and} \quad \left\|c^{\frac{-\alpha_n}{\omega}} f(t + \alpha_n) - f(t)\right\| \leq \epsilon. \tag{5}$$

The equation (5) motivates us to introduce the following concepts of (ω, c)-uniform recurrence and (ω, c)-almost periodicity [it is not clear how we can do that for (compact) (ω, c)-almost automorphicity in a satisfactory way].

Definition 3.1. *Suppose that $f : I \to E$ is continuous, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*

(i) We say that $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) if and only if there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| f(t+\alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t) \Big\| = 0 \ \left(\lim_{n\to+\infty} \sup_{t\in I} \Big\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \Big\| = 0 \right).$$

(ii) We say that $f(\cdot)$ is (ω, c)-almost periodic of type 1 (type 2) if and only if for each $\epsilon > 0$ the set

$$\Big\{ \tau > 0 : \sup_{t\in I} \Big\| f(t+\tau) - c^{\frac{\tau}{\omega}} f(t) \Big\| < \epsilon \Big\} \ \left(\Big\{ \tau > 0 : \sup_{t\in I} \Big\| c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t) \Big\| < \epsilon \Big\} \right)$$

is relatively dense in $[0, \infty)$.

By $UR_{\omega,c,i}(I : E)$ and $AP_{\omega,c,i}(I : E)$, we denote the space of all (ω, c)-uniformly recurrent functions of type i and the space of all (ω, c)-almost periodic functions of type i, respectively ($i = 1, 2$).

It is clear that the set $\{n\omega : n \in \mathbb{N}\}$ is relatively dense in $[0, \infty)$. Taking into account this observation, it follows that the space $P_{\omega,c}(I : E)$ is contained in the spaces $UR_{\omega,c,i}(I : E)$ and $AP_{\omega,c,i}(I : E)$, for $i = 1, 2$; moreover, $UR_{\omega,c,i}(I : E) \supseteq AP_{\omega,c,i}(I : E)$ for $i = 1, 2$ and it is clear that for any $t \in I$ and $\tau \geq 0$ we have

$$\begin{aligned} \Big\| c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t) \Big\| &= \Big\| c^{\frac{-\tau}{\omega}} \Big[f(t+\tau) - c^{\frac{\tau}{\omega}} f(t) \Big] \Big\| \\ &= |c|^{\frac{-\tau}{\omega}} \Big\| f(t+\tau) - c^{\frac{\tau}{\omega}} f(t) \Big\|. \end{aligned}$$

Therefore, in the case that $|c| = 1$, it simply follows that the (ω, c)-almost periodicity of type 1 (type 2) is equivalent with the usual almost periodicity as well as that the notion of (ω, c)-uniform recurrence of type 1 is equivalent with the notion of (ω, c)-uniform recurrence of type 2.

But, in the case that $|c| \neq 1$, the concepts introduced in Definition 3.1 are not satisfactory to a great extent. Before stating the corresponding result which justifies this fact, let us denote by $M_{\omega,c}(I : E)$ the space consisting of all functions $f : I \to E$ such that $c^{-\cdot/\omega} f(\cdot) \in P(I : E)$. It is clear that $M_{\omega,c}(I : E)$ is not a vector space together with the usual operations.

Theorem 3.2. *Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*

(i) Suppose that $|c| > 1$. Then $UR_{\omega,c,i}(I : E) = AP_{\omega,c,i}(I : E) = M_{\omega,c}(I : E)$ for $i = 1, 2$.

(ii) Suppose that $|c| < 1$ and $I = \mathbb{R}$. Then $UR_{\omega,c,i}(I : E) = AP_{\omega,c,i}(I : E) = M_{\omega,c}(I : E)$ for $i = 1, 2$.

Before giving the proof of Theorem 3.2, we will state two lemmas. The first one is simple and follows almost immediately from Definition 3.1:

Lemma 3.3. *Suppose that $f : I \to E$ is continuous, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*

(i) If $|c| \geq 1$, then $UR_{\omega,c,1}(I : E) \subseteq UR_{\omega,c,2}(I : E)$ and $AP_{\omega,c,1}(I : E) \subseteq AP_{\omega,c,2}(I : E)$.

(ii) If $|c| \leq 1$, then $UR_{\omega,c,1}(I : E) \supseteq UR_{\omega,c,2}(I : E)$ and $AP_{\omega,c,1}(I : E) \supseteq AP_{\omega,c,2}(I : E)$.

(iii) In the case that $I = [0, \infty)$ and $|c| \geq 1$, then $UR_{\omega,c,2}(I : E) \subseteq UR_{\omega,c}(I : E)$ and $AP_{\omega,c,2}(I : E) \subseteq AP_{\omega,c}(I : E)$.

Lemma 3.4. *(see also Proposition 2.5) Suppose that $I = \mathbb{R}$ and $f : \mathbb{R} \to E$. Then $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) [(ω, c)-almost periodic of type 1 (type 2)] if and only if the function $\check{f}(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent of type 2 (type 1) [(ω, c)-almost periodic of type 2 (type 1)].*

Proof. The proof simply follows by observing that, for every $\tau > 0$ and $\epsilon > 0$, we have:

$$\sup_{t\in I}\Big\|f(t+\tau)-c^{\frac{\tau}{\omega}}f(t)\Big\|<\epsilon \Leftrightarrow \sup_{t\in I}\Big\|f(-t+\tau)-c^{\frac{\tau}{\omega}}f(-t)\Big\|<\epsilon$$
$$\Updownarrow$$
$$\sup_{t\in I}\Big\|\check{f}(t-\tau)-c^{\frac{\tau}{\omega}}\check{f}(t)\Big\|<\epsilon \Leftrightarrow \sup_{t\in I}\Big\|\check{f}(t)-c^{\frac{\tau}{\omega}}\check{f}(t+\tau)\Big\|<\epsilon$$
$$\Updownarrow$$
$$\sup_{t\in I}\Big\|(1/c)^{-\frac{\tau}{\omega}}\check{f}(t+\tau)-\check{f}(t)\Big\|<\epsilon.$$

□

Proof of Theorem 3.2. Keeping in mind Lemma 3.4, it suffices to prove (i). Towards this end, we recognize two cases: $I=[0,\infty)$ and $I=\mathbb{R}$. In the first case, it suffices to show that $UR_{\omega,c,2}([0,\infty):E)\subseteq M_{\omega,c}([0,\infty):E)$ and $M_{\omega,c}([0,\infty):E)\subseteq AP_{\omega,c,1}([0,\infty):E)$. So, let $f\in UR_{\omega,c,2}([0,\infty):E)$. This implies that there exist a finite constant $M>0$ and a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\sup_{t\in I,n\in\mathbb{N}}\Big\|c^{\frac{-\alpha_n}{\omega}}f(t+\alpha_n)-f(t)\Big\|\leq M.$$

Since $f(t)=c^{t/\omega}f_{\omega,c}(t)$, $t\geq 0$, the above implies

$$\Big\|f_{\omega,c}(t+\alpha_n)-f_{\omega,c}(t)\Big\|\leq |c|^{-(t/\omega)}M,\quad t\geq 0,\ n\in\mathbb{N}.$$

Hence, for every $n\in\mathbb{N}$, we have $\lim_{t\to+\infty}[f_{\omega,c}(t+\alpha_n)-f_{\omega,c}(t)]=0$. On the other hand, Lemma 3.3(iii) yields that, for every $n\in\mathbb{N}$, we have that the function $f_{\omega,c}(\cdot+\alpha_n)-f_{\omega,c}(\cdot)$ is uniformly recurrent; hence, for every $n\in\mathbb{N}$, we have $f_{\omega,c}(\cdot+\alpha_n)\equiv f_{\omega,c}(\cdot)$ and therefore $f_{\omega,c}(\cdot)$ belongs to the space $P([0,\infty):E)$, as claimed (cf. [16, points (ix) and (xi)]). To see that $M_{\omega,c}([0,\infty):E)\subseteq AP_{\omega,c,1}([0,\infty):E)$, suppose that $f_{\omega,c}(t+T)=f_{\omega,c}(t)$ for all $t\geq 0$ and some $T>0$. This simply implies that $f(t+nT)=c^{nT/\omega}f(t)$ for all $n\in\mathbb{N}$ so that $f\in AP_{\omega,c,1}([0,\infty):E)$ because the set $\{nT : n\in\mathbb{N}\}$ is relatively dense in $[0,\infty)$. Suppose now that $I=\mathbb{R}$. Similarly as above, it follows that $UR_{\omega,c,i}(\mathbb{R}:E)\supseteq AP_{\omega,c,i}(\mathbb{R}:E)\supseteq M_{\omega,c}(\mathbb{R}:E)$ for $i=1,2$. Therefore, it suffices to show that $UR_{\omega,c,2}(\mathbb{R}:E)\subseteq M_{\omega,c}(\mathbb{R}:E)$. Let $f\in UR_{\omega,c,2}(\mathbb{R}:E)$. Since the restriction of $f(\cdot)$ on $[0,\infty)$ belongs to the space $UR_{\omega,c,2}([0,\infty):E)$, it readily follows that there exists a number $T>0$ such that $f_{\omega,c}(t+T)=f_{\omega,c}(t)$ for all $t\geq 0$. To complete the proof, it suffices to prove that this equality holds for all real numbers $t<0$. Let $\epsilon>0$ be fixed. Due to our assumption, we have the existence of an integer $n_0\in\mathbb{N}$ such that $t+\alpha_n>0$ as well as that

$$\Big\|c^{t/\omega}f_{\omega,c}(t+\alpha_n)-c^{t/\omega}f_{\omega,c}(t)\Big\|\leq\epsilon$$
$$\text{and }\Big\|c^{(t+T)/\omega}f_{\omega,c}(t+T+\alpha_n)-c^{(t+T)/\omega}f_{\omega,c}(t+T)\Big\|\leq\epsilon,$$

i.e.,

$$\Big\|c^{t/\omega}f_{\omega,c}(t+\alpha_n)-c^{t/\omega}f_{\omega,c}(t)\Big\|\leq\epsilon$$
$$\text{and }\Big\|c^{t/\omega}f_{\omega,c}(t+\alpha_n)-c^{t/\omega}f_{\omega,c}(t+T)\Big\|\leq\epsilon|c|^{-T/\omega}.$$

This implies

$$\Big\|c^{t/\omega}\Big[f_{\omega,c}(t+T)-f_{\omega,c}(t)\Big]\Big\|\leq\epsilon\Big(1+|c|^{-T/\omega}\Big).$$

Letting $\epsilon\to 0+$, we get $f_{\omega,c}(t+T)=f_{\omega,c}(t)$, as claimed. □

Corollary 3.5. *Suppose that $i = 1, 2$, $|c| < 1$, $\omega > 0$ and $f \in AP_{\omega,c,i}([0,\infty) : E)$. Then there exists a function $F \in AP_{\omega,c,i}(\mathbb{R} : E)$ such that $F(t) = f(t)$ for all $t \geq 0$ if and only if $f \in M_{\omega,c}([0,\infty) : E)$.*

Further on, the points [4., 5., 6., 7.] can be restated as follows:

4′. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent function $f(\cdot)$ of type 1 (type 2), we have that the function $\|f(\cdot)\|$ is $(\omega, |c|)$-uniformly recurrent of type 1 (type 2).

5′. The spaces $UR_{\omega,c,i}(I : E)$ and $AP_{\omega,c,i}(I : E)$ are invariant under pointwise multiplications with scalars ($i = 1, 2$).

6′. The spaces $UR_{\omega,c,i}(I : E)$ and $AP_{\omega,c,i}(I : E)$ are translation invariant ($i = 1, 2$).

7′. If $I = [0,\infty)$, $|c| \geq 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c,2}(I : E)$ converges uniformly to a function $f : I \to E$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c,2}(I : E)$. Furthermore, if $I = [0,\infty)$, $|c| \leq 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c,1}(I : E)$ $(AP_{\omega,c,1}(I : E))$ converges uniformly to a function $f : I \to E$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c,1}(I : E)$ $(AP_{\omega,c,1}(I : E))$.

Now we will prove the following

Proposition 3.6. *Suppose that $i = 1, 2$, $|c| < 1$, $\omega > 0$ and $f \in AP_{\omega,c,i}(I : E)$. Then the function $f_{\omega,c}(\cdot)$ is bounded and $\lim_{t\to+\infty} f(t) = 0$.*

Proof. By Theorem 3.2(ii) and Lemma 3.3(iv) it suffices to consider the case $I = [0,\infty)$ and the class $AP_{\omega,c,1}([0,\infty) : E)$. Let $\epsilon = 1$. Then there exists a finite number $l > 0$ such that any subinterval I' of $[0,\infty)$ contains a point τ such that $\|c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t)\| < 1$ for all $t \geq 0$. Suppose that $t \in [nl, (n+1)l]$ for some $n \in \mathbb{N}$. Then there exists $\tau_n \in [(n-1)l, nl]$ such that $\|c^{\frac{-\tau_n}{\omega}} f(t'+\tau_n) - f(t')\| < 1$ for all $t' \geq 0$. In particular, $t - \tau_n = t' \in [0, 2l]$ and the above implies $\|f(t)\| \leq (1+M)|c|^{\tau_n/\omega} \leq (1+M)[\max_{t''\in[0,2l]} |c|^{-t''/\omega}]|c|^{t/\omega}$, where $M := \sup_{x\in[0,2l]} \|f(x)\|$. This yields the required limit equality. □

Example 3.7. *The first example of an even, unbounded, uniformly continuous, uniformly recurrent function $f : \mathbb{R} \to \mathbb{R}$ has been constructed in [13, Theorem 2.1]. Denote its restriction to the non-negative real axis by the same symbol. Then Proposition 3.6 implies that the function $c^{-\cdot/\omega} f(\cdot)$ cannot belong to the space $AP_{\omega,c,i}([0,\infty) : \mathbb{C})$ for $i = 1, 2$. On the other hand, it is clear that $c^{-\cdot/\omega} f(\cdot) \in UR_{\omega,c}([0,\infty) : \mathbb{C}) \subseteq UR_{\omega,c,i}([0,\infty) : \mathbb{C})$ for $i = 1, 2$.*

Corollary 3.8. *Suppose that $|c| < 1$ and $\omega > 0$. Then $f \in AP_{\omega,c,1}([0,\infty) : E)$ if and only if the function $f_{\omega,c}(\cdot)$ is bounded and continuous.*

Proof. Due to Proposition 3.6, it suffices to show that the boundedness of function $f_{\omega,c}(\cdot)$ implies that $f \in AP_{\omega,c,1}([0,\infty) : E)$. If so, then we need to prove that for each $\epsilon > 0$ the set consisting of all positive reals $t > 0$ such that

$$\left\| c^{(t+\tau)/\omega} f_{\omega,c}(t+\tau) - c^{(t+\tau)/\omega} f_{\omega,c}(t) \right\| \leq \epsilon, \quad t \geq 0$$

is relatively dense in $[0,\infty)$. But, this simply follows from the fact that this set contains a ray $[a(\epsilon), \infty)$ for a sufficiently large real number $a(\epsilon) > 0$, which can be proved by using the boundedness of $f_{\omega,c}(\cdot)$ and the simple inequality $|c|^{t/\omega} \leq 1$, $t \geq 0$. □

Remark 3.9. *Suppose that $|c| < 1$ and $\omega > 0$. Using Corollary 3.8, we can simply prove that $f \in AP_{\omega,c,1}([0,\infty) : E)$ if and only if for every (there exists) strictly increasing sequence (α_n) of positive reals tending to plus infinity such that $\lim_{n\to+\infty} \sup_{t\geq 0} \|f(t+\alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\| = 0$.*

Example 3.10. *Suppose that $f(t) := 2^{-t}[1 + (1/\ln(2+t))]$, $t \geq 0$. Due to Corollary 3.8, this function belongs to the space $AP_{1,1/2,1}([0,\infty) : \mathbb{C}) \subseteq UR_{1,1/2,1}([0,\infty) : \mathbb{C})$. On the other hand, $f(\cdot)$ does not belong to the space $UR_{1,1/2,2}([0,\infty) : \mathbb{C})$. Otherwise, we would have the existence of an arbitrarily large positive real number $\alpha > 0$ such that*

$$\sup_{t\geq 0} \left| 2^{-t} \frac{\ln(1 + (\alpha/(1+t)))}{\ln(2+t) \cdot \ln(2+t+\alpha)} \right| \leq \epsilon.$$

Taking $t = 0$, this simply leads us to a contradiction.

The class $UR_{\omega,c,1}([0,\infty) : E)$ is also extremely non-interesting due to the following characterization:

Proposition 3.11. *Suppose $c \in \mathbb{C} \setminus \{0\}$, $|c| < 1$ and $\omega > 0$. Then $UR_{\omega,c,1}([0,\infty) : E) = C_0([0,\infty) : E)$.*

Proof. If $f \in C_0([0,\infty) : E)$, then for each strictly increasing sequence (α_n) tending to plus infinity and for each real number $\epsilon > 0$ we can always find an integer $n_0 \in \mathbb{N}$ such that $\|f(t+\alpha_n) - c^{\alpha_n/\omega} f(t)\| \leq (\epsilon/2) + |c|^{\alpha_n/\omega}\|f(t)\| \leq (\epsilon/2) + |c|^{\alpha_n/\omega}\|f\|_\infty \leq \epsilon$, $t \geq 0$, $n \geq n_0$, which implies $f \in UR_{\omega,c,1}([0,\infty) : E)$. To prove the converse, let us first show that the assumption $f \in UR_{\omega,c,1}([0,\infty) : E)$ implies the boundedness of $f(\cdot)$. If (α_n) satisfies the requirements of definition of space $UR_{\omega,c,1}([0,\infty) : E)$, then we may assume without loss of generality that $\alpha_{n+1} - \alpha_n > 3$ for all $n \in \mathbb{N}$ and

$$\left\|f(t+\alpha_n)\right\| \leq 1 + |c|^{\alpha_n/\omega}\|f(t)\|, \quad t \geq 0,\ n \in \mathbb{N}. \tag{6}$$

Let $n \in \mathbb{N}$ be fixed and let $M_0 := \max_{t\in[0,\alpha_n]}\|f(t)\|$. Then (6) inductively implies that for arbitrary $T \in (0,\alpha_n]$ and for arbitrary $k \in \mathbb{N}$ we have

$$\left\|f(T+k\alpha_n)\right\| \leq \sum_{j=0}^{k-1} |c|^{\alpha_n j/\omega} + |c|^{k\alpha_n/\omega} M_0 \leq \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0.$$

Therefore, $\|f(t)\| \leq \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0$, $t \geq 0$, as claimed. The remainder of proof is simple; since the function $f(\cdot)$ is bounded, then we have the existence of an integer $n_1 \in \mathbb{N}$ such that

$$\left\|f(t+\alpha_n)\right\| \leq |c|^{\alpha_n/\omega}\|f\|_\infty + (\epsilon/2) < \epsilon, \quad t \geq 0,\ n \geq n_1,$$

and therefore $f \in C_0([0,\infty) : E)$. □

Now we will prove the following result:

Proposition 3.12. *Suppose that $|c| < 1$ and $\omega > 0$. Then $f \in AP_{\omega,c,2}([0,\infty) : E)$ if and only if the function $f_{\omega,c}(\cdot)$ is bounded continuous and for each $\epsilon > 0$ and $N > 0$ the set of all positive real numbers $\tau > 0$ such that*

$$\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leq \epsilon, \quad t \in [0, N] \tag{7}$$

is relatively dense in $[0,\infty)$.

Proof. Suppose first that $f \in AP_{\omega,c,2}([0,\infty) : E)$. Due to Proposition 3.6, the function $f_{\omega,c}(\cdot)$ is bounded. Let $\epsilon > 0$ and $N > 0$ be fixed, and let $\epsilon_0 > 0$ be such that $\epsilon_0|c|^{-N/\omega} \leq \epsilon$. By our assumption, the set of all positive reals $\tau > 0$ such that $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leq \epsilon_0|c|^{-t/\omega}$, $t \geq 0$ is relatively dense in $[0,\infty)$. If τ belongs to this set, then we have $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leq \epsilon_0|c|^{-t/\omega} \leq \epsilon$, $t \in [0,N]$. For the converse, it suffices to consider the case in which $f_{\omega,c} \neq 0$. Fix a number $\epsilon > 0$. In this case, we can find a number $N > 0$ such that $|c|^{t/\omega} \leq \epsilon/(2(1 + \|f_{\omega,c}\|_\infty))$ for all $t \geq N$. For this $\epsilon > 0$ and $N > 0$ we can find a relatively dense set of positive reals τ satisfying (7). If τ belongs to this set, then there exist two possibilities: $t \geq N$ or $t \in [0,N)$. In the first case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau) - f_{\omega,c}(t)]\| \leq \epsilon|c|^{t/\omega} \leq \epsilon$; in the second case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau) - f_{\omega,c}(t)]\| \leq (2\epsilon\|f_{\omega,c}\|_\infty)/(2(1 + \|f_{\omega,c}\|_\infty)) < \epsilon$. Summa summarum, we have that $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leq \epsilon_0|c|^{-t/\omega}$, $t \geq 0$. The proof of the proposition is thereby complete. □

Remark 3.13. *(i) Let us recall that any Levitan N-almost periodic function $f_{\omega,c} : [0,\infty) \to E$ satisfies that for each $\epsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that* (7) *holds is relatively dense in* $[0,\infty)$ *(cf. [20, Definition 2, p. 53]). In particular, the restriction of any almost automorphic function $f_{\omega,c} : \mathbb{R} \to E$ to $[0,\infty)$ satisfies this condition. Denote by $AA_{[0,\infty)}(E)$ the vector space consisting of such functions; thus, $c^{\cdot/\omega}AA_{[0,\infty)}(E) \subseteq AP_{\omega,c,2}([0,\infty) : E)$. Recall also that the function $t \mapsto 1/(2 + \cos t + \cos(\sqrt{2}t))$, $t \geq 0$ is Levitan N-almost periodic and unbounded.*

(ii) *According to [20, Definition 2, p. 80], a continuous function $f : I \to E$ is called recurrent if and only if for each $\epsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that* (7) *holds is relatively dense in* $[0, \infty)$ *(the case $I = \mathbb{R}$ has been considered in [20], only). The Stepanov generalizations of recurrent functions can be also introduced but then it is not clear how one can consider the invariance of recurrence under the action of infinite convolution product given by* (14) *since the methods proposed in the proof of [15, Proposition 2.6.11] and related results do not work in this framework. Note also that we can extend the notion of (ω, c)-almost automorphicity by requiring that the function $f_{\omega,c}(\cdot)$ is recurrent.*

(iii) *Due to Corollary 3.8, $AP_{\omega,c,1}([0,\infty) : E)$ is the vector space together with the usual operations. This is not longer true for the space $AP_{\omega,c,2}([0,\infty) : E)$, which can be deduced from Proposition 3.12 and a counterexample constructed by Veech (see e.g., [4, Example 2.8], and the corresponding example given in the pioneering paper [19] by Levin). In particular, $AP_{\omega,c,2}([0,\infty) : E) \subseteq UR_{\omega,c,2}([0,\infty) : E)$ strictly contains $c^{\cdot/\omega}AA_{[0,\infty)}(E)$. On the other hand, the compactly almost automorphic function constructed by Fink in [10] is not asymptotically uniformly recurrent, as shown in [16, Example 2.23]. This implies that there exists a function $f \in c^{\cdot/\omega}AA_{[0,\infty)}(E)$ such that $f_{\omega,c}(\cdot)$ is not uniformly recurrent; in particular, $UR_{\omega,c,2}([0,\infty) : E)$ strictly contains $UR_{\omega,c}([0,\infty) : E)$.*

(iv) *As shown by Bohr by a counterexample constructed on [6, pp. 113-118], there exist two bounded, even, uniformly continuous, uniformly recurrent functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that its sum is not uniformly recurrent. Furthermore, we can choose $f(\cdot)$ and $g(\cdot)$ such that $f(0) = g(0) = 1$ and $|f(t) + g(t)| \le 1$ for $|t| \ge 1$. Denote the restrictions of such functions to the non-negative real axis by the same symbols, and define after that $F(t) := 2^{-t}f(t)$, $t \ge 0$ and $G(t) := 2^{-t}g(t)$, $t \ge 0$. Then $F, G \in UR_{1,1/2}([0,\infty) : \mathbb{C}) \subseteq UR_{1,1/2,2}([0,\infty) : \mathbb{C})$ but $F + G \notin UR_{1,1/2,2}([0,\infty) : \mathbb{C})$. If we suppose the contrary, then we would have the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\ge 0} \Big| 2^{-t}\Big[f(t+\alpha_n) + g(t+\alpha_n)\Big] - 2^{-t}[f(t) + g(t)]\Big| = 0,$$

which is impossible because for each $n \in \mathbb{N}$ such that $\alpha_n \ge 1$ we have that $\sup_{t\ge 0} |2^{-t}[f(t+\alpha_n) + g(t+\alpha_n)] - 2^{-t}[f(t) + g(t)]| \ge |f(0) + g(0) - [f(\alpha_n) + g(\alpha_n)]| = |2 - [f(\alpha_n) + g(\alpha_n)]| \ge 1$. *In particular, this example can be used to show that the set $UR_{\omega,c,2}([0,\infty) : \mathbb{C})$ does not form a vector space together with the usual operations.*

(v) *Using Proposition 3.12, as well as the arguments contained in the proofs of Proposition 2.12 and [5, Theorem 8°, pp. 3-4], it follows that $AP_{\omega,c,2}([0,\infty) : E)$ is a complete metric space equipped with the distance $d(\cdot,\cdot) := \|\cdot - \cdot\|_{\omega,c}$.*

Keeping in mind the proved results, we will consider the following notion for Stepanov classes, only:

Definition 3.14. *Let $p \in [1,\infty)$, $c \in \mathbb{C}\setminus\{0\}$, $|c| \le 1$ and $\omega > 0$. Then it is said that a function $f \in L^p_{loc}([0,\infty) : E)$ is Stepanov (p, ω, c)-uniformly recurrent of type 2, resp. Stepanov (p, ω, c)-almost periodic of type 2 if and only if*

$$\lim_{n\to+\infty} \sup_{t\ge 0} \int_t^{t+1} \Big\| c^{\frac{-\alpha_n}{\omega}} f(s+\alpha_n) - f(s) \Big\|^p ds = 0,$$

resp. for each $\epsilon > 0$ the set

$$\Big\{ \tau > 0 : \sup_{t\ge 0} \int_t^{t+1} \Big\| c^{\frac{-\tau}{\omega}} f(s+\tau) - f(s) \Big\|^p ds < \epsilon \Big\}$$

is relatively dense in $[0, \infty)$.

By $S^pUR_{\omega,c,2}([0,\infty) : E)$ and $S^pAP_{\omega,c,2}([0,\infty) : E)$ we denote the space of all Stepanov (p, ω, c)-uniformly recurrent functions of type 2 and the space of all Stepanov (p, ω, c)-almost periodic functions of type 2, respectively.

If $1 \le p \le q < \infty$ and $f \in S^qUR_{\omega,c,2}([0,\infty) : E)$, resp. $f \in S^qAP_{\omega,c,2}([0,\infty) : E)$, then $f \in S^pUR_{\omega,c,2}([0,\infty) : E)$, resp. $f \in S^pAP_{\omega,c,2}([0,\infty) : E)$; furthermore, the space $S^pUR_{\omega,c,2}([0,\infty) : E)$, resp. $S^pAP_{\omega,c,2}([0,\infty) : E)$, contains the space $UR_{\omega,c,2}([0,\infty) : E)$, resp. $AP_{\omega,c,2}([0,\infty) : E)$. It is simply verified that the space $S^pUR_{\omega,c,2}([0,\infty) : E)$, resp. $S^pAP_{\omega,c,2}([0,\infty):E)$, consists of those locally p-integrable functions $f:I\rightarrow E$ for which $\hat{f}(\cdot)$ belongs to the space $UR_{\omega,c,2}([0,\infty):L^p([0,1]:E))$, resp. $AP_{\omega,c,2}([0,\infty):L^p([0,1]:E))$. Keeping in mind this observation, it is straightforward to transfer the previously proved results and the points [4'.-7'.] for the introduced Stepanov classes; details can be omitted. Note, finally, that $S^pAP_{\omega,c,2}([0,\infty):E)$ is a complete metric spaces equipped with the distance $d(\cdot,\cdot):=\|\cdot-\cdot\|_{p,\omega,c}$.

3.1. *Composition principles for (ω,c)-almost periodic type functions*

In [17], the second named author has recently investigated composition principles for uniformly recurrent functions and $\odot_g$-almost periodic functions. The methods established in this paper enable one to formulate a great number of composition principles for the classes introduced in Definition 2.1, Definition 2.6 and Definition 2.9-Definition 2.10. We will explain this fact only in the case of consideration of [17, Theorem 2.9] for Stepanov uniformly recurrent functions. So, let us assume that the function $F:I\times X\rightarrow E$ is continuous and the function $f_{\omega,c}(\cdot)$ is Stepanov p-uniformly recurrent, i.e., the function $f(\cdot)$ is Stepanov (p,ω,c)-almost periodic ($p>1$, $\omega>0$, $c\in\mathbb{C}\setminus\{0\}$). Define the function $G:I\times X\rightarrow E$ by

$$G(t,x):=c_1^{-\frac{t}{\omega_1}}F\Big(t,c^{t/\omega}x\Big),\quad t\in I,\ x\in X,$$

where $c_1\in\mathbb{C}\setminus\{0\}$ and $\omega_1>0$. If the requirements of the above-mentioned theorem hold with the functions $f(\cdot)$ and $F(\cdot,\cdot)$ replaced respectively with the functions $f_{\omega,c}(\cdot)$ and $G(\cdot,\cdot)$, then the resulting function

$$t\mapsto G\Big(t,f_{\omega,c}(t)\Big)=c_1^{-t_1/\omega_1}F(t,f(t)),\quad t\in I$$

is Stepanov q-uniformly recurrent so that the function $t\mapsto F(t,f(t))$, $t\in I$ is Stepanov (q,ω_1,c_1)-uniformly recurrent. More precisely, we have:

Theorem 3.15. *Let $I=\mathbb{R}$ or $I=[0,\infty)$. Suppose that the following conditions hold:*

(i) The function $G:I\times X\rightarrow E$ is Stepanov p-uniformly recurrent, with $p>1$, and there exist a number $r\geq\max(p,p/p-1)$ and a function $L_G\in L_S^r(I)$ such that:

$$\|G(t,x)-G(t,y)\|\leq L_G(t)\|x-y\|_X,\quad t\in I,\ x,\ y\in X. \tag{8}$$

(ii) The function $f_{\omega,c}:I\rightarrow X$ is Stepanov p-uniformly recurrent and there exists a set $\mathrm{E}\subseteq I$ with $m(\mathrm{E})=0$ such that $K:=\{f_{\omega,c}(t):t\in I\setminus\mathrm{E}\}$ is relatively compact in X.

(iii) For every compact set $K\subseteq X$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\rightarrow+\infty}\sup_{t\in I}\sup_{u\in K}\int_0^1\Big\|G(t+s+\alpha_n,u)-G(t+s,u)\Big\|^p\,ds=0 \tag{9}$$

and (2) *holds with the function $f_{\omega,c}(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f_{\omega,c}}(\cdot)$ and the norm $\|\cdot\|_{L^p([0,1]:X)}$ therein.*

Then $q:=pr/p+r\in[1,p)$ and $F(\cdot,f(\cdot))$ is Stepanov (q,ω_1,c_1)-uniformly recurrent.

In the remainder of this subsection, we will state and prove some composition principles for (ω,c)-uniformly recurrent functions of type 2; see also Corollary 3.8 and Proposition 3.11 (we can simply re-formulate these results for (ω,c)-almost periodic functions of type 2). Hence, in the continuation of this subsection, we will assume that $|c|\leq 1$, $I=[0,\infty)$ and $i=2$.

Suppose that $F:I\times X\rightarrow E$ is a continuous function and there exists a finite constant $L>0$ such that

$$\|F(t,x)-F(t,y)\|\leq L\|x-y\|_X,\quad t\in I,\ x,\ y\in X. \tag{10}$$

Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. We will use the following estimate ($\tau \geq 0$, $\omega > 0$, $c \in \mathbb{C} \setminus \{0\}$, $t \in I$):

$$\begin{aligned}
&\Big\| c^{(-\tau)/\omega} F(t+\tau, f(t+\tau)) - F(t, f(t)) \Big\| \\
&\leq \Big\| c^{(-\tau)/\omega} F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega} f(t+\tau)\Big) \Big\| \\
&+ \Big\| F\Big(t, c^{(-\tau)/\omega} f(t+\tau)\Big) - F(t, f(t)) \Big\| \\
&\leq \Big\| c^{(-\tau)/\omega} F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega} f(t+\tau)\Big) \Big\| + L \Big\| c^{(-\tau)/\omega} f(t+\tau) - f(t) \Big\|. \qquad (11)
\end{aligned}$$

Remark 3.16. *Albeit we will not employ this estimate henceforth, it should be noticed that we also have*

$$\begin{aligned}
&\Big\| F(t+\tau, f(t+\tau)) - c^{\tau/\omega} F(t, f(t)) \Big\| \\
&\leq \Big\| F(t+\tau, f(t+\tau)) - F\Big(t+\tau, c^{\tau/\omega} f(t)\Big) \Big\| + \Big\| F\Big(t+\tau, c^{\tau/\omega} f(t)\Big) - c^{\tau/\omega} F(t, f(t)) \Big\| \\
&\leq L \Big\| f(t+\tau) - c^{\tau/\omega} f(t) \Big\| + \Big\| F\Big(t+\tau, c^{\tau/\omega} f(t)\Big) - c^{\tau/\omega} F(t, f(t)) \Big\|.
\end{aligned}$$

Using the proof of [15, Theorem 3.29] and (11), we can simply deduce the following result:

Theorem 3.17. *Suppose that $F : I \times X \to E$ is a continuous function and there exists a finite constant $L > 0$ such that* (10) *holds.*

(i) Suppose that $f : I \to X$ is (ω, c)-uniformly recurrent of type 2*. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \Big\| = 0$$

and

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| c^{(-\alpha_n)/\omega} F\Big(t+\alpha_n, f(t+\alpha_n)\Big) - F\Big(t, c^{(-\alpha_n)/\omega} f(t+\alpha_n)\Big) \Big\| = 0,$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is (ω, c)-uniformly recurrent of type 2*.*

(ii) Suppose that $f : I \to X$ is (ω, c)-almost periodic of type 2*. If for each $\epsilon > 0$ the set of all positive real numbers $\tau > 0$ such that*

$$\sup_{t\in I} \Big\| c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t) \Big\| < \epsilon,$$

and

$$\sup_{t\in I} \Big\| c^{(-\tau)/\omega} F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega} f(t+\tau)\Big) \Big\| < \epsilon$$

is relatively dense in $[0, \infty)$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is (ω, c)-almost periodic of type 2*.*

We can similarly reformulate the statements of [15, Theorem 3.30, Theorem 3.31] in our context (cf. also [2, Theorem 2.11] and [9, Theorem 2.11]).

Now we will provide two results for Stepanov classes of (ω, c)-uniformly recurrent functions of type 2, thus continuing the analysis raised in [17, Theorem 2.9, Theorem 2.10, Theorem 2.11]. We will first state an analogue of the last mentioned theorem:

Theorem 3.18. *Let $I = [0, \infty)$, $|c| \leq 1$, $\omega > 0$, $p,\ q \in [1, \infty)$, $r \in [1, \infty]$, $1/p = 1/q + 1/r$ and the following conditions hold:*

(i) The function $F : I \times X \to E$ is Stepanov p-uniformly recurrent and there exists a function $L_F \in L^r_S(I)$ such that (8) *holds with the function $G(\cdot, \cdot)$ replaced with the function $F(\cdot, \cdot)$ therein.*

(ii) There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \Bigl(\sup_{u\in R(f)} \Bigl\| c^{-\alpha_n/\omega} F\bigl(s+\alpha_n, u\bigr) - F\bigl(s, c^{\alpha_n/\omega} u\bigr) \Bigr\| \Bigr)^p \, ds = 0 \tag{12}$$

and

$$\lim_{n\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \Bigl\| c^{\frac{-\alpha_n}{\omega}} f(s+\alpha_n) - f(s) \Bigr\|^q \, ds = 0.$$

Then the function $F(\cdot, f(\cdot))$ is Stepanov (p, ω, c)-uniformly recurrent of type 2. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov p-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov p-bounded, as well.

Proof. We will only provide the main details of proof since it is very similar to the proof of [22, Theorem 2.2]. Using the arguments contained for proving the estimate (11), we get that ($t \geq 0$, $n \in \mathbb{N}$):

$$\begin{aligned} &\Bigl\| c^{-\alpha_n/\omega} F\bigl(t+\alpha_n, f(t+\alpha_n)\bigr) - F(t, f(t)) \Bigr\| \\ &\leq \Bigl\| c^{(-\alpha_n)/\omega} F(t+\alpha_n, f(t+\alpha_n)) - F\bigl(t, c^{(-\alpha_n)/\omega} f(t+\alpha_n)\bigr) \Bigr\| \\ &+ L_F(t) \Bigl\| c^{(-\alpha_n)/\omega} f(t+\alpha_n) - f(t) \Bigr\|. \end{aligned} \tag{13}$$

Keeping in mind (13), we can repeat almost verbatim the arguments given in the proof of [22, Theorem 2.2] so as to conclude that there exists a finite constant $c_p > 0$ such that ($n \in \mathbb{N}$):

$$\begin{aligned} &\sup_{t\geq 0} \int_t^{t+1} \Bigl\| c^{-\alpha_n/\omega} F\bigl(s+\alpha_n, f(s+\alpha_n)\bigr) - F(s, f(s)) \Bigr\|^p \, ds \\ &\quad \leq \Bigl\| L_F(\cdot) \Bigr\|_{S^r}^p \cdot \sup_{t\geq 0} \Bigl(\int_t^{t+1} \Bigl\| c^{-\alpha_n/\omega} f\bigl(s+\alpha_n\bigr) - f(s) \Bigr\|^q \, ds \Bigr)^{p/q} \\ &\quad + \sup_{t\geq 0} \int_t^{t+1} \Bigl(\sup_{u\in R(f)} \Bigl\| c^{-\alpha_n/\omega} F\bigl(s+\alpha_n, u\bigr) - F(s, u) \Bigr\| \Bigr)^p \, ds. \end{aligned}$$

By (12), this yields that the function $F(\cdot, f(\cdot))$ is Stepanov (p, ω, c)-uniformly recurrent of type 2. If the function $F(\cdot, 0)$ is Stepanov p-bounded, then the arguments given on [22, p. 6, l. -1-l. -5] enable one to see that the function $F(\cdot, f(\cdot))$ is Stepanov p-bounded, as claimed. □

We can simply formulate a consequence of this result with the usual Lipshitzian condition used (see [17, Theorem 2.10]). Similarly, we can prove the following analogue of [17, Theorem 2.9]:

Theorem 3.19. *Let $I = [0, \infty)$, $|c| \leq 1$, $\omega > 0$ and the following conditions hold:*

(i) The function $F : I \times X \to E$ is Stepanov p-uniformly recurrent with $p > 1$, and there exist a number $r \geq \max(p, p/p-1)$ and a function $L_F \in L^r_S(I)$ such that (8) holds with the function $G(\cdot, \cdot)$ replaced with the function $F(\cdot, \cdot)$ therein.

(ii) There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \Bigl(\sup_{u\in R(f)} \Bigl\| c^{-\alpha_n/\omega} F\bigl(s+\alpha_n, u\bigr) - F(s, c^{\alpha_n/\omega} u) \Bigr\| \Bigr)^p \, ds = 0$$

and

$$\lim_{n\to+\infty} \sup_{t\geq 0} \int_t^{t+1} \Bigl\| c^{\frac{-\alpha_n}{\omega}} f(s+\alpha_n) - f(s) \Bigr\|^p \, ds = 0.$$

Then $q := pr/p + r \in [1, p)$ and the function $F(\cdot, f(\cdot))$ is Stepanov (q, ω, c)-uniformly recurrent of type 2. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov q-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov q-bounded, as well.

Remark 3.20. *Concerning Theorem 3.18 and Theorem 3.19, it should be noticed that we do not require that there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that the set $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact. For Stepanov (p, ω, c)-uniformly recurrent functions of type 2, we cannot assume, in (12), a slightly weaker condition*

$$\lim_{n\to+\infty} \sup_{t\geq 0} \sup_{u\in R(f)} \int_t^{t+1} \Big\| c^{-\alpha_n/\omega} F\big(s+\alpha_n, u\big) - F(s, c^{\alpha_n/\omega}u)\Big\|^p \, ds = 0.$$

See also [22, Lemma 2.1].

4. (ω, c)-Almost periodic properties of convolution products and applications to integro-differential equations

In the first part of this section, we will examine the invariance of (ω, c)-almost periodic properties of the infinite convolution product

$$F(t) := \int_{-\infty}^{t} R(t-s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{14}$$

where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(E, X)$ satisfies certain assumptions. As already mentioned, the consideration is simple for the (ω, c)-uniformly recurrent functions, (ω, c)-almost periodic functions and (compactly) (ω, c)-almost automorphic functions because we then need to examine when the function $t \mapsto c^{-(t/\omega)}F(t)$, $t \in \mathbb{R}$ is uniformly recurrent, almost periodic or (compactly) almost automorphic, respectively. But, we have

$$c^{-\frac{t}{\omega}} F(t) = \int_{-\infty}^{t} \Big[c^{-\frac{t-s}{\omega}} R(t-s)\Big]\Big[c^{-\frac{s}{\omega}} f(s)\Big]\, ds, \quad t \in \mathbb{R},$$

so that the statements of [16, Proposition 3.1, 3.2] (uniform recurrence), [15, Proposition 2.6.11] (almost periodicity) and [15, Proposition 3.5.3] (almost automorphicity), for instance, can be simply reformulated in our context by replacing respectively the operator family $(R(t))_{t>0}$ and the function $f(\cdot)$ by the operator family $(c^{-\frac{t}{\omega}} R(t))_{t>0}$ and the function $c^{-\frac{\cdot}{\omega}} f(\cdot)$. We will do this only in the case of the last mentioned result:

Proposition 4.1. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(E, X)$ is a strongly continuous operator family satisfying that*

$$M := \sum_{k=0}^{\infty} \Big\| c^{-\frac{\cdot}{\omega}} R(\cdot)\Big\|_{L^q[k,k+1]} < \infty.$$

If $c^{-\frac{\cdot}{\omega}} f : \mathbb{R} \to X$ is S^p-almost automorphic, then the function $F : \mathbb{R} \to X$, given by (14), is well defined and (ω, c)-almost automorphic.

It is worth noting that this result can be applied in both cases $|c| > 1$ and $|c| < 1$, under suitable conditions. It is straightforward to incorporate the above results in the study of the existence and uniqueness of (ω, c)-almost periodic type solutions for the various classes of abstract inhomogeneous integro-differential equations. Keeping in mind Theorem 3.2, we will skip all related details concerning the invariance of (ω, c)-uniform recurrence of type 1 (type 2) (ω, c)-almost periodicity of type 1 (type 2) under the actions of infinite convolution products.

Consider now the finite convolution product

$$H(t) := \int_0^t R(t-s) f(s)\, ds, \quad t \geq 0. \tag{15}$$

Due to the fact that

$$c^{-\frac{t}{\omega}}\int_0^t R(t-s)f(s)\,ds=\int_0^t\Big[c^{-\frac{t-s}{\omega}}R(t-s)\Big]\Big[c^{-\frac{s}{\omega}}f(s)\Big]\,ds,\quad t\geq 0, \tag{16}$$

we can similarly analyze the invariance of asymptotical (ω, c)-uniform recurrence, asymptotical (ω, c)-almost periodicity and asymptotical (compact) (ω, c)-almost automorphicity under the actions of finite convolution products; the interested reader may try to reformulate the statements of [15, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem 2.9.15] in our new context.

If $|c| < 1$ and $\omega > 0$, then it is worth noting that the (ω, c)-uniform recurrence of type 2 and the (ω, c)-almost periodicity of type 2 cannot be so simply retained after the actions of finite convolution products. The situation is much simpler for the classes $AP_{\omega,c,1}([0,\infty) : E)$ and $UR_{\omega,c,1}([0,\infty) : E)$ ($S^pAP_{\omega,c,1}([0,\infty) : E)$ and $S^pUR_{\omega,c,1}([0,\infty) : E)$), where $p \geq 1$, because in this case we can apply Corollary 3.8, Proposition 3.11 and (16).

In the remainder of this section, we will provide a few applications to the abstract integro-differential equations and inclusions in Banach spaces.

1. We start by observing that the examples and results presented by Zaidman [26, Examples 4, 5, 7, 8; pp. 32-34], which have been employed by numerous authors so far, for various purposes, can be used to provide certain applications of our results. For example, in the case of consideration of [26, Example 4], we know that the unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, accompanied with the initial condition $u(x, 0) = f(x)$, is given by

$$u(x,t):=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(x-s)^2}{4t}}f(s)\,ds,\quad x\in\mathbb{R},\ t\geq 0.$$

Let the number $t_0 > 0$ be fixed, let $c \in \mathbb{C}\setminus\{0\}$, $\omega > 0$ and let the function $c^{-\cdot/\omega}f(\cdot)$ be bounded uniformly recurrent (almost periodic, (compactly) almost automorphic). If $c^{-\frac{\cdot}{\omega}}e^{-\cdot^2/4t_0} \in L^1(\mathbb{R})$, then we can apply Proposition 2.8 in order to see that the solution $x \mapsto u(x, t_0)$, $x \in \mathbb{R}$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(compactly) (ω, c)-almost automorphic). See also [2, Example 2.9].

2. The results about the invariance of various kinds of (ω, c)-almost type periodicity, introduced in Section 2, under the actions of infinite convolution products can be applied in the qualitative analysis of solutions to the following fractional Cauchy inclusion

$$D_{t,+}^{\gamma}u(t)\in\mathcal{A}u(t)+f(t),\ t\in\mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0, 1]$, $f : \mathbb{R} \to E$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator. Furthermore, the results about the invariance of various kinds of asymptotical (ω, c)-almost type periodicity, introduced in Section 2, under the actions of finite convolution products can be applied in the qualitative analysis of solutions to the following fractional Cauchy inclusion

$$(\mathrm{DFP})_{f,\gamma}:\begin{cases}\mathbf{D}_t^{\gamma}u(t)\in\mathcal{A}u(t)+f(t),\ t\geq 0,\\ u(0)=x_0,\end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order $\gamma \in (0, 1]$, $x_0 \in E$, $f : [0,\infty) \to E$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator (see [15] for more details). Applying Theorem 3.15 and the methods established in [15], we can analyze the existence and uniqueness of (ω, c)-uniformly recurrent solutions for various classes of abstract semilinear abstract Volterra integro-differential equations and inclusions (the existence and uniqueness of asymptotically (ω, c)-uniformly recurrent solutions can be analyzed similarly; details can be left to the interested readers). It is also clear that Theorem 3.2 can be applied in the analysis of a wide class of the abstract Volterra integro-differential equations with periodic solutions; see also Proposition 3.12.

3. It is worth noting that the notion from Definition 3.1 and Definition 3.14 can be introduced with the intervals $I = [-a, \infty)$, where $a > 0$ is an arbitrary real number. To explain the importance of this observation, we will reexamine [26, Example 5]. It is well known that the unique regular solution of the wave equation $u_{xx}(x,t) = u_{tt}(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, accompanied with the initial conditions $u(x, 0) = f(x)$, $x \in \mathbb{R}$, $u_t(x, 0) = g(x)$, $x \in \mathbb{R}$, is given by the famous d'Alembert formula

$$u(x,t) := \frac{1}{2}\Big[f(x+t) + f(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds, \quad x \in \mathbb{R},\ t \geq 0.$$

Here we would like to note the following fact about the term

$$H_{t_0}(x) := \frac{1}{2}\int_{x-t_0}^{x+t_0} g(s)\, ds, \quad x \in \mathbb{R},$$

where $t_0 > 0$ is a fixed real number. Suppose that the function $g : [-t_0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2, for example (the same comment applies to all other classes of functions introduced in Definition 3.1). Then there exists a strictly increasing sequence (α_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{t\geq -t_0} \Big| c^{-\alpha_n/\omega} g(t+\alpha_n) - g(t) \Big| = 0.$$

If $\epsilon > 0$ is given, this implies the existence of an integer $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\Big| c^{-\alpha_n/\omega} H_{t_0}(x+\alpha_n) - H_{t_0}(x) \Big| \leq \int_{-t_0}^{t_0} \Big| c^{-\alpha_n/\omega} g(x+s+\alpha_n) - g(x+s) \Big| \, ds \leq 2t_0\epsilon, \quad x \geq 0.$$

Hence, the function $H_{t_0} : [0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2.

It would be very enticing to provide certain applications of composition principles established in Subsection 3.1 in the qualitative analysis of solutions to the abstract semilinear Cauchy inclusions which belongs to the classes $AP_{\omega,c,2}([0,\infty)$ and $UR_{\omega,c,2}([0,\infty)$.

5. Conclusions and final remarks

Following the recent researches [2] by Alvarez, Gómez and Pinto and [3] by Alvarez, Castillo and Pinto, in this paper we have introduced and systematically analyzed several new classes of (ω, c)-almost periodic type functions and their Stepanov extensions. The corresponding classes of two-parameter (ω, c)-almost periodic type functions and related composition principles have been also analyzed; some applications to the abstract Volterra integro-differential equations in Banach spaces have been also given. The case $|c| \neq 1$ is still unexplored in the theory of almost periodic functions and we feel it is our duty to say that the classes of (ω, c)-almost periodic type functions with $|c| \neq 1$ have some very unusual and unpleasant features.

In the remainder of paper, we would like to note a few useful comments and observations about problematic considered so far.

1. Concerning the notion introduced in Section 2, we would like to note that we can similarly analyze the classes consisting of those functions $f(\cdot)$ for which the function $f_{\omega,c}(\cdot)$ is (equi)-Weyl-p-almost periodic, Besicovitch p-almost periodic or Besicovitch-Doss p-almost periodic (see [15] for the notion).

2. A similar comment can be applied to the notion considered in Section 3. For example, we can introduce and analyze the following notions of (equi)-Weyl-(p, ω, c)-almost periodicity of type 1 (2) and (equi)-Weyl-(p, ω, c)-uniform recurrence of type 1 (2):

Definition 5.1. *Let $1 \leq p < \infty$, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $f \in L^p_{loc}(I : E)$.*

(i) We say that the function $f(\cdot)$ is equi-Weyl-(p,ω,c)-almost periodic of type 1*, resp.* 2*, if and only if for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that*

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-c^{\tau/\omega}f(t)\right\|^p dt\right]^{1/p} \leq \epsilon,$$

resp.

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|c^{-\tau/\omega}f(t+\tau)-f(t)\right\|^p dt\right]^{1/p} \leq \epsilon.$$

(ii) We say that the function $f(\cdot)$ is Weyl-(p,ω,c)-almost periodic of type 1*, resp.* 2*, if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that*

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-c^{\tau/\omega}f(t)\right\|^p dt\right]^{1/p} \leq \epsilon,$$

resp.

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|c^{-\tau/\omega}f(t+\tau)-f(t)\right\|^p dt\right]^{1/p} \leq \epsilon,$$

Definition 5.2. *Let $1 \leq p < \infty$, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $f \in L^p_{loc}(I : E)$.*

(i) We say that the function $f(\cdot)$ is equi-Weyl-(p,ω,c)-uniformly recurrent of type 1*, resp.* 2*, if and only we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and*

$$\lim_{n\to+\infty}\sup_{x\in I}\left[\frac{1}{l_n}\int_x^{x+l_n}\left\|f(t+\alpha_n)-c^{\alpha_n/\omega}f(t)\right\|^p dt\right]^{1/p} = 0,$$

resp.

$$\sup_{x\in I}\left[\frac{1}{l_n}\int_x^{x+l_n}\left\|c^{-\alpha_n/\omega}f(t+\alpha_n)-f(t)\right\|^p dt\right]^{1/p} = 0.$$

(ii) We say that the function $f(\cdot)$ is Weyl-(p,ω,c)-uniformly recurrent of type 1*, resp.* 2*, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and*

$$\lim_{n\to+\infty}\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\alpha_n)-c^{\alpha_n/\omega}f(t)\right\|^p dt\right]^{1/p} = 0,$$

resp.

$$\lim_{n\to+\infty}\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|c^{-\alpha_n/\omega}f(t+\alpha_n)-f(t)\right\|^p dt\right]^{1/p} = 0.$$

The class of Doss-(p,ω,c)-almost periodic functions of types 1 (2) and the class of Doss-(p,ω,c)-uniformly recurrent functions of types 1 (2) can be also introduced following the same idea (cf. [15, Definition 2.13.2(iii)]).

3. The notion introduced in [2]-[3] depends on two parameters, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$. It is worth observing that we can also analyze the notion depending on only one parameter, $c \in \mathbb{C} \setminus \{0\}$. For example, of concern is the following notion:

Definition 5.3. *Let $c \in \mathbb{C} \setminus \{0\}$. Then a continuous function $f : I \to E$ is said to be c-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and*

$$\lim_{n\to+\infty} \left\| f(\cdot + \alpha_n) - cf(\cdot) \right\|_\infty = 0.$$

If $c = -1$, then we also say that the function $f(\cdot)$ is uniformly anti-recurrent.

Further on, let $f : I \to E$ be a continuous function and let a number $\epsilon > 0$ be given. We call a number $\tau > 0$ an $\epsilon - (\omega, c)$-period for $f(\cdot)$ if $\|f(t + \tau) - cf(t)\| \leq \epsilon$ for all $t \in I$. By $\vartheta_c(f, \epsilon)$ we denote the set consisting of all $\epsilon - (\omega, c)$-periods for $f(\cdot)$.

Definition 5.4. *It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta_c(f, \epsilon)$ is relatively dense in $[0, \infty)$.*

For more details about these classes of functions, we refer the reader to our recently published paper [14].

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