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LIOUVILLE'S FORMULA FOR CERTAIN CLASSES OF LINEAR EQUATIONS AND ITS APPLICATIONS

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Abstarct. Liouville's formula for certain class of linear operator equations of order n is derived. The result is applied to solving a linear second order operator equation. A number of examples which illustrate the results are given.

0. Introduction.

Let V be a commutative algebra over R (or C) and let L be a linear operator on V, which belongs to the class $D_{\alpha}(V)$, where $\alpha \in \ker L$ (see [1] - [4]), i.e.

$$L(uv) = uLv + vLu + \alpha LuLv, \text{ for every } u,v \in V.$$

In [1] it was shown that the only interesting cases are $\alpha = 0$ and $\alpha = 1$.

The properties of the linear equation

$$(0.1) \quad \left(\sum_{k=o}^{n} p_k L^{n-k} \right) x = q \quad (p_o = 1)$$

where $p_1,\dots,p_n,q \in V$ were investigated in [1]-[3].

For the linear homogeneous equation

$$(0.2) \quad \left(\sum_{k=o}^{n} p_k L^{n-k} \right) x = 0 \quad (p_o = 1, p_1,\dots,p_n \in V)$$

a generalization of Wronskian is defined in [1] by the following

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$$\sum_{k=1}^{n}(-1)^{n+k}(L^{n-1}x_k)LW_k = \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-3}x_1+\alpha L^{n-2}x_1 & \dots & L^{n-3}x_n+\alpha L^{n-2}x_n \\ L^{n-1}x_1 & \dots & L^{n-1}x_n \\ L^{n-1}x_1 & \dots & L^{n-1}x_n \end{vmatrix} = 0$$

and also

$$\sum_{k=1}^{n}(-1)^{n+k}\alpha(L^{n}x_k)LW_k = \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-3}x_1+\alpha L^{n-2}x_1 & \dots & L^{n-3}x_n+\alpha L^{n-2}x_n \\ L^{n-1}x_1 & \dots & L^{n-1}x_n \\ L^{n}x_1 & \dots & L^{n}x_n \end{vmatrix}.$$

Since

$$\sum_{k=1}^{n}(-1)^{n+k}(L^{n}x_k)W_k = \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-3}x_1+\alpha L^{n-2}x_1 & \dots & L^{n-3}x_n+\alpha L^{n-2}x_n \\ L^{n-2}x_1 & \dots & L^{n-2}x_n \\ L^{n}x_1 & \dots & L^{n}x_n \end{vmatrix}.$$

from (1.2) and the above we obtain (1.1) which completes the proof of lemma.

THEOREM 1.1. Let $L \in D_\alpha(V)$ and let $x_1,\dots,x_n$ be (kerL)-linearly independent solutions of (0.2) with the generalized Wronskian W. Then W satisfies the first order equation

$$LW(x_1,\dots,x_n) + \Big(\sum_{k=1}^{n}(-\alpha)^{k-1}p_k\Big)W(x_1,\dots,x_n) = 0. \tag{1.3}$$

PROOF. Since $x_1,\dots,x_n$ are solutions of (0.2) then from (1.1) we find

$$LW(x_1,\dots,x_n) + \Big(\sum_{k=1}^{n}(-\alpha)^{k-1}p_k\Big)W(x_1,\dots,x_n)$$

$$= \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-2}x_1+\alpha L^{n-1}x_1 & \dots & L^{n-2}x_n+\alpha L^{n-2}x_n \\ -\sum_{k=1}^{n} p_k L^{n-k}x_1 & \dots & -\sum_{k=1}^{n} p_k L^{n-k}x_n \end{vmatrix}$$

$$+ \sum_{k=1}^{n} \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-2}x_1+\alpha L^{n-1}x_1 & \dots & L^{n-2}x_n+\alpha L^{n-1}x_n \\ (-\alpha)^{k-1}p_k L^{n-1}x_1 & \dots & (-\alpha)^{k-1}p_k L^{n-1}x_n \end{vmatrix}$$

$$= \sum_{k=1}^{n} (-p_k) \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-2}x_1+\alpha L^{n-1}x_1 & \dots & L^{n-2}x_n+\alpha L^{n-1}x_n \\ L^{n-k}x_1-(-\alpha)^{k-1}L^{n-1}x_1 & \dots & L^{n-k}x_n-(-\alpha)^{k-1}L^{n-1}x_n \end{vmatrix}$$

Since

$$L^{n-k}x_j-(-\alpha)^{k-1}L^{n-1}x_j = \sum_{i=o}^{k-2} (-\alpha)^i (L^{n-k+i}x_j+ \alpha L^{n-k+i+1}x_j)$$

we conclude that (1.4) is valid.

REMARKS AND EXAMPLES. (i) If $\alpha = 0$ (1.4) becomes

(1.5) $\quad LW(x_1,\dots,x_n) + p_1 W(x_1,\dots,x_n) = 0$

and in the case $\alpha = 1$ we obtain

(1.6) $\quad LW(x_1,\dots,x_n) + (\sum_{k=1}^{n} (-1)^{k-1}p_k) W(x_1,\dots,x_n) = 0.$

(ii) Let $L = d/dx \in D_o(V)$, where V is the set of n-times differentiable real functions, $\ker L = R$. Then (1.5) reduces to the well-known Liouville's formula for differential equations

$$W(y_1,\dots,y_n) = C\exp(-\int p_1(x)dx),$$

C is arbitrary constant (see e.g. [5]), $y_1,\dots,y_n$ are linearly independent solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = 0.$$

(iii) Let $L = f(x,y)\partial/\partial x + g(x,y)\partial/\partial y$, where f,g are n-th order differentiable functions. In this case V is the set of n-th order differentiable real functions in two variables, $\alpha = 0$

and we have

$$W(u_1,\dots,u_n) = F(h(x,y))\exp(-\int p_1(h,\theta)d\theta),$$

where F is an arbitrary differentiable function, $h(x,y) = C$ is the general solution of $y' = -f(x,y)/g(x,y)$ and θ is a particular solution of $f\theta_x + g\theta_y = 1$, $p_1 = p_1(x,y) = p_1(h,\theta)$.

In this case equation (0.2) reduces to n-th order linear partial differential equation, with particular solutions $u_1,\dots,u_n$. This result can be obtain also, using the results from [1].

(iv) Let $L = D \in D_o(V)$, where V is a set of complex functions, such that real and imaginary part are n-th order differentiable functions, D is a Kolosoff's operator, defined by

$$Dw = (u_x - v_y + i(u_y + v_x))/2, \ (w = u + iv),$$

kerL is a set of analytic functions. Then the Liouville's formula has the form

$$W(w_1,\dots,w_n) = F(z)\exp(- \int p_1(z,\bar{z})d\bar{z})$$

where F is an arbitrary analytic function, $w_1,\dots,w_n$ are particular solutions of

$$D^n w(z,\bar{z}) + p_1(z,\bar{z})D^{n-1}w(z,\bar{z}) + \dots + p_n(z,\bar{z})w(z,\bar{z}) = 0.$$

This, also, follows from [11].

(v) Let $L = \Delta \in D_1(V)$, $\Delta f = f(x+1) - f(x)$ where V is the set of real functions. Then we have:

$$W(y_1,\dots,y_n) = P(x) \prod_{j=o}^{x-1} (1+ \sum_{k=1}^{n} (-1)^k p_k(j)),$$

where P is an arbitrary periodic function with period 1, $y_1,\dots,y_n$ are particular solutions of the difference equation

$$\Delta^n y + p_1(x)\Delta^{n-1}y + \dots + p_n(x)y = 0.$$

This is nothing else but the well-known result for difference equations (see e.g. [6] - [9])

(vi) Let $L \in D_1(V)$ be defined by $Lf = f(qx) - f(x)$, where $q = \text{const}$, $q > 0$, $q \neq 1$, V is the set of real functions.

Then we have:

$$W(y_1,\dots,y_n) = P(\log_q x) \prod_{j=o}^{\log_q x-1} (1+ \sum_{k=1}^{n} (-1)^k p_k(q^k)),$$

where P is an arbitrary periodic function with period 1.

$$(0.3)\qquad W(x_1,\dots,x_n) = \begin{vmatrix} x_1 & \dots & x_n \\ Lx_1 & \dots & Lx_n \\ \vdots & & \\ L^{n-1}x_1 & \dots & L^{n-1}x_n \end{vmatrix}$$

where $x_1,\dots,x_n$ are particular solutions of (0.2). Particular solutions $x_1,\dots,x_n$ are (kerL)-linearly dependent if and only $W(x_1,\dots,x_n) = 0$ (see [1]).

REMARK 0.1. In the case of differential equations, (0.3) is the standard Wronskian, but in the case of difference equations this is an equivalent form of Casoratian [6] - [10].

In this note we will derive a generalization of the well-known Liouville's formula for the equation (0.2). The obtained result will be applied to solving second-order equations. As examples we will considered differential, difference and functional equations.

1. Liouville's formula.

LEMMA 1.1. Let $L \in D_\alpha(V)$ and let W be given by (0.3). Then

$$(1.1)\qquad LW(x_1,\dots,x_n) = \begin{vmatrix} x_1+\alpha Lx_1 & \dots & x_n+\alpha Lx_n \\ \vdots & & \\ L^{n-2}x_1+\alpha L^{n-1}x_1 & \dots & L^{n-2}x_n+\alpha L^{n-1}x_n \\ L^n x_1 & \dots & L^n x_n \end{vmatrix}$$

where $x_1,\dots,x_n \in V$.

PROOF. We will prove (1.1) by using the induction. For $n=2$ it is trivial. Supposing that (1.1) holds for $n-1$, we obtain

$$(1.2)\qquad LW(x_1,\dots,x_n) = L\left(\sum_{k=1}^{n} (-1)^{n+k} (L^{n-1}x_k) W_k \right)$$
$$= \sum_{k=1}^{n} (-1)^{n+k} \left((L^n x_k) W_k + (L^{n-1}x_k) LW_k + \alpha (L^n x_k) LW_k \right),$$

where W_k is the determinant of order $n-1$ obtained from $W(x_1,\dots,x_n)$ by missing the n-th row and k-th collumn.

Using the induction hypothesis we find

(vii) Let $L \in D_1(V)$ be defined by $Lf = f(wx) - f(x)$, where w is given function, $wx = w(x)$, $w^k x = w(w^{k-1}x)$, V is the set of real functions. Then, for the linear functional equation

$$f(w^n x) + p_1(x)f(w^{n-1}x) + \dots + p_n(x)f(x) = 0,$$

the Wronskian satisfies the first order equation

$$W(f_1(wx),\dots,f_n(wx)) + p_1(x)W(f_n(x),\dots,f_1(x)) = 0,$$

where $f_1,\dots,f_n$ are particular solutions of the above equation.

2. Second order equations.

In this section we will consider the second-order equation

(2.1) $\quad L^2x + pLx + qx = 0,\ p,q \in V$

where $L \in D_\alpha(V)$. Then the Liouville's fomula becomes

(2.2) $\quad LW(u,v) + (p - \alpha q)W(u,v) = 0,$

where $W(u,v)$ is generalized Wronskian and u,v are particular solutions of (2.1).

THEOREM 2.1. Let $L \in D_\alpha(V)$ and let u be a nontrivial particular solution of (2.1) and $u + \alpha Lu$ is not proper zero devisor in V. If v and w are nontrivial particular solutions of of the system

(2.3) $\quad uLv - vLu = w, \quad Lw + (p - \alpha q)w = 0,$

then

(i) v satisfies (2.1);

(ii) u, v are ($\ker L$)-linearly independent;

(iii) The general solution of (2.1) is $x = au + bv$, where $a,b \in \ker L$ are arbitrary.

PROOF. (i) Since u is the nontrivial solution of (2.1) then $u + \alpha Lu \neq 0$, and we find $(u+\alpha Lu)L^2v = (v+\alpha Lv)L^2u + Lw$, which gives $(u+\alpha Lu)(L^2v + pLv + qv) = 0$, wherefrom follows that v satisfies the equation (2.1).

(ii) Since w is a nontrivial solution of (2.3), i.e. $w = 0$, we obtain that u, v are ($\ker L$)-linearly independent (see [1], Theorem 1).

(iii) From the above and Theorem 5 from [1] the form of general solution of (2.1) follows immediately which compeltes the proof of the theorem.

REMARKS AND EXAMPLES. The general solutions of the equations

$$(2.4)\quad y'' + p(x)y' + q(x)y = 0,$$

$$(2.5)\quad f^2 z_{xx} + 2fg z_{xy} + g^2 z_{yy} + (pf + ff_x + gf_y) z_x + (pg + fg_x + gg_y) z_y + qz = 0,$$

$$(2.6)\quad D^2 w(z,\bar{z}) + p(z,\bar{z}) Dw(z,\bar{z}) + q(z,\bar{z}) w(z,\bar{z}) = 0,$$

$$(2.7)\quad \Delta^2 y + p(x)\Delta y + q(x) y = 0,$$

$$(2.8)\quad y(q^2 x) + P(x) y(qx) + Q(x) y(x) = 0,$$

$$(2.9)\quad y_{n+2} + P_n y_{n+1} + Q_n y_n = 0,$$

$$(2.10)\quad f(w^2 x) + p(x) f(wx) + q(x) f(x) = 0,$$

are given by

$$(2.4')\quad y = C_1 u + C_2 u \int \exp(-\int p\,dx) u^{-2} dx,$$

(C_1, C_2 are arbitrary constants);

$$(2.5')\quad z = F_1(h(x,y))u + F_2(h(x,y))u \int \exp(-\int p(h,\theta)d\theta) u^{-2} d\theta,$$

where F_1, F_2 are arbitrary differentiable functions, $h(x,y) = C$ is the general solution of $y' = -f(x,y)/g(x,y)$ and θ is a particular solution of $f\theta_x + g\theta_y = 1$, $p = p(x,y) = p(h,\theta)$;

$$(2.6')\quad w(z,\bar{z}) = F_1(z)u + F_2(z)u \int \exp(-\int p(z,\bar{z})d\bar{z}) u^{-2} d\bar{z},$$

where F_1 and F_2 are arbitrary analytic functions;

$$(2.7')\quad y = P_1(x)u + P_2(x)u \sum_{k=o}^{x-1} \left(\prod_{j=o}^{k} (1-p(j)+q(j)) \right) / u(k)u(k+1),$$

where P_1, P_2 are arbitrary periodic functions with period 1. This formula can be found, for example, in [7], [8].

$$(2.8')\quad y = P_1(\log_q x)u + P_1(\log_q x)u \sum_{k=o}^{\log_q x-1} \left(\prod_{j=o}^{k} Q(q^j) \right) / u(q^k) u(q^{k+1}),$$

where P_1, P_2 are arbitrary periodic functions with period 1.

$$(2.9')\quad y_n = C_1 u_n + C_1 u_n \sum_{k=o}^{n-1} \left(\prod_{j=o}^{k} Q_j \right) / u_k u_{k+1})$$

(C_1, C_2 are arbitrary constants);

$$(2.10')\quad f(x) = C_1(x)u(x) + C_2(x)v(x)$$

where C_1 and C_2 are arbitrary solutions of $C(wx) - C(x) = 0$, and u, v, v' satisfy the system

$$v(wx) - u(wx)v(x)/u(x) = v'(x)/u(x), \quad v'(wx) + p(x)v'(x) = 0.$$

In the above examples u is the particular nontrivial solution of the equations (2.4) - (2.10).

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LIOUVILLEOVA FORMULA ZA NEKE KLASE LINEARNIH JEDNACINA

I NJENE PRIMENE

Liouvilleova formula za neke klase linearnih operatorskih jednačina je izvedena. Rezultat je primenjen na rešavanje linearne jednačine drugog reda. Više primera ilustruje dobijene rezultate.

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