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ON HYPERSPACE OF COMPACT SUBSETS OF k-SPACES

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Abstract. Let X be a Hausdorff space and $\mathcal{Z}(X)$ be the hyperspace of compact subsets of X. For a Hausdorff space X let kX be the space on the set X generated by the family of k-closed subsets of the space X. In this paper we esta - blish som of the properties of the spaces: $\mathcal{Z}(kX)$, $k\mathcal{Z}(X)$, $\mathcal{Z}^{(n)}(kX)$ and so on. Among other, we prove that te following conditions are equivalent: $\mathcal{Z}(X)$ is a k-space. $\mathcal{Z}^{(n)}(X)$ is a k-space. $\mathcal{Z}^{(\omega)}(X)$ is a k-space.

For a space X, let exp(X) be the collection of all non-empty closed subsets of X with the Vietoris exponential topology.

Troughout this paper all spaces are assumed to be Hausdorff.

The Vietoris topology on exp (X) is the one generated by collection of the form

$$\langle U_1, U_2, \ldots, U_n\rangle = \left\{F \in \exp(X) : F \subset \bigcup_1^n U_i, F \cap U_i \neq \emptyset, 1 \leq i \leq n\right\},$$

with $U_1, U_2, \ldots, U_n$, open subsets of X.

Let $\mathcal{Z}(X)$ denote the subspace of exp (X) consisting of all non-empty compact subsets of X. Take $\mathcal{Z}(X)$ with Vietoris topology, then $\mathcal{Z}(X)$ is called the hyper- space of compact sets of X(|2|, |3|,|6|).

The set of all non-empty compact subsets of a space X will be denoted by $\mathcal{Z}(X)_{set}$.

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Put formally $\mathcal{Z}(X) = \mathcal{Z}^{(1)}(X)$, and for $n \geq 1$, let

$$\mathcal{Z}^{(n)}(X) = \mathcal{Z}(\mathcal{Z}^{(n-1)}(X)).$$

An element of $\mathcal{Z}(X)$ will be denoted by K and an element of $\mathcal{Z}^{(n)}(X)$ by $K^{(n-1)}$ (|4|,|5|).

The union mapping

$$u = u^{(1)}: \mathcal{Z}^{(2)}(X) \to \mathcal{Z}(X),$$

given by $u(K^{(1)}) = \bigcup\{K: K \in K^{(1)}\}$, is well defined and continuous (see|3|, |6|).

For $n \geq 1$, let

$$u^{(n)}: \mathcal{Z}^{(n+1)}(X) \to \mathcal{Z}^{(n)}(X) .$$

So we get an inverse sequence

$$(\mathcal{Z}^{(n)}(X), u^{(n)}, n \in N),$$

and let

$$\mathcal{Z}^{(\omega)}(X) = \varprojlim(\mathcal{Z}^{(n)}(X), u^{(n)}, n \in N),$$

be the limit space (see |5|, of a compact space X).

Remark. Using the continuity of the functor $\mathcal{Z}$ (see |9|,|2|), it is easy to see that the limit space $\mathcal{Z}^{(\omega)}(X)$ has the property $\mathcal{Z}(\mathcal{Z}^{(\omega)}(X)) \approx \mathcal{Z}^{\omega}(X)$.

The following definiton gives the notion of k-space.

1. DEFINITION. A subset A of a space X is called <u>k-closed</u> if $A \cap K$ is relatively closed in K for every compact $K \subset X$. A space X is a <u>k-space</u> if every k-closed subset of X is closed (for some equivalent definitions see |1|, |2|, |7|).

For every Hausdorff space X there exists exactly one k-spece that has the same underlying set and the same compact subspaces (see |1|, T.2.2. and |2|). Let $\mathcal{Y}$ be the family of all k-closed subsets of X. The set X with the topology generated by the family $\mathcal{Y}$ of a closed subsets will be denoted by kX. Clearly, a subset of kX is open if and only if its intersection with any compact subspace K of the space X is open in K. The topology of kX is finer that the topology of X. The spaces X and kX have the same compact subspaces, $\mathcal{Z}(X)_{set} = \mathcal{Z}(kX)_{set}$, and the same topology on those spaces. The formula $\varkappa_X(x) = x$ defines a contiruous k-mapping $\varkappa_X: kX \to X$ ($f: X \to Y$ is a k-mapping if for every compact sets $K \subset X, Z \subset Y, F(K)$ and $f^{-1}(Z)$ are compact sets).

If Y is a k-space and $F:Y \to X$ is a continuous one to one k-maping, then $Y \approx kX$(|1|, T.2.2.).

Let X be a k-space. Then is not necessarily that the hyperspace $\mathcal{Z}(X)$ is a k-space. In |8| V.V. Popov gave some examples of k-spaces whose hyperspace of compact sets are not k-space.

Another examples can be obtained in this way:

2. **EXAMPLE.** Let X_1 and X_2 be two k-spaces whose Cartesian product $X_1 \times X_2$ is not a k-space (see |2|, Example 3.3.29.). The sum $Y = X_1 \oplus X_2$ is a k-space. Let

$$\mathcal{X} = \{\{x_1, x_2\} : x_1 \in X_1, x_2 \in X_2\} \subset \mathcal{Z}(X).$$

It is easy to see that the set $\mathcal{X}$ is a closed subspace of $\mathcal{Z}(X)$ and $\mathcal{X}$ is homeomorphic to the Cartesian product $X_1 \times X_2$. Therefore $\mathcal{X}$ is not k-space. Since $\mathcal{X}$ is a closed subspace of $\mathcal{Z}(X)$, then $\mathcal{Z}(X)$ also is not a k-space.

Let $J_i(X) = \{K \in \mathcal{Z}(X) : \mathrm{card}K \leq i, i \in N\}$. Then $J_i(X)$ is a closed subspace of the hyperspace $\mathcal{Z}(X)$ and $J_1(X) \approx X$ (see |6|). The mapping $j_i : X^i \to J_i(X)$, defined by $j_i((x_1, x_2, \dots, x_i)) = \{x_1, x_2, \dots, x_i\}$, is a perfect mapping(1).

In |1|, Arhangel'skii (see also |2|, T. 3.7.25.) has proved the following statement.

3. **THEOREM.** Let $f:X \to Y$ be a perfect mapping. Then X is a k-space if and only if Y is a k-space.

From the last theorem we deduce the following.

4. **COROLLARY.** Let X be a space. Then

(i) The space X^i is a k-space iff $J_i(X)$ is a k-space.

(ii) If X^i is not k-space, then $\mathcal{Z}(X)$ is not a k-space.

For a space X we have the following set - rellations:

$$\mathcal{Z}(X)_{set} = \mathcal{Z}(kX)_{set} = k\mathcal{Z}(X)_{set} = k\mathcal{Z}(kX)_{set},$$

and the topological inclusions

$$\mathcal{Z}(X) \leqslant \mathcal{Z}(kX) \leqslant k\mathcal{Z}(X) \leqslant k\mathcal{Z}(kX).$$

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(1) A continuous mapping $f:X \to Y$ is perfect if f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

5. PROPOSITION. Let X be a space. Then

$$k\,\mathcal{Z}(X) \approx k\,\mathcal{Z}(kX).$$

PROOF. First, we prove that the spaces $\mathcal{Z}(X)$ and $\mathcal{Z}(kX)$ have the same compact subsets, i. e., $\mathcal{Z}^{(2)}(kX)_{set} = \mathcal{Z}^{(2)}(X)_{set}$. Since $\mathcal{Z}(X) \leq \mathcal{Z}(kX)$, then $\mathcal{Z}^{(2)}(kX)_{set} \subsetneq \mathcal{Z}^{(2)}(X)_{set}$. Let $\mathcal{K} \subset \mathcal{Z}(X)$ be a compact set. Then the set $|\mathcal{K}| = \bigcup\{K\colon K \in \mathcal{K}\}$ is a compact subset in X and in kX. The mapping i: $\mathcal{Z}(kX) \to \mathcal{Z}(X)$ defined by $\forall K \in \mathcal{Z}(kX)$, $i(K) = K \in \mathcal{Z}(X)$, is a continuous mapping. The set $i^{-1}(\mathcal{K}) = \mathcal{K} \subset \mathcal{Z}(kX)$ is closed, and $\mathcal{K} \subset \langle|\mathcal{K}|\rangle \subset \mathcal{Z}(kX)$. Hence, $\mathcal{K}$ is a compact subset of $\mathcal{Z}(kX)$.

We shall prove that a set $\mathcal{A}$ is closed in $k\,\mathcal{Z}(kX)$ if and only if $\mathcal{A}$ is closed in $k\,\mathcal{Z}(X)$.

Let $\mathcal{A}$ be a closed subset of $k\,\mathcal{Z}(X)$. Since $k\,\mathcal{Z}(X) \leq k\,\mathcal{Z}(kX)$, then $\mathcal{A}$ is closed in $k\,\mathcal{Z}(kX)$.

Suppose that $\mathcal{A}$ is not closed in $k\mathcal{Z}(X)$. Then the set $\mathcal{A}$ is not k-closed in $\mathcal{Z}(X)$. Thus, there exists a compact set $\mathcal{K} \subset \mathcal{Z}(X)$ such that $\mathcal{A} \cap \mathcal{K}$ is not relatively compact in $\mathcal{K}$. The set $\mathcal{K}$ is also compact in $\mathcal{Z}(kX)$. Since $\mathcal{A} \cap \mathcal{K}$ is not relatively closed in $\mathcal{K} \subset \mathcal{Z}(kX)$, then $\mathcal{A}$ is not closed in $k\,\mathcal{Z}(kX)$.

For a space X, let

$$(k\,\mathcal{Z})^{(2)}(X) = k\,\mathcal{Z}(k\,\mathcal{Z}(X))$$

and

$$(k\,\mathcal{Z})^{(n)}(X) = k\,\mathcal{Z}((k\,\mathcal{Z}^{(n-1)}(X)).$$

In this notation we have the following

6. COROLLARY. Let X be a space. Then

$$(k\,\mathcal{Z})^{(n)}(X) \approx k\,\mathcal{Z}^{(n)}(kX) \approx k\,\mathcal{Z}^{(n)}(X).$$

7. THEOREM. The following properties of a space X are equivalent:

(i) $\mathcal{Z}(X)$ is a k-space.

(ii) $\mathcal{Z}^{(n)}(X)$ is a k-space, for $n \geq 2$.

(iii) $\mathcal{Z}^{(\omega)}(X)$ is a k - space.

PROOF. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The space $\mathcal{Z}(X)$ is homeo-morphic to the closed subspace of $\mathcal{Z}^{(n)}(X)$, and $\mathcal{Z}^{(n)}(X)$, $n \in N$, is homeomorphic to the closed subspace of $\mathcal{Z}^{(\omega)}(X)$. Therefore, if $\mathcal{Z}^{(n)}(X)$ is a k-space, then $\mathcal{Z}(X)$ is a k-space, and if $\mathcal{Z}^{(\omega)}(X)$ is a k-space, then $\mathcal{Z}^{(n)}(X)$, $n \in N$, is a k-space.

(i) $\Rightarrow$(ii). Let $\mathcal{Z}(X)$ be a k-space . It is enought to prove that $\mathcal{Z}^{(2)}(X)$ is a k-space.

We are going to prove that the union mapping u: $\mathcal{Z}^{(2)}(X) \longrightarrow \mathcal{Z}(X)$ is a continuous k-mapping. It is easy to see that the mapping u is a continuous mapping (for a compact space X, see for example |4|).

Let $K \in \mathcal{Z}(X)$. Then

$$u^{-1}(K) = \{c^{(1)} \in \mathcal{Z}^{(2)}(X) : u(c^{(1)}) = |c^{(1)}| = K\},$$

and

$$u^{-1}(K) \subset \exp^{(2)}(K) = \mathcal{Z}^{(2)}(K) = \langle\langle K \rangle\rangle .$$

The space $\mathcal{Z}^{(2)}(K)$ is a compact subspace of $\mathcal{Z}^{(2)}(X)$. Since the mapping u is a continuous mapping, then the set $u^{-1}(K)$ is a closed subset of $\mathcal{Z}^{(2)}(K) \subset \mathcal{Z}^{(2)}(X)$. It follows that $u^{-1}(K)$ is a compact subset of $\mathcal{Z}^{(2)}(X)$.

Let $\mathcal{K} \subset \mathcal{Z}(X)$ be a compact set. Then $|\mathcal{K}| = \bigcup \{K : K \in \mathcal{K}\}$ is a compact subset of X, $|\mathcal{K}| \in \mathcal{Z}(X)$. The set $u^{-1}(\mathcal{K})$ is a closed subset of $\mathcal{Z}^{(2)}(X)$. Since $u^{-1}(\mathcal{K}) \subset \mathcal{Z}^{(2)}(|\mathcal{K}|) = \langle\langle |\mathcal{K}| \rangle\rangle$, we have that $u^{-1}(\mathcal{K})$ is a compact subset of $\mathcal{Z}^{(2)}(X)$.

Hence, the union mapping u is a continuous k-mapping.

From the assumption that the hyperspace $\mathcal{Z}(X)$ is a k-space, we have that the union mapping u: $\mathcal{Z}^{(2)}(X) \longrightarrow \mathcal{Z}(X)$ is a perfect mapping(see |1|, T.2.1. and |2|, T.3.7.18.).

From the theorem 3. it follows that $\mathcal{Z}^{(2)}(X)$ is a k-space.

(i) $\Longrightarrow$ (iii). Let $\mathcal{Z}(X)$ be a k-space. Since all bonding mapping $u^{(n)}$ of the inverse sequence ($\mathcal{Z}^{(n)}(X)$, $u^{(n)}$, $n \in N$) are perfect, then all projections $p_n : \mathcal{Z}^{(\omega)}(X) \rightarrow \mathcal{Z}^{(n)}(X)$ also are perfect mappings (see |2|,T.3.7.12.). Therefore $\mathcal{Z}^{(\omega)}(X)$ is a k - space.

From this theorem we have the following result:

8.COROLLARY. Let X be a space. Then:

(i) If $\mathcal{Z}(kX) \approx k\,\mathcal{Z}(X)$, then $\mathcal{Z}^{(n)}(kX) \approx k\,\mathcal{Z}^{(n)}(X)$.

(ii) If $\mathcal{Z}(kX) \approx k\,\mathcal{Z}(X)$, then $\mathcal{Z}^{(\omega)}(kX) \approx k\,\mathcal{Z}^{(\omega)}(X)$.

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O HIPERPROSTORU KOMPAKTNIH PODSKUPOVA k-PROSTORA

Neka je X Hausdorff-ov prostor i $\mathcal{Z}(X)$ hiperprostor kompaktnih podskupova prostora X. Za Hausdorff-ov prostor X, sa kX je označen prostor na skupu X koji je generisan familijom k-zatvorenih podskupova prvobitnog prostora X. U ovom radu ispituju se neka svojstva prostora: $\mathcal{Z}(kX)$, $k\mathcal{Z}(X)$, $\mathcal{Z}^{(n)}(kX)$ i sličnih. Pored ostalog dokazuje se da su sledeći iskazi ekvivalentni: $\mathcal{Z}(X)$ je k-prostor. $\mathcal{Z}^{(n)}(X)$ je k-prostor. $\mathcal{Z}^{(\omega)}(X)$ je k-prostor.

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