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ON A CATEGORY FOR FUZZY TOPOLOGY

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Abstract. Patterned after Rodabaugh's category FUZZ [12] a new general category of fuzzy topological spaces GFT is defined. The category FUZZ, our category FT and some other known categories of Fuzzy Topology are identified with special subcategories of GFT. Some properties of GFT are discussed.

The notion of a fuzzy set was introduced by Zadeh in 1965 [18] but three years later the first attempt (approach) to graft this notion on General Topology was undertaken. Chang [2], the author of this approach, defined a fuzzy (topological) space as a pair (X,T) where X is a set and T is a family of its fuzzy subsets (i.e. $T\subset I^X$, $I:=[0,1]$), satisfying the following axioms: (1C) $0,1\in T$; (2C) if $U,V\in T$, then $U\wedge V\in T$, and (3C) if $U_s\in T$ for all $s\in S$, then $\bigvee_S U_s\in T$. A mapping $f\colon X\to Y$ where (X,T_X) and (Y,T_Y) are fuzzy spaces is called continuous if $f^{-1}(V)\in T_X$ for each $V\in T_Y$. The category of Chang fuzzy spaces and continuous mappings will be denoted CFT.

In [6] Goguen has noted that some special properties of the unit interval I are often unessential and sometimes also too burdensome for the theory of fuzzy sets and their applications and proposed a more general notion of an L-fuzzy set where L is a complete completely distributive lattice with a maximal 1 and a minimal 0 elements, $0\neq 1$. In the sequel when speaking about L-fuzzy sets and derived notions we shall always understand by L such a lattice. (It is necessary to mention however, that on the one hand

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some authors working in Fuzzy Topology consider more general (semi)lattices (see e.g.[13]) and on the other, to develop an appropriate concrete theory one has to demand sometimes special properties of L (such as the existence of an order-reversing involution on L, L to be a chain or, conversely, L to be orthocomplemented; see e.g. [1],[3],[4],[5],[14],[12]e.a.).

In [7] Goguen has defined the notion of an L-fuzzy (topological) space as a pair (X,T) where X is a set and T is an L-fuzzy topology on it (i.e. $T \subset L^X$ and T satisfies the axioms completely analogous to the axioms (1C) - (3C) of Chang's definition). In the sequel such spaces will be called Chang L-fuzzy spaces and the corresponding category will be denoted CFT(L). Note that CFT(I) is just the category CFT of Chang fuzzy spaces.

The overwhelming majority of the papers devoted to the theory of (L-)fuzzy topological spaces and related problems are concerned with the case of a fixed lattice L. The exception are some papers by Rodabaugh [11],[12],[13], e.a., Hutton [8], Eklund [3],[4]e.a. where the category FUZZ and some other categories of Chang L-fuzzy spaces with different lattices L are defined and studied.

In [14]-[16]e.a. we proposed a different approach to "fuzzification" of Topology. According to [14]-[16] a fuzzy (topological) space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}: I^X \to I$ is a mapping such that (1) $\mathcal{T}(0) = \mathcal{T}(1) = 1$; (2) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$ for arbitrary $U, V \in I^X$, and (3) $\mathcal{T}(\bigvee_s U_s) \geq \bigwedge_s \mathcal{T}(U_s)$ for arbitrary $U_s \in I^X$, $s \in S$. A mapping $f: X \to Y$ where $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are fuzzy topological spaces is called continuous if $\mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V)$ for each $V \in I^Y$. The corresponding category of fuzzy spaces and continuous mappings will be denoted FT(I). Substituting a'la Goguen a unit interval I for an arbitrary lattice L in the above definition one comes naturally to the category FT(L) of L-fuzzy (topological) spaces.

The aim of this paper is to define a new category GFT (General Category of Fuzzy Topological Spaces), to describe its elementary properties and to identify the above mentioned and so other known categories of Fuzzy Topology with suitable subcategories of GFT.

1. Definition of the category GFT. The objects of GFT are triples $(X,L,\mathcal{T})$ where X is a set, L is a lattice and $\mathcal{T}:L^X \to L$ is a mapping such that (1) $\mathcal{T}(0)=\mathcal{T}(1)=1$; (2) $\mathcal{T}(U\wedge V)\geq \mathcal{T}(U)\wedge\mathcal{T}(V)$ for any $U,V\in L^X$, and (3) $\mathcal{T}(\bigvee_s U_s)\geq \bigwedge_s \mathcal{T}(U_s)$ for any $U_s\in L^X$, $s\in S$. The morphisms from $(X_1,L_1,\mathcal{T}_1)$ to $(X_2,L_2,\mathcal{T}_2)$ are pairs (f,φ) where $f: X_1 \to X_2$ and $\varphi^{-1}: L_2 \to L_1$ are mappings and besides φ^{-1} is nondecreasing (i.e. $s\leq t$ implies $\varphi^{-1}(s)\leq\varphi^{-1}(t)$ for all $s,t\in L_2$) and preserves suprema and infima, and $\mathcal{T}_1(\varphi^{-1}\circ N\circ f)\geq\mathcal{T}_2(N)$ for each $N\in L^Y$. The composition of morphisms $(f,\varphi):(X_1,L_1,\mathcal{T}_1)\to(X_2,L_2,\mathcal{T}_2)$ and $(g,\psi):(X_2,L_2,\mathcal{T}_2)\to(X_3,L_3,\mathcal{T}_3)$ is defined as $(g\circ f,\psi\circ\varphi)$: $(X_1,L_1,\mathcal{T}_1)\to(X_3,L_3,\mathcal{T}_3)$ and (id_X,id_L): $(X,L,\mathcal{T})\to(X,L,\mathcal{T})$ is the identical morphism.

(Let Lat denote the category of lattices (as always we consider only complete completely distributive lattices with 0 and 1) and nondecreasing mappings preserving infima and suprema; let Lat^{op} be its opposite category and let Set be the category of sets and mappings. Using terminology of Eklund and Gähler [5] one can say that $Set\times Lat^{op}$ is taken as the ground category for GFT and GFT is a topological framing of $Set\times Lat^{op}$. (cf: Top is the topological framing of the ground category Set!))

The final object of GFT is $(X,2,\mathcal{T}_0)$ where $|X|=1$, $2=\{0,1\}$, and $\mathcal{T}_0$: $2^X\to\{1\}$.

2. Initial structures in GFT. Let X,Y be sets, L,K be lattices, $\mathcal{T}$: $K^Y\to K$ be a K-fuzzy topology on Y and $(f,\varphi):(X,L)\to(Y,K)$ be a morphism in $Set\times Lat^{op}$ (i.e. $f:X\to Y$ and $\varphi^{-1}:K\to L$ are mappings and besides φ^{-1} is nondecreasing and preserves infima and suprema). In accordance with general tradition the weakest L-fuzzy topology $\mathcal{S}$ on X for which (f,φ): $(X,L,\mathcal{S})\to(Y,K,\mathcal{T})$ is continuous, i.e. is a morphism in GFT is called initial for (f,φ).

It is obvious that the initial fuzzy topology exists and is unique. To describe it in an effective way consider for each $\alpha\in K^+(:=K\setminus\{0\})$ the family of K-fuzzy sets $\mathcal{T}_\alpha:=\{V: V\in K^Y, \mathcal{T}(V)\geq\alpha\}$ and let $\mathcal{S}^{\varphi^{-1}(\alpha)}:=\{\varphi^{-1}\circ V\circ f: V\in\mathcal{T}_\alpha\}$. Then according to [16], the equality $\mathcal{S}(U):=\bigvee\{\mathcal{S}^{\varphi^{-1}\alpha}(U)\wedge\varphi^{-1}\alpha : \alpha\in K^+\}$, where $U\in L^X$, defines a fuzzy topology $\mathcal{S}$ on (X,L). It is easy to notice now that $\mathcal{S}$ is desired initial L-fuzzy topology.

Let now $\{(Y_\gamma, L_\gamma, \mathcal{T}_\gamma): \gamma\in\Gamma\}$ be a family of objects of GFT and let $\{(f_\gamma, \varphi_\gamma): (X,L) \to (Y_\gamma, L_\gamma) : \gamma\in\Gamma\}$ be a family of morphisms in $\mathrm{Set}\times\mathrm{Lat}^{op}$. The weakest L-fuzzy topology $\mathcal{T}$ on X for which all $(f_\gamma, \varphi_\gamma)$ are morphisms in GFT is called initial for the family $\{(f_\gamma, \varphi_\gamma): \gamma\in\Gamma\}$. From [16] it easily follows that $\mathcal{T}$ can be defined by the equality $\mathcal{T} = \sup_\gamma \mathcal{S}_\gamma$ where $\mathcal{S}_\gamma$ is the initial L-fuzzy topology for $(f_\gamma, \varphi_\gamma)$.

3. Product in GFT. We begin with recalling the operation of product in Lat^{op} considered by Hutton [8].

Let $\{L_\gamma : \gamma\in\Gamma\}$ be a family of lattices. Consider a set $L = \bigotimes_\gamma L_\gamma$ the elements of which are the subsets $a \subset \prod\{L_\gamma^+ : \gamma\in\Gamma\}$ such that 1) if $t\in a$ and $s \leq t$, then $s\in a$ (for elements $s,t\in L$ we let $s\leq t$ iff $s_\gamma \leq t_\gamma$ in L_γ for each $\gamma\in\Gamma$) and 2) if $b_\gamma \subset L_\gamma^+$ and $b = \prod b_\gamma \subset a$, then $\beta := (\sup b_\gamma) \in a$. For $a,b\in L$ we let $a\leq b$ iff $a\subset b$. Define a mapping $\pi_\gamma^{-1}: L_\gamma \to L$ by the equality $\pi_\gamma^{-1}(t_\gamma) = \{s\in \prod L_\gamma^+ : s_\gamma \leq t_\gamma\}$ $(\in L)$.

Now let $\{(X_\gamma, L_\gamma, \mathcal{T}_\gamma): \gamma\in\Gamma\}$ be a family of fuzzy spaces, i.e. a family of objects of GFT. Let $X := \prod_\gamma X_\gamma$ be the product in Set and $p_\gamma : X \to X_\gamma$ be the corresponding projection. Let $L := \bigotimes_\gamma L_\gamma$ and $\pi_\gamma: L \to L_\gamma$ be defined as above. Let $\mathcal{T}$ denote the initial fuzzy topology for the family $\{(f_\gamma, \pi_\gamma): \gamma\in\Gamma\}$ of morphisms in $\mathrm{Set}\times\mathrm{Lat}^{op}$. Applying proposition 2 of [8] it is easy to show that $(X,L,\mathcal{T})$ is the product of $\{(X_\gamma, L_\gamma, \mathcal{T}_\gamma): \gamma\in\Gamma\}$ in GFT.

Dually one can define the notion of the final fuzzy topology for a family of morphisms $\{(f_\gamma, \varphi_\gamma): (X_\gamma, L_\gamma, \mathcal{T}_\gamma) \to (Y,K) : \gamma\in\Gamma\}$ in $\mathrm{Set}\times\mathrm{Lat}^{op}$ and the operation of coproduct (or direct sum) in GFT.

We pass now to the discussion of the main subcategories of the category GFT.

4. Category GCFT. Let GCFT be the full subcategory of the category GFT the objects of which are Chang L-fuzzy spaces (for all lattices L). It is obvious that GCFT in a natural way can be identified with the category FUZZ introduced and studied by Rodabaugh [11],[13] e.a.

The category GCFT is both epireflective and epicoreflective in GFT. Really, it is easy to notice that for each fuzzy space $(X,L,\mathcal{T})$ $(\in \mathrm{Ob(GFT)})$ the identical morphisms in $\mathrm{Set}\times\mathrm{Lat}^{op}$

$(id_X, id_L): (X,L,\mathcal{T}) \to (X,L,\mathcal{T}_1)$ ($\in Ob(GCFT)$) and
$(id_X, id_L): (X,L,\mathcal{T}_o) \to (X,L,\mathcal{T})$ ($(X,L,\mathcal{T}_o) \in Ob(GCFT)$) where
$\mathcal{T}_1 := \{ U \in L^X : \mathcal{T}(U)=1 \}$ and $\mathcal{T}_o := \{ U \in L^X : \exists \{V_s : s \in S\}, \mathcal{T}(V_s) > 0, U = \vee V_s \}$
are respectively an epireflexion and epicoreflexion.

5. Category GLFT. Let GLFT be a full subcategory of the category GFT the objects of which are laminated L-fuzzy spaces. (We call a fuzzy space $(X,L,\mathcal{T})$ laminated if $\mathcal{T}(c)=1$ for each constant $c \in L$ (cf also [10]).) An important feature of GLFT is that if $(c,\varphi): (X,L) \to (Y,K)$ is a morphism in $Set \times Lat^{op}$ and $c: X \to Y$ is a constant mapping then $(c,\varphi): (X,L,\mathcal{T}) \to (Y,K,\mathcal{S})$ is a morphism in GLFT, too for any laminated L-fuzzy topology $\mathcal{T}$ on X and any laminated K-fuzzy topology $\mathcal{S}$ on Y.

The category GLFT has products, but they are not the products of GFT, i.e. GLFT is not closed under formation of products in GFT.(Really, it is not difficult to notice that the product of infinitely many copies of a fuzzy space $(X,L,\mathcal{T})$ where $|X| \geq 1$, $|L| \geq 3$ and $\mathcal{T}(M)=1$ iff M=const, otherwise $\mathcal{T}(M)=0$, is not laminated.)Hence GLFT is not reflective in GFT. On the other hand it is not difficult to show that GLFT is epicoreflective in GFT (cf [13],[17]).

It is easy to notice that the full subcategory GLCFT := := GLFT $\cap$ GCFT of GFT is isomorphic to the category $\mathcal{C}_k$ considered in [11].

6. Categories GFT(L) and FT(L). Let L be a fixed lattice and let GFT(L) be the full subcategory of the category GFT, the objects of which are of the type $(X,L,\mathcal{T})$ where L is the given lattice. Notice, that GFT(L) is not isomorphic to the category FT(L) (except for the case $L = 2$; the both categories GFT(2) and FT(2) are isomorphic to the category Top of topological spaces. In addition, in GFT(L) as distinct from FT(L) the product does not generally exist.

Let $\varphi^{-1}: L \to L$ be a fixed mapping which preserves suprema and infima. Then one can consider a subcategory $GFT(L,\varphi)$ of GFT(L) having the same objects as GFT(L) and pairs of the type (f,φ^n) with the given φ and $n \in N$ as morphisms. It is easy to notice that the categories $GFT(L, id_L)$ and FT(L) are isomorphic.

The category GCFT(L,id):=GFT(L,id) $\cap$ GCFT is isomorphic to the category CFT(L) of Chang L-fuzzy topological spaces and the category GLCFT(I,id):=GCFT(I,id) $\cap$ GLFT is isomorphic to the category of Lowen fuzzy topological spaces.

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O JEDNOJ KATEGORIJI ZA FAZI TOPOLOGIJU

U radu se definiše jedna nova kategorija fazi topoloških prostora, nazvana opšta kategorija fazi topoloških prostora i označena sa GFT. Neke poznate kategorije fazi topologija su ustvari specijalne podkategorije kategorije GFT. Diskutovane su neke osobine kategorije GFT.

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