Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 2(1988), 47-53

Biljana Č. Popović

SOME PREDICTIONS OF THE FIRST ORDER EXPONENTIAL TIME SERIES*

(Received 28.8.1988.)

Abstract. Time series with non-Gaussian marginal distribution are of great interest nowdays. The time series with exponential marginal distribution are considered in this paper. The method of linear prediction is applicated to the exponential moving average EMA(1) and the exponential autoregressive EAR(1) time series. Some difficulties of inverting the matrix of the system of linear equations for solving the unknown coefficients of the prediction are presented.

1. Introduction

One of the important problems in time series analysis is the following: Given n observations on a realization, predict the $(n+s)$th observation in the realization where s is a positive integer. This problem was investigated by many authors and for different types of time series. In this paper, we shall investigate linear predictors for the time series with exponential marginal distribution. As it is necessary to have a criterion by which the performance of a predictor is measured, we adopt the usual one, the mean square error of the predictor. The solution for the Gaussian time series is well known (see for instance [1]). Theoretically, there is no difficalty to apply the above method in predicting the stationary time series with exponential marginals. So, the chapter 2 is refered to the general case, but two following chapters are refered to the first order moving average and the first order autoregressive time series with exponential marginal distribution with parameter λ.

The last chapter is about the matrices of the systems of linear equations for solving the coefficients of the predictions.

AMS Subject Classification (1980): 62M20

*Presented at the 18th European Meeting of Statisticians, Berlin, GDR, August 22-26, 1988.

2. General Case

Let us have n observations on a realization $x=(X_1,\dots,X_n)^T$ of the stationary time series $\{X_i : i\in(0,\pm1,\pm2,\dots)\}$ with exponential marginal distribution with parameter λ (real and positive). The best linear prediction of the process $\{X_i\}$ for s periods ahead (s being a positive integer) according to the mean square error method is $\hat{X}_{n+s}(X_1,\dots,X_n)=x^T a_s$, where a_s is the unknown vector of coefficients of the linear prediction.

Let us set the theorem:

THEOREM. Let $\{X_i : i\in(0\pm1,\pm2,\dots)\}$ be a stationary real valued time series with exponential marginal distribution and known autocovariance structure.

The best linear prediction of the process $\{X_i\}$ in the moment $n+s$ (n and s being positive integers), using n observations on the realization of $\{X_i\}$ according to the mean square error method is given by

$$(2.1)\qquad \hat{X}_{n+s}(X_1,X_2,\dots,X_n) = x^T a_s$$

where

$$(2.2)\qquad x^T = (X_1,X_2,\dots,X_n)$$

(2.3) $\quad a_s = V_{nn}^{+} V_{ns}$, V_{nn}^{+} is the Moore-Penrose generalized invers of V_{nn}

$$(2.4)\qquad V_{nn} = E(xx^T)$$

$$(2.5)\qquad V_{ns} = E(xX_{n+s})$$

PROOF. The expression for the mean square error is

$$(2.6)\qquad M = E[(X_{n+s} - x^T a_s)^T (X_{n+s} - x^T a_s)] = E(X^2_{n+s}) - V^T_{ns} a_s - a^T_s V_{ns} + a^T_s V_{nn} a_s.$$

To minimize it with respect to the vector of linear coefficients a_s, it should be derived and equated with zero. It follows that

$$a_s^T V_{nn} = V_{ns}^T$$

Finally

$$a_s = V_{nn}^{+} V_{ns}$$

because of the simetry of the matrix V_{nn}.

Let us remark that the prediction of the exponential time series will not depend on the parameter of its exponential distribution.

3. The Prediction of the EMA(1) Time Series

It is not difficult to apply the result of the chapter 2 in predicting the EMA(1) time series defined by Lawrance and Lewis [2] 1977.

We consider the time series $\{X_i\}$ defined as

$$(3.1) \qquad X_i=\begin{cases}\beta\varepsilon_i & \text{with probability } \beta \\ \beta\varepsilon_i+\varepsilon_{i-1} & \text{with probability } 1-\beta\end{cases}\quad \begin{matrix}0\le\beta\le 1\\ \\ i=0,\pm1,\pm2,\ldots\end{matrix}$$

where $\{\varepsilon_i\}$ is the sequence of the independent identically distributed random variables with exponential distribution of parameter λ. It is well known that X_i has the exponential marginal distribution with the same parameter λ for each $i=0\pm1,\ldots.$

As the covariance structure of the process is given as

$$(3.2) \qquad \mathrm{Cov}(X_i,X_j)=\begin{cases}\frac{1}{\lambda^2} & , |i-j|=0\\ \frac{1}{\lambda^2}\beta(1-\beta), & |i-j|=1\\ 0 & , |i-j|>1\end{cases},$$

it follows that

$$(3.3) \qquad E(X_iX_j)=x_{ij}=\begin{cases}\frac{2}{\lambda^2} & ,|i-j|=0\\ \frac{B}{\lambda^2} & ,|i-j|=1\\ \frac{1}{\lambda^2} & ,|i-j|>1\end{cases}$$

That is the vector of n components V_{ns} and the square matrix V_{nn} are

$$(3.4) \qquad V_{ns}^T=\begin{cases}\frac{1}{\lambda^2}(1,1,\ldots,1,B) \text{ for } s=1\\ \frac{1}{\lambda^2}(1,1,\ldots,1,1) \text{ for } s>1\end{cases},\quad V_{nn}=[x_{ij}]=\frac{1}{\lambda^2}W_{nn}^{(B)},\quad B=1+\beta-\beta^2$$

where square matrix $W_{nn}^{(B)}$ depends only on the parameter β, i.e. does not depend on λ.

$$(3.5) \qquad W_{nn}^{(B)}=\begin{bmatrix}2 & B & 1 & 1 & \ldots & 1 & 1 & 1\\ B & 2 & B & 1 & \ldots & 1 & 1 & 1\\ 1 & B & 2 & B & \ldots & 1 & 1 & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & 1 & 1 & 1 & \ldots & 2 & B & 1\\ 1 & 1 & 1 & 1 & \ldots & B & 2 & B\\ 1 & 1 & 1 & 1 & \ldots & 1 & B & 2\end{bmatrix}$$

It is obvious that

$$
(3.6) \qquad \hat{X}_{n+s} = \begin{cases} \Sigma_{i=1}^{n}\Sigma_{j=1}^{n-1}X_i w_{ij}^{(B)} + B\Sigma_{i=1}^{n}X_i w_{in}^{(B)} & \text{for } s=1 \\ \Sigma_{i=1}^{n}\Sigma_{j=1}^{n}X_i w_{ij}^{(B)} & \text{for } s>1 \end{cases}
$$

does not depend on λ, where $w_{ij}^{(B)}$ is to be the element in the ith row and jth column of the inverse matrix of the matrix $W_{nn}^{(B)}$.

It is interesting to discuss the inversion of the matrix $W_{nn}^{(B)}$. (See the Appendix.)

EXAMPLE 3.1. Suppose that we have two observations on the realization

$$x^T=(X_1,X_2) \quad .$$

The matrix $W_{22}^{(B)}$ is not singular for $\beta\in[0,1]$, so, the prediction for the EMA(1) process in the above sense is

$$
X_{2+s} = \begin{cases} (4-B^2)^{-1}((2-B^2)X_1+BX_2) & \text{for } s=1 \\ (2+B)^{-1}(X_1+X_2) & \text{for } s>1 \end{cases}
$$

The error of the prediction (3.6) is

$$
(3.7) \qquad E(|X_{n+s}-\hat{X}_{n+s}|^2) =
$$

$$
= \begin{cases} \frac{2}{\lambda^2}(1-\Sigma_{i=1}^{n-1}\Sigma_{j=1}^{n-1}w_{ij}^{(B)}-2B\Sigma_{j=1}^{n-1}w_{nj}^{(B)}-B^2w_{nn}^{(B)})+\Sigma_{i=1}^{n}\Sigma_{k=1}^{n}x_{ik}(\Sigma_{j=1}^{n-1}w_{ij}^{(B)}+2Bw_{kn}^{(B)})+B^2w_{in}^{(B)}w_{kn}^{(B)}), & s=1 \\ \frac{2}{\lambda^2}(1-\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}w_{ij}^{(B)})+\Sigma_{i=1}^{n}\Sigma_{k=1}^{n}x_{ik}(\Sigma_{j=1}^{n}\Sigma_{l=1}^{n}w_{ij}^{(B)}w_{kl}^{(B)}), & s>1 \end{cases}
$$

4. The Prediction of the EAR(1) Time Series

Another type of exponential time series defined by Lawrance and Lewis [3] 1980. is the first order autoregressive exponential time series. It was defined as

$$
(4.1) \qquad X_i = \begin{cases} \rho X_{i-1} & \text{with probability } \rho \\ \rho X_{i-1}+E_i & \text{with probability } 1-\rho \end{cases} \quad \begin{matrix} 0\le\rho<1 \\ i=0,\pm1,\pm2,\dots \end{matrix}
$$

where $\{E_i\}$ was the sequence of the independent identically distributed random variables with exponential distribution of parameter λ.

In this case the matrix $\mathbf{V}_{nn}$ has the form

$$(4.2)\qquad \mathbf{V}_{nn}=[x_{ij}]=\frac{1}{\lambda^2}\mathbf{W}_{nn}^{(R)}\quad ,R=(1+2\varrho-\varrho^2)(1+\varrho)^{-1}$$

where the matrix $\mathbf{W}_{nn}^{(R)}$ has the same form as the matrix $W_{nn}^{(B)}$ defined in (3.5) except the elements B above and under the main diagonal which are now replaced by the function R (4.2) because of the expectation

$$(4.3)\qquad E(X_iX_j)=x_{ij}=\begin{cases}\frac{2}{\lambda^2}\ , & |i-j|=0\\ \frac{R}{\lambda^2}\ , & |i-j|=1\\ \frac{1}{\lambda^2}\ , & |i-j|>1\end{cases}$$

The problem of iverting the matrix $\mathbf{W}_{nn}^{(R)}$ is also interesting and will be discussed in the Appendix.

The vector $\mathbf{V}_{ns}$ will be

$$(4.4)\qquad \mathbf{V}_{ns}^{T}=\begin{cases}\frac{1}{\lambda^2}(1,1,\dots,1,R) & \text{for}\quad s=1\\ \frac{1}{\lambda^2}(1,1,\dots,1,1) & \text{for}\quad s>1\end{cases}$$

As it can be seen, the prediction will not depend on λ and will be

$$(4.5)\qquad \hat{X}_{n+s}=\begin{cases}\Sigma_{i=1}^{n}\Sigma_{j=1}^{n-1}X_i w_{ij}^{(R)}+R\Sigma_{i=1}^{n}X_i w_{in}^{(R)} & \text{for } s=1\\ \Sigma_{i=1}^{n}\Sigma_{j=1}^{n}X_i w_{ij}^{(R)} & \text{for } s>1\end{cases}$$

The error of the prediction of the EAR(1) time series after the prediction in the sense of the mean square error method, which was used above, is

$$(4.6)\qquad E(|X_{n+s}-\hat{X}_{n+s}|^2)=$$

$$=\begin{cases}\frac{2}{\lambda^2}(1-\Sigma_{i=1}^{n-1}\Sigma_{j=1}^{n-1}w_{ij}^{(R)}-2R\Sigma_{j=1}^{n-1}w_{nj}^{(R)}-R^2w_{nn}^{(R)})+\Sigma_{i=1}^{n}\Sigma_{k=1}^{n}x_{ik}(\Sigma_{j=1}^{n-1}w_{ij}^{(R)}(\Sigma_{l=1}^{n-1}w_{kl}^{(R)}+2Rw_{kn}^{(R)})+\\ \qquad +R^2w_{in}^{(R)}w_{kn}^{(R)})\ , & s=1\\ \Sigma_{i=1}^{n}\Sigma_{j=1}^{n}w_{ij}^{(R)})+\Sigma_{i=1}^{n}\Sigma_{k=1}^{n}x_{ik}(\Sigma_{j=1}^{n}\Sigma_{l=1}^{n}w_{ij}^{(R)}w_{kl}^{(R)})\ , & s>1\ .\end{cases}$$

5. Appendix

The point of the linear prediction by the mean square error method is the inverting of the matrix V_{nn}. This problem is to be discussed in the connection with every concrete type of time series with exponential marginals, as well as with every order of the considered type.

Two special time series mentioned in this paper have the matrices $W_{nn}^{(B)}$ and $W_{nn}^{(R)}$ to be inverted. The concerning determinants are equal to n+1 when $\beta=0$ and $\rho=0$ [4]. We should like to know if there is any value for β and ρ for which the matrices would be singular and Moore-Penrose inverse would be needed. In order to make some approach to this problem, we shall set some numerical results.

Let us consider $W_{nn}^{(B)}$ first. This matrix and its determinant depends on the polynomial B (3.4) which graphic representation is given approximately on the figure 1. So, $B\in[1, 1.25]$ for $\beta\in[0,1]$.It is important to remark that we may cosider only $\beta\in[0, 0.5]$ because of the symmetry of the function B. The approximate numerical values for the determinant of the matrix $W_{nn}^{(B)}$ for β with step 0.1 and for several different values of n can be found in the table 1. It is obvious that the determinant of the matrix $W_{nn}^{(B)}$ is near to zero for large enough n and β near to 0.5 and we must take care of that if we need approximate values of the determinant.

The orther matrix $W_{nn}^{(R)}$ depends on the rational function R (4.2) which approximate graphic representation is given on the figure 2. We can calculate that $R\in[1, 4-2\sqrt{2}]$ for $\rho\in[0, 1)$. It is remarkable that it is also enough to watch only the part of the interval for ρ , i.e. $\rho\in[0, \sqrt{2}-1|$. Some of the approximate numerical values for the determinant of the matrix $W_{nn}^{(R)}$ for ρ with step 0.1 and different values for n are given in the table 2.

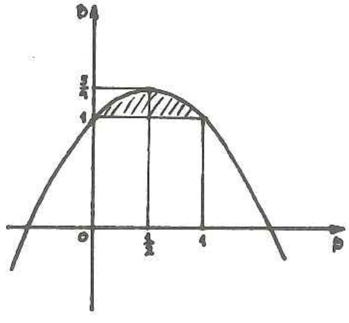

Fig. 1

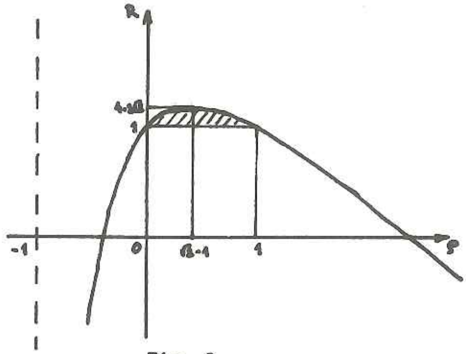

Fig. 2

Table 1

n \ ρ	0	0.1	0.2	0.3	0.4	0.5
5	6	5.21	4.50	3.96	3.63	3.52
10	11	8.93	6.92	5.43	4.56	4.28
20	21	15.48	9.87	6.25	4.46	3.93
30	31	20.94	11.06	5.68	3.45	2.86
40	41	25.45	11.16	4.65	2.40	1.88
50	51	29.12	10.60	3.59	1.58	1.16
60						0.69
70						0.40
80						0.23
90						0.13
100						0.07
104						0.06

Table 2

n \ ρ	0	0.1	0.2	0.3	0.4
2	3	2.83	2.72	2.65	2.63
3	4	3.66	3.43	3.30	3.26
4	5	4.48	4.41	3.91	3.84
5	6	5.28	4.78	4.48	4.38
10	11	9.15	7.71	6.87	6.58
20	21	16.11	12.00	9.75	8.99
30	31	22.12	14.65	10.87	9.66
40					9.35
50					8.52
90					4.42
100					3.60

REFERENCES

[1] FULLER,A.W.(1976).Introduction to statistical time series. John Wiley&Sons, New York.London.Sydney.Toronto.

[2] LAWRANCE,A.J. and LEWIS,P.A.W.(1977). An exponential moving-average sequence and point process (EMA 1). J.Appl.Prob. 14, 98-113.

[3] LAWRANCE,A.J. and LEWIS,P.A.W.(1980). The exponential autoreressive-moving average EARMA(p,q) process. J.R.Statist.Soc.B, 42, No.2, pp 150-161.

[4] PROSKURJAKOV,I.V.(1984). Sbornik zadač po lineinoii algebre, Nauka. Moskva .

Biljana Č. Popović

PROGNOZE EKSPONENCIJALNIH VREMENSKIH SERIJA PRVOG REDA

U novije vreme veliko interesovanje pobuduju negausovske vremenske serije. U ovom radu se posmatraju vremenske serije sa eksponencijalnom marginalnom raspodelom. Primenjuje se metod linearnog prognoziranja na eksponencijalne pokretne sredine EMA(1) i eksponencijalne autoregresivne vremenske serije EAR(1). Posebno je istaknut problem inverzije matrice sistema linerarnih jednačina za odredivanje nepoznatih koeficijenata prognoze.

Filozofski fakultet
18 000 Niš
Jugoslavija