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SOME REMARKS ON n-GROUPS

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Abstract. In this note we give some facts about (algebraic and topological) n-groups.

0. The notation and terminology in this note are standard and follow the ones from [1],[5] and [3]. However, we shall give a few known definitions and facts. An n-group $(G,[\,])$ is generated by a binary group $(G,\circ)$ if

$$x_1,x_2,\dots,x_n \in G \Rightarrow [x_1,x_2,\dots,x_n] = x_1\circ x_2\circ\dots\circ x_n .$$

An element $e\in G$ is called an identity (element) of $(G,[\,])$ if for each $x\in G$ we have

$$[x,e,\dots,e] = [e,x,\dots,e] = \dots\dots = [e,e,\dots,x] = x \quad ([2]).$$

It is known that if $(G,[\,])$ is an n-group with an identity e, then it can be generated by a binary group $(G,\circ)$ where

$$x\circ y = [x,y,e,\dots,e] \quad , \; x,y\in G.$$

Let M denote the set $\{m_k\in N:\ m_k = k(n-1)+1,\ k\in N\}$. For any $x\in(G,[\,])$ one defines the <u>n-order</u> of x, denoted $\mathrm{Ord}_n(x)$, as the least integer $m\in M$ such that $[x^m] = x$; if no such integer exists we write $\mathrm{Ord}_n(x) = \infty$. An element of finite n-order is called a torsion element of G. We put

$$T^{(m)} = \{x\in G:\mathrm{Ord}_n(x) = m\} \quad \text{and} \quad T = \{x\in G: x \text{ is a torsion element of } G\}.$$

G is a torsion n-group if $T = G$. When $x\in G$ is regarded as an element of $(G,\circ)$, then $\mathrm{ord}(x)$ denotes the (usual binary) order of x.

An n-group $(G,[\,])$ together with a topology given on it is said to be a topological n-group if $[\,]:G^n\to G$ is a continuous mapping [4].

1.1. THEOREM. Let $(G,[\,])$ be an n-group with an identity e and let $(G,\circ)$ be a binary group generating $(G,[\,])$. An element $x\neq e$ in G is an identity for $(G,[\,])$ if and only if it commutes with all elements in $(G,\circ)$ and its order r (in $(G,\circ)$) satisfies $n = kr+1$ for some $k\in N$.

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PROOF. ($\Rightarrow$) Let $x \neq e$ be an identity in $(G,[\,])$. Then we have

$$[y,x,x,\ldots,x] = [x,y,x,\ldots,x] \quad \text{for every } y \in G.$$

Since $(G,[\,])$ is generated by $(G,\circ)$ one has

$$(\ldots((y\circ x)\circ x\circ\circ\circ x) = (\ldots((x\circ y)\circ x\circ\circ\circ x)$$

i.e. $y\circ x = x\circ y$.

On the other hand,

$$y = [y,x,\ldots,x] = (\ldots((y\circ x)\circ x)\circ\circ\circ x) = y\circ x^{n-1},$$

so that $x^{n-1} = e$. From this it follows $n-1 = kr$, i.e. $n = kr+1$, for some $k \in N$

($\Leftarrow$) Conversely, if $x \neq e$ satisfies the conditions of the theorem, then for every $y \in G$ and every $i = 0,1,\ldots,n-1$ we have

$$[x^i,y,x^{n-i-1}] = x^i\circ y\circ x^{n-i-1} = x^{n-1}\circ y = e\circ y = y,$$

which means that x is an identity for $(G,[\,])$. The theorem is proved.

1.2. COROLLARY. In an n-group $(G,[\,])$ with an identity all elements are identity elements if and only if $(G,[\,])$ is commutative and the least common multiple K of orders of elements of $(G,\circ)$ satisfies $n = kK+1$ for some $k \in N$.

2. Let now $(G,[\,],\mathcal{T})$ be a T_2 topological n-group. Then the following statement is true:

2.1. PROPOSITION. The set $T^{(m)}$ is closed in G for every $m \in M$.

PROOF. Let $x \notin T^{(m)}$, i.e. $[x^m] \neq x$. Since G is a T_2 space there are neighbourhoods U of x and V of $[x^m]$ with $U \cap V = \emptyset$. The continuity of $[\,]$ implies the existence of W, a neighbourhood of x, such that $[W^m] \subset V$. Let us put $H = W \cap U$. Then $H \cap T^{(m)} = \emptyset$ because otherwise there exists $y \in H$ with $[y^m] \in T^{(m)}$, i.e. $y = [y^m] \in [H^m] \subset V$ which is impossible.

2.2. COROLLARY. The sets $A = \{x \in G : [x^m] \neq x\}$ and $B = \{x \in G : \mathrm{Ord}_n(x) > m \in M\}$ are open.

Let us note that $\mathrm{Ord}_n : G \to M \subset R$ is a real-valued function defined on G. In this connection we have

2.3. PROPOSITION. Ord_n is a lower semi-continuous function on G.

PROOF. Let x be a torsion element of G with $\mathrm{Ord}_n(x) = m_k \in M$. The set $U = \{y \in G : \mathrm{Ord}_n(y) > m_{k-1}\}$ is open by Corollary 2.2 and contains x, which means that U is a neighbourhood of x. Since $\mathrm{Ord}_n(y) \geqslant \mathrm{Ord}_n(x)$ for each $y \in U$, one concludes that Ord_n is lower semi-continuous in this case.

Now let $\mathrm{Ord}_n(x) = \infty$. If $p \in M$ is an arbitrary element, then the set

$U = \{y \in G : Ord_n(y) > p\}$ is open (by 2.2), contains x and satisfies $Ord_n(y) > p$ for each $y \in U$. So, Ord_n is lower semi-continuous in this case, too. The proposition is proved.

Let C denote the set of all points in G at which Ord_n is continuous. Let us note that Ord_n is locally constant on the set $T \cap C$. Indeed, let $x \in C \cap T^{(m)}$. As Ord_n is continuous at x, there exists a neighbourhood U of x such that

$$Ord_n(U) \subset \{m\}, \text{ i.e. } Ord_n(y) = m \text{ for each } y \in U.$$

We also have $U \subset C$. Therefore, as a corollary we have: If G is a torsion T_2 topological n-group, then C is open in G.

2.4. THEOREM. If G is a T_2 topological n-group with the Baire property (i.e. the intersection of countably many open dense sets is dense), then $\overline{C} = G$

PROOF. We shall use one result of Fort (see 1.7.14 in [3]) which states that if $f : X \rightarrow R$ is a lower semi-continuous function on a space X, then there exists $U \subset X$ which is the intersection of countably many dense open subsets of X, such that f is continuous at every point of U. In our case, as G has the Baire property, the set U is dense in G and since $U \subset C$ the theorem follows.

In particular the preceding theorem holds in the following cases: G is Čech-complete; G is countably compact; G is pseudocompact.

2.5. COROLLARY. If G is a torsion T_2 topological n-group with the Baire property, then the set G-C is nowhere dense in G.

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NEKE PRIMEDBE O n-GRUPAMA

U radu se daje potreban i dovoljan uslov da neki element n-grupe sa jedinicom i sam bude jedinica. Ispitana su i neka svojstva n-reda elementa u topološkim n-grupama.

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