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ON THE CATEGORY OF ORDERED SETS

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Abstract. In this paper the category of ordered sets and isotone maps is considered as well as presheaves and sheaves of such sets. Also, some classes of ordered topological spaces are considered as the class of A-ordered, S- and strongly S-ordered topological spaces and the class of O_i-topological spaces for $i = 0,1,2$. (see[1] and [6])

A direct system of ordered sets indexed by a directed set $\bigwedge$ is $(G_\alpha, \rho_{\alpha\beta})_{\alpha\leq\beta}$ where G_α is an ordered (means quasi-ordered) set and for $\alpha\leq\beta$, $\rho_{\alpha\beta}:G_\alpha \longrightarrow G_\beta$ an isotone map. There exists a limit of a direct system of ordered sets. (see[5])

Let X be a topological space. A presheaf $\mathcal{F}$ of ordered sets on X is an ORD-valued presheaf, where ORD is a category of all ordered sets and isotone maps. (see [5])

Fix $x \in X$. The $\mathcal{F}(U)$, as U runs through all open sets such that $U \ni x$, form a directed system with maps $\rho_V^U:\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ whenever $U \supset V \ni x$. The stalk $\mathcal{F}(x)$ of $\mathcal{F}$ at x is an ordered set, as in set´s case this is $\varinjlim \mathcal{F}(U)$. This comes equipped with isotone maps $\mathcal{F}(U) \rightarrow \mathcal{F}(x):s \rightarrow s_x$, whenever an open $U \ni x$.

1. PROPOSITION. (a) Each germ $t \in \mathcal{F}(x)$ arises as $t = s_x$, for some open neighboorhood U of x.
(b) Two germs s_x, $t_x \in \mathcal{F}(x)$ (with $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$) say $s_x \leq t_x \leftrightarrow$ open $W \subset U \cap V$ such that $\rho_V^U(s) \leq \rho_W^V(t)$.

PROOF is just a restatement of 4.2, in [7].

1. EXAMPLE. Let A be any given ordered set. Then the constant presheaf A_X on X is given by:

$$A_X(U) = A \text{ for U open in X,}$$
$$\rho_V^U = id_A : A_X(U) \longrightarrow A_X(V) \text{ for } V \subset U \text{ open in X.}$$

A_X is a presheaf of ordered sets.

2. EXAMPLE. Let C^Y be the presheaf of continuous Y-valued functions on X. If in addition Y has the ordering structure, so has each $C^Y(U)$ with pointwise order of functions. The C^Y is thus the presheaf of ordered sets.

3. EXAMPLE. Let $\mathcal{K}$ be a presheaf of sets given by: (a) $\mathcal{K}(U) = \{f \; f:U \longrightarrow X, \text{ f open and continuous}\}$, (b) $\rho_V^U : \mathcal{K}(U) \longrightarrow \mathcal{K}(V) : f \in \mathcal{K}(U) \longrightarrow f|V \in \mathcal{K}(V)$ for $V \subset U$ open in X. Put $f \leq g$ for $f, g \in \mathcal{K}(U)$ iff $\exists h \in \mathcal{K}(f(U))$ such that is $g = h \circ f$. $\mathcal{K}$ is a presheaf of ordered sets. (see[3])

A morphism of presheaves F and G, this is a corresponding functor's morphism $f : F \longrightarrow G$ given by isotone maps $f(U)$, with $f(U) : F(U) \longrightarrow G(U)$ for each open U of X, such that whenever is $U \supset V$ are open in the diagram:

$$\begin{array}{ccc} F(U) & \xrightarrow{f(U)} & G(U) \\ \rho_V^u \downarrow & & \downarrow \rho_V^{\prime u} \\ F(V) & \xrightarrow{f(V)} & G(V) \end{array}$$

commutes i.e., $\rho_V^{\prime U} \circ f(U) = f(V) \circ \rho_V^U$.

Given a morphism of presheaves of ordered sets $f : F \longrightarrow G$ on X for each point $x \in X$ we can produce an isotone morphism of stalks $f_x : F(x) \longrightarrow G(x)$ in such a way that whenever $F \xrightarrow{f} G \xrightarrow{g} H$ we have $(g \circ f)_x = g_x \circ f_x$.

This fact is a special case of a generality concerning "maps of direct system" (see[7] for example)

Let X be a topological space and F a presheaf of ordered sets over X, then F is a sheaf of ordered sets if it is a sheaf of sets satisfying additional condition:

MO) Suppose that U is an open set of X and U = $= \bigcup U_i$ is an open covering of U and $s, s' \subset F(U)$ are two section of F such

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that:

$$\forall i, i \in I \quad \rho^U_{U_i}(s) \leqslant \rho^U_{U_i}(s') \text{ then } s \leqslant s'.$$

All examples 1.,2.,3., of presheaves of ordered sets are sheaves of ordered sets.

For examples 1., and 2., it is obvious. Let us prove the condition MO) for example 3. Really, let $U = \cup U_i$ is an open covering of U and $s,s' \in \mathcal{K}(U)$ are two sections of $\mathcal{K}$ such that $\forall i, \rho^U_{U_i}(s) \leqslant = \rho^U_{U_i}(s')$ it means there exists $p_i \in \mathcal{K}(\rho^U_{U_i}(s)(U))$ such that is:

$$p_i \rho^U_{U_i}(s) = \rho^U_{U_i}(s').$$

The $\cup \rho^U_{U_i}(s)(U_i)$ is an open covering of $s(U)$, so p_i and p_j agree on $\rho^U_{U_i}(s)(U_i) \cap \rho^U_{U_i}(s)(U_j)$, then there exists $p:s(U) \rightarrow \rightarrow s'(U)$ with $p|\rho^U_{U_i}(s)(U)^i = p_i$ such that is $p \circ s = s'$; it means $s \leqslant s'$.

2. PROPOSITION. If X is a topological space and F a sheaf of ordered sets on X, then for any open U and $s,s' \in F(U)$ we have $s \leqslant s' \longleftrightarrow \forall x, x \in U, s_x \leqslant s'_x$.

PROOF. $\longrightarrow$ is clear. Conversely, given $s,s' \in F(U)$ such that $\forall x \in U, s_x \leqslant s'_x$ then there exists $U_x \ni x$ such that is:

$$\rho^U_{U_x}(s) \leqslant \rho^U_{U_x}(s^i).$$

Applying MO) to $\{U_x\}_{x \ U}$ we have $s \leqslant s'$.

1. DEFINITION. Let X be a topological space. A sheaf space of ordered sets over X is a coresponding sheaf space of sets (E,p) such that holds:

(a) For any $x \in X$, $p^{-1}(x)$ is an ordered set.

(b) For every $U \subset X$ open in X, $\Gamma(U,E)$ is an ordered set by pointwise ordering.

3. PROPOSITION. If (E,p) is a sheaf space of ordered sets, then the stalk of E at $x \ X$ is (up to ordering isomorphism) just the fibre $p^{-1}(x)$ of p over x.

PROOF. We let, for U open in X, $\Gamma(U,E) = = \{\sigma : \sigma :U \rightarrow E, p \circ \sigma = id_U, \sigma \text{ is a continuous map}\}.$

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The presheaf $\Gamma E: U \longrightarrow \Gamma(U,E)$ is the presheaf of ordered sets by (b) in definition 1., where all maps $\rho_V^U: \Gamma(U,E) \longrightarrow$ $\longrightarrow \Gamma(V,E)$ are isotone maps. The presheaf ΓE is a sheaf of ordered sets; really, it holds the condition MO): Let $U = \cup U_i$ be an open covering of U and $s, s' \in \Gamma(U,E)$ such that is $\rho_{U_i}^U(s) \leq \rho_{U_i}^U(s')$. Put $x \in U$, then $\exists i \in I$ such that $x \in U_i$. It will be:

$$\rho_{U_i}^U(s)(x) \leq \rho_{U_i}^U(s')(x) \text{ e.g., } s|U_i(x) \leq s^i|U_i(x),$$

then $s(x) \leq s'(x), \forall x \in U$, it means $s \leq s'$.

We can prove $p^{-1}(x)$ is isomorphic with $\varinjlim \Gamma(U,E)$ as in the set case, All maps $\Gamma(U,E) \longrightarrow p^{-1}(x): \sigma \longrightarrow \sigma(x)$ are isotone maps so $\varinjlim \Gamma(U,E)$ is a limit of direct system of ordered sets, unique up to order isomorphism.

For each presheaf F of ordered sets on X we can construct a sheaf space of ordered sets $\lfloor F$ and also as in a set`s case we can prove the next theorem.

1. THEOREM. If E is a sheaf space of ordered sets over X, then $\lfloor \Gamma E$ is isomorphic to E as sheaf space over X .

Similarly, we can make a procedure of sheafifi-cation of a presheaf of ordered sets in the same way as in the case of sets or abelian groups.

If beside a category ORD of ordered sets we con-sider a category of ordered groups or category of ordered topological spaces we can ask ourself is the presiding consideration holds.

For ordered groups we can see [2]. The situation in the case of ordered topological spaces is more complicated. The con-struction of limit of direct system carries not much of properties of topological ordered sets. In fact the next theorems holds.

2. THEOREM. Every direct system of A-ordered topological spaces $(U_\alpha, \rho_{\alpha\beta})_{\alpha \leq \beta}$ has a direct limit $(U, \mathcal{T}_\alpha)_{\alpha \in \Lambda}$ where U is A-ordered topological space and $\mathcal{T}_\alpha: U_\alpha \longrightarrow U$ an isotone, antitone and open map for every $\alpha \in \Lambda$.

3. THEOREM. Every direct system of S-ordered and strongly S-ordered topological space $(U_\alpha, \mathcal{S}_{\alpha\beta})_{\alpha \leq \beta}$ has a direct limit $(U, \mathcal{T}_\lambda)_{\lambda \in \Lambda}$, where U is S-ordered or strongly S-ordered topological space and $\mathcal{T}_\lambda : U_\lambda \to U$ is an isotone, antitone and open maps for every $\alpha \in \Lambda$.

4. THEOREM. Every direct system of A-ordered O_0-topological spaces $(U_\alpha, \mathcal{S}_{\alpha\beta})_{\alpha \leq \beta}$ has a direct limit $(U, \mathcal{T}_\lambda)$, where U is A-ordered O_i-topological space (i =1,2,0) and $\mathcal{T}_\lambda : U_\lambda \to U$ an isotone antitone and open map for every $\alpha \in \Lambda$.

We can prove for example Theorem 2., other proofs are similar.

PROOF of Theorem 2. Let U be a limit of corresponding direct system of sets and maps. On U we can define an ordering structure with $[u_\alpha] \leq [v_\beta]$ iff $\exists \gamma \in \Lambda, \alpha \leq \gamma . \beta \leq \gamma$ with $\mathcal{S}_{\alpha\gamma}(u_\alpha) \leq \mathcal{S}_{\beta\gamma}(v_\beta)$. (see[4] for example) Also, we can define a topological structure as in [4]. So, we must prove compatibility of this structures, it means that U is an A-ordered topological space.

Really, let us consider a point $u \in U$, its neighbourhood $M \subset U$ and a point $v \in U$ such that $v \notin M$ with $u \leq v$. Choose elements $u_\alpha \in U_\alpha$ and $v_\beta \in U_\beta$ such that is $u=[u_\alpha]$, $v=[v_\beta]$ .For $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. Elements $u_\gamma = \mathcal{S}_{\alpha\gamma}(v_\alpha)$ aredifferent in U_γ. Let M^* be a qusi-neighbourhood of u_γ with $p(M^*)=M$. The set $M^* = \bigcup_{\mu \leq \lambda} \mathcal{S}_{\mu\lambda}(M_\mu)$. We can supose $\mu = \gamma$, and we have that is $M^* = \bigcup_{\gamma \leq \lambda} \mathcal{S}_{\gamma\lambda}(M_\gamma)$, where M_γ is $\mathcal{T}_\gamma$-neighbourhood of u_γ, and $v_\gamma \notin M_\gamma$, because $v \notin M$. Topological and ordering structure are compatible on U_γ, A-compatible, and there exists a neighbourhood N_γ of a point u_γ orderably separated from $\{v_\gamma\}$, so that is $u_\gamma \bar{\mathcal{S}} v_\gamma$ implies $N_\gamma \bar{\mathcal{S}} v_\gamma$. From our propose $u \bar{\mathcal{S}} v$ on U we can take $u_\gamma \bar{\mathcal{S}} v_\gamma$ on U_γ. Let N^* be aquasi-neighbourhood of $u_\gamma \in U_\gamma$ in form $N^* = \bigcup_{\gamma \leq \lambda} \mathcal{S}_{\gamma\lambda}(N_\gamma)$ and put $N=p(N^*)$. From $N_\gamma \subset M_\gamma$ will be $N \subset M$, and from $u_\gamma \in N_\gamma$ willbe $u \in N \subset M$. Let us prove that the set N is an orderably separated from $\{v\} \subset U$. Really, if for every element $x \in N$ holds $x \| v$, N is orderably separated from $\{v\}$. If we propose that there exists $x \in N$ comparable with the element $v \in N$, it can be, in this case, $x \rho v$ or $v \rho x$. Since, $N=p(N^*)$ willbe $x=p(x_\gamma)$ for $x \in N^* = \bigcup_{\gamma \leq \lambda} \mathcal{S}_{\gamma\lambda}(N_\gamma)$ and $x_\gamma \in \mathcal{S}_{\gamma,\nu}(N_\gamma)$ for some index $\nu \in \Lambda$ with $\gamma \leq \nu$. So, $x_\gamma = \mathcal{S}_{\gamma,\nu}(x_\gamma)$ for some element x_γ in N_γ. It holds $N_\gamma \bar{\mathcal{S}} v_\gamma$ in U_γ thus can be $x_\gamma \bar{\mathcal{S}} v_\gamma$ or $x_\gamma \| v_\gamma$. If it is $x_\gamma \bar{\mathcal{S}} v_\gamma$

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will be ($\mathcal{G}_{\beta',\gamma}:U_{\beta'} \to U_{\gamma}$ is an isotone map) $x_{\gamma}\mathcal{G}v_{\gamma}$, where $v_{\gamma} = \mathcal{G}_{\beta',\gamma}(v_{\beta'})$ and also, $x \mathcal{G} v$ in $\bar{U}$. If it is $x_{\beta'}||v_{\beta'}$ in $U_{\beta'}$, x can be not comparable with v, beaucose all maps $\mathcal{G}_{\beta'\lambda}$ are antitone for every index $\lambda \in \Lambda$ with $\beta' \le \lambda$. So, it holds $N \bar{\mathcal{G}} v$. Let us suppose that there exists an element $y \in N$ with $v \mathcal{G} y$. From $y \in N = p(N^{*})$ will be $y = p(y_{\eta})$ for some index $\eta \in \Lambda$ with $\beta' \le \eta$. Put $v_{\eta} \mathcal{G} y_{\eta}$, where $v_{\eta} = \mathcal{G}_{\beta'\eta}(v_{\beta'})$. The map is an antitone map thus must be $v_{\beta'} \mathcal{G} y_{\beta'}$, with $y_{\beta'} \in N_{\beta'}$ such that is $y_{\eta} = \mathcal{G}_{\beta'\eta}(y_{\beta'})$. For every element $x_{\beta'}$, $x_{\beta'} \in N_{\beta'}$ holds $x_{\beta'} \bar{\mathcal{G}} v_{\beta'}$ can be not $v_{\beta'} \mathcal{G} y_{\beta'}$, also $v_{\eta} \bar{\mathcal{G}} y_{\eta}$ and so $v \bar{\mathcal{G}} y$. So, we proved it holds $x \mathcal{G} v$. The set N is an orderebly separated from $\{v\}$ and it holds the next inplication $u \bar{\mathcal{G}} v \to N \bar{\mathcal{G}} v$.

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O KATEGORIJI UREĐENIH SKUPOVA

U radu se proučava kategorija uredenih skupova
i izotonih preslikavanja. Proučavaju se takode predpramenovi i prameno-
vi takvih skupova. Posmatraju se i neke klase uredenih topoloških pros-
tora, kao što su A-uredeni, S-uredeni, strogo S-uredeni topološki pros-
tori kao i O_i-uredeni topološki prostori za i 0 0,1,2.

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$$k' = c + \log_4 k'$$

where $c = \log_4(nk) + 2\log_4(B \cdot \ln 4/A)$. Then, since $c \geq 4$ for sufficiently large n, an iteration

$$k'_0 = c, \quad k'_1 = c + \log_4 c, \quad k'_2 = c + \log_4(c + \log_4 c)$$

converges. Since

$$k'_{m+1} - k' = \log_4(k'_m/k') \leq (k'_m - k')/(4 \ln 4)$$

the convergence is very fast. As a rough approximaton, we shall take k'_1. Then, with a future approximation, the solution of the problem 1 is.

(3.1) $k' = (1.66)(x + \log_{10} x) + 0.84$

where $x = \log_{10}(nk) + 2\log_{10}(B/A)$.

Similarly, for the problem 2, we have an approximation for the solution:

(3.2) $k'' = (3.32)(y + \log_{10} y) + 2.20$

where $y = -\log_{10}(e/B)$.

Since (3.1) and (3.2) are very rough approximations, we need numeric solutions for (2.1) and (2.2). First, we shall search for solutions in the form $k' = k'(x)$ and $k'' = k''(y)$ for $0.5 \leq x \leq 9.5$ and $0.5 \leq y \leq 6$ with steps 0.5. An unexpected fact is the linearity of those functions:

(3.3) $k' = (1.82)x + 1.24$
(3.4) $k'' = (3.71)y + 3.05$

with the mean-square errors 0.05 and 0.17 respectively, and the maximum errors in the given ranges 0.48 and 0.38 respectively

In the second try, we shall search for the solutions in the forms $k' = k'(x + \log_{10} x)$ and $k'' = k''(y + \log_{10} y)$, as in (3.1) and (3.2). For that case, the lines of best fit are

$$k' = (1.63)(x + \log_{10} x) + 1.22$$
$$k'' = (3.16)(y + \log_{10} y) + 3.50$$

with the mean-square errors 0.05 and 0.16 respectively, and the maximum errors 0.13 and 0.39 respectively.

As we see, the errors for both cases are of the same order, but (3.3) and (3.4) are more practical.

4. Example

The values of A and B depend on concrete problems, but as an illustration, we can take A = B = 1. Then $x = \log_{10}(nk)$, $y = -\log_{10}e$. With k=32, what is usual in many calculations, from (3.3) and (3.4) it follows

$$k' = (1.82) \log_{10}n + 3.98$$
$$k'' = (3.71) \log_{10}(1/e) + 3.05$$

For $n = 10^2$, 10^3, 10^4 this gives $k = 7.62$, 9.44, 11.26. Also, for $e = 10^{-1}$, 10^{-2} we have $k'' = 6.76$, 10.47. Those values are not too far from 8, so we can set $k'=k''=8$. Since the splitting of a 32-bits number into four 8-bits number is a relative simple operation, such choice seems to be reasonable. Then, $e(k')/e(k) = 1/2$ and $n''/n = 1/4$, so we can make the error about two times less then usual, or the number of generated random numbers about four times less.

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DEOBA SLUČAJNIH BROJEVA U METODI MONTE-KARLO

Razmatran je uticaj dužine (broj cifara) na grešku u metodi Monte-Carlo. Na osnovu osnovne formule (1.1) za ukupnu grešku, razmatrana su dva problema i dati predlozi za njihovo rešavanje: prvi, minimiziranje ukupne greške i drugi, minimiziranje broja slučajnih brojeva pri zadanoj grešci. Dat je i numerički primer da bi se videla efikasnost.

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