Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 2(1988),41-46

Milutin Obradović, M.K. Aouf and Shigeyoshi Owa

A CHARACTERIZATION OF THE CLASS OF STARLIKE FUNCTIONS OF COMPLEX ORDER

(Received 25.4.1988)

Abstract. Let $S(b)$ ($b\neq 0$, complex) denote the class of functions f analytic in $|z|<1$, normalized so that $f(0)=f'(0)-1=0$ and starlike of order b. A necessary and sufficient condition that $f\in S(b)$ is given.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U=\{z:|z|<1\}$.

A function $f\in A$ is said to be a starlike function of order b if and only if $f(z)/z\neq 0$, $z\in U$, and

$$\mathrm{Re}\{1+\frac{1}{b}\,(z\,\frac{f'(z)}{f(z)}-1)\} > 0 \tag{1}$$

for some b ($b\neq 0$, complex) and for all $z\in U$. We denote by $S(b)$ the class of all such functions. The class $S(b)$ was introduced by Nasr and Aouf in [3]. In the same paper they investigated certain properties of the class $S(b)$.

We note that for b equal to 1, $1-\alpha$ ($0\leq\alpha<1$), $\cos\lambda e^{-i\lambda}$ (λ is real and $|\lambda|<\Pi/2$) and $(1-\alpha)\cos\lambda e^{-i\lambda}$ ($0\leq\alpha<1$, $|\lambda|<\Pi/2$) we have the class of starlike functions (S^*), starlike functions of order α ($S^*(\alpha)$), spirallike functions (S^{λ}) and spirallike functions of order α (S^{λ}_{α}). All of these special subclasses are the subclasses of univalent functions in U. More about these classes we can see in [1].

Let f and g be analytic in the unit disc U. The function f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if g is univalent,

AMS Subject Classification (1980): 30C45

$f(0)=g(0)$ and $f(U) \subset g(U)$.

For the proof of our result in the part 2 of this paper we need the following lemma given by Miller and Mocanu in [2] (see Remark on the page 191).

LEMMA 1. Let q be univalent in U and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and suppose that Q is starlike (univalent) in U (i.e. $Q(0)=0$, $Q'(0) \neq 0$ and $\operatorname{Re}\{zQ'(z)/Q(z)\}>0$, $z \in U$). If p is analytic in U, with $p(0)=q(0)$, $p(U) \subset D$ and

$$(2) \qquad zp'(p)\phi(p(z)) \prec zq'(z)\phi(q(z)) = Q(z),$$

then $p \prec q$, and q is the best dominant of (2).

In the end of this part, we note that the univalent function q is said to be a dominant of the differential subordination (2) if $p \prec q$ for all p satisfying (2). If $\tilde{q}$ is a dominant of (2) and $\tilde{q} \prec q$ for all dominants q of (2), then $\tilde{q}$ is said to be the best dominant of (3) (for more details see [2]).

2. A characterization of the class $S(b)$

In this part we give a necessary and sufficient condition for a function $f \in A$ to be in the class $S(b)$.

THEOREM 1. Let $f \in A$. A necessary and sufficient condition that $f \in S(b)$ ($b \neq 0$, complex) is that for each real number k, $-1<k<1$, the function defined by

$$(3) \qquad F_k(z) = \left(\frac{kf(z)}{f(kz)}\right)^{1/2b}, \quad F_k(0) = 1, \quad F_0(z) = \left(\frac{f(z)}{z}\right)^{1/2b}$$

be analytic in U and subordinate to $\frac{1-kz}{1-z}$, $z \in U$, or equivalently that

$$(4) \qquad \operatorname{Re}\{F_k(z)\}>(1+k)/2, \quad z \in U.$$

PROOF. First, let $f \in S(b)$ ($b \neq 0$, complex). In Lemma 1 we choose $q(z)=\frac{1-kz}{1-z}$ $(-1<k<1)$, and $\phi(w)=\frac{1}{w}$. Then q is univalent in U and

$$Q(z) = zq'(z)\phi(q(z)) = z\frac{q'(z)}{q(z)} = \frac{(1-k)z}{(1-z)(1-kz)} \text{ is starlike in U.}$$

Really,

$$\mathrm{Re}\{z\frac{Q'(z)}{Q(z)}\} = \frac{1}{2}\mathrm{Re}\{\frac{1+z}{1-z}+\frac{1+kz}{1-kz}\} > 0, \quad z \in U.$$

Therefore, the conditions of Lemma 1 are satisfied and we have that for p analytic in U, with $p(0)=1$, the following implication

$$(5) \qquad \frac{zp'(z)}{p(z)} \prec \frac{(1-k)z}{(1-z)(1-kz)} = Q(z) \Rightarrow p(z) \prec \frac{1-kz}{1-z} = q(z)$$

is true, and q is the best dominant.

Now, let show that for the function $p(z)=F_k(z)=(\frac{kf(z)}{f(kz)})^{1/2b}$, where $f \in S(b)$, the left side of implication (5) is satisfied. In that sense, let show that the function

$$z\frac{F_k'(z)}{F_k(z)} = \frac{1}{2b}[z\frac{f'(z)}{f(z)} - kz\frac{f'(kz)}{f(kz)}]$$

is subordinate to the function $Q(z)$. Really, since $f \in S(b)$, from (1) we have

$$(6) \qquad 1+\frac{1}{b}(z\frac{f'(z)}{f(z)}-1) \prec \frac{1+z}{1-z};$$

and from there and subordinate principle,

$$(7) \qquad 1+\frac{1}{b}(kz\frac{f'(kz)}{f(kz)}-1) \prec \frac{1+kz}{1-kz}, \ -1<k<1.$$

Further, $f_1 \prec g_1$ and $f_2 \prec g_2$ imply that $f_1-f_2 \prec g_1-g_2$ if g_1-g_2 is univalent. In our cases, (6) and (7), because

$$\frac{1+z}{1-z} - \frac{1+kz}{1-kz} = \frac{2(1-k)z}{(1-z)(1-kz)}$$

is starlike as we have shown previously, we get

$$\frac{1}{b}\left[z\frac{f'(z)}{f(z)} - kz\frac{f'(kz)}{f(kz)}\right] \prec \frac{2(1-k)z}{(1-z)(1-kz)},$$

i.e.

$$\frac{1}{2b}\left[z\frac{f'(z)}{f(z)} - kz\frac{f'(kz)}{f(kz)}\right] = z\frac{F_k'(z)}{F_k(z)} \prec Q(z),$$

which was to be proved. The result follows from (5).

We note that the previous proof of necessert part of Theorem 1 is different than those given in [5] and [6] for similar situation.

For the proof of sufficient part of Theorem 1 we employ the manner due to Robertson [5]. Namely, let F_k defined by (3) satisfy the condition (4). Then it is equivalent to condition

$$(8) \qquad \left|\frac{1+k}{F_k(z)} - 1\right| < 1, \quad z \in U.$$

If we put

$$(9) \qquad \Psi(z) = \frac{1+k}{F_k(z)} - 1 = (1+k)\left(\frac{f(kz)}{kf(z)}\right)^{1/2b} - 1,$$

then we conclude that the function Ψ is analytic in U and $|\Psi(z)|<1$, for all k, $0<k<1$. Let show that the function f satisfies (1). Let $t=1-k$, $0<t<1$. Since

$$f(kz) = f(z-tz) = f(z) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!}(-tz)^n, \quad |tz|<1-|z|,$$

we have

$$\Psi(z) = (2-t)\left[\frac{f(z-tz)}{(1-t)f(z)}\right]^{1/2b} - 1$$

$$= 1 - \left[1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right)\right]t + \mathcal{O}(t)$$

as $t\to 0$, $|z|<1$. From there, since $|\Psi(z)|<1$, we obtain

$$1-\mathrm{Re}\left\{1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right)\right\}t + \mathcal{O}(t) < 1,$$

which implies

$$-\mathrm{Re}\left\{1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right)\right\} + \mathcal{O}(1) < 0,$$

i.e.

$$\mathrm{Re}\{1+\frac{1}{b}\ (z\,\frac{f'(z)}{f(z)}\ -\ 1)\}>0,\ z\in U.$$

For z=0 the function $1+\frac{1}{b}(z\frac{f'(z)}{f(z)}-1)$ does not vanish so we have $\mathrm{Re}\{1+\frac{1}{b}(z\frac{f'(z)}{f(z)}-1)\}>0$ and $f\in S(b)$.

Putting $b=(1-\alpha)\cos\lambda e^{-i\lambda}$ in Theorem 1 we have

COROLLARY 1. Let $f\in A$ and α and λ be real numbers satisfying $0\leq\alpha<1$, $|\lambda|<\Pi/2$. A necessary and sufficient condition that $f\in S_{\alpha}^{\lambda}$ is that for each real number k, $-1<k<1$,

$$\mathrm{Re}\{[\frac{kf(z)}{f(kz)}]^{\delta(1+itg\lambda)}\}>\frac{k+1}{2},\quad z\in U,$$

where

$$\delta=\frac{1}{2(1-\alpha)}\ .$$

This is the earlier result given by Selinger in [6]. In a special case when b=1, we have that the condition

$$\mathrm{Re}\{[\frac{kf(z)}{f(kz)}]^{1/2}\}>\frac{k+1}{2},\quad z\in U,$$

for each k, is necessery and sufficient for $f\in A$ to be in S^*. This is the former result obtained by Robertson [5].

Letting $k\to-1$ in Theorem 1, we get

COROLLARY 2. Let $f\in S(b)$ ($b\neq0$, complex). Then

$$\mathrm{Re}\{(\frac{f(z)}{-f(-z)})^{1/2b}\}>0,\quad z\in U.$$

In the end, for k=0 in Theorem 1, we obtain

COROLLARY 3. If $f\in S(b)$ ($b\neq0$, complex), then

$$\mathrm{Re}\{(\frac{f(z)}{z})^{1/2b}\}>\frac{1}{2},\quad z\in U,$$

or equivalently,

$$\left(\frac{f(z)}{z}\right)^{1/2b} \prec \frac{1}{1-z} .$$

We note that from the last relation we have also

$$\left(\frac{z}{f(z)}\right)^{1/2b} \prec 1-z,$$

and from there

$$\left|\left(\frac{z}{f(z)}\right)^{1/2b} - 1\right| \leq |z|, \quad z \in U,$$

which is the result given in [4].

REFERENCES

[1] A.W.GOODMAN, Univalent functions, Vol. I,II (1983), Tampa , Florida

[2] S.S.MILLER and P.T.MOCANU, On some classes of first-order differential subordinations, Michigan Math. J. 32 (1985), 185-195.

[3] M.A.NASR and M.K.AOUF, Starlike function of complex order, Journal of Natural Sci. and Math., Vol. 25, No. 1 (1985), 1-12.

[4] M.OBRADOVIC, M.K.AOUF and S.OWA, On some results for starlike functions of complex order, to appear.

[5] M.S.ROBERTSON, A characterization of the class of starlike univalent functions, Michigan Math. J. 26 (1979), 65-69.

[6] V.M.SELINGER, A characterization of the class of spirallike univalent functions, Mathematica (Cluj), Vol. 28 (51), No. 2, 1986, 181-183.

Department of Mathematics
Faculty of Technology and Metallurgy
4 Karnegieva Street
11000 Belgrade
Yugoslavia

Department of Mathematics
Faculty of Science
University of Qatar
P.O.Box 2713
Doha - Qatar

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan