Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 2(1988), 31-32

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REDUCED PRODUCTS MODULO MEASURES

(Received 15.2.1988.)

The Rudin-Keisler order $\leq_{RK}$ of ultrafilters over a given index set is a complexity hierarchy which classifies a lot of diferent ultrafilters.

A number of ultrafilter properties can be transfered to other kinds of measures. In such a way diferent and unusual real valued measures are obtained. Similarly to ultrafilters there is a need for classification. One, rather natural way, for which we belive to be promissing, is the following.

For measures $\mu, \nu : P(k) \to [0,1]$ (patial or total, but nontrivial) we define the order relation:

$$\nu \leq_J \mu \quad \text{iff } (\exists f \in {}^k k)(\forall y \in P(k))(\nu(y) = \mu(f^{-1}(y))).$$

If $\nu \leq_J \mu$ and $\mu \leq_J \nu$, we write $\nu \simeq_J \mu$. Clearly, $\simeq_J$ is an equivalence relation in the set

$$M(k) = \{\mu : \mu : P(k) \to [0,1] \text{ and } \mu \text{ is a measure}\}$$

and $\leq_J$ induces a partial order in the quotient structure which can be denoted with the same symbol. Let $\tau(\mu) = \{\nu : \nu \simeq_J \mu\}$ and $\beta(k)$ be the set of ultrafilters over k. Then the following facts could be established:

1. $\forall \mu \in M(k)\ |\tau_{\simeq_J}(\mu)| \leq 2^k$,
2. $|M(k)/\simeq_J| = 2^{2^k}$,
3. $\mu \in \beta(k) \Rightarrow \tau_{\simeq_{RK}}(\mu) = \tau_{\simeq_J}(\mu)$,
4. $\beta(k)/\simeq_{RK} \subsetneq M(k)/\simeq_J$ (initial segment),
5. $\tau_J(\nu) \leq_J \tau_J(\mu) \Rightarrow \text{add}(\nu)\ \ \text{add}(\mu)$ and $\|\nu\| \leq \|\mu\|$.

Measure additivity and the norm are defined as follows

$$\text{add}(\mu) = \min\{|x| : x \subseteq \text{dom}(\mu) \text{ and } \mu(\bigcup x) > 0 \text{ and } (\forall y \in x)(\mu(y) = 0\},$$

$$\|\mu\| = \min\{|x| : x \in \text{dom}(\mu) \text{ and } \mu(x) > 0\}.$$

The position of an ultrafilter in the Rudin-Keisler order is nicely related to the ultrafilter's combinatory properties and to the structure of ultrapowers of its index with some well ordering, motivating to expect the similar for measures. Hence, we introduce the reduced products modulo measures.

AMS Subject Classification (1980): 04A20

Let $\mathcal{M}$ be a partial or total measure over some index set S and let $X_i, i \in S$ be some sets. For $f, g \in \prod_{i \in S} X_i$ define : $f =_{\mathcal{M}} g$ iff $\mathcal{M}\{i \in S : f(i) = g(i)\} = 1$. $=_{\mathcal{M}}$ is an equivalence relation, so we denote the class of f with $f_{\mathcal{M}}$. The set $\prod_{i \in S} X_i /=_{\mathcal{M}}$ we call the (reduced if $\mathcal{M}$ is partial) measure product of sets X_i, modulo measure $\mathcal{M}$. If all $X_i = X$, the product is power. As in the case of reduced products we can define the reduced measure products of models. Let $\mathfrak{U}_i = \{A_i, G_i, \ldots R_i, \ldots, a_i, \ldots\}$, for $i \in S$ be models of a language L. Define G, R and a with

$G(\vec{f}_{\mathcal{M}}) = \langle G_i(\vec{f}(i)) : i \in S \rangle_{\mathcal{M}}$; $R(\vec{f}_{\mathcal{M}})$ iff $\{i \in S : R_i(\vec{f}(i))\} = 1$ and
$a = \langle a_i : i \in S \rangle_{\mathcal{M}}$.

Then we can define a (reduced if $\mathcal{M}$ is partial) measure product of models $\mathfrak{U}_i$, for $i \in S$, modulo $\mathcal{M}$ with: $\prod_{\mathcal{M}} \mathfrak{U}_i = \langle \prod_{\mathcal{M}} A_i, G, \ldots, R, \ldots, a, \ldots \rangle$ (which is a model for L).

The full Theorem of Los does not hold in the general case, i.e. for arbitrary measure $\mathcal{M}$, which limits the possible application of this model construction. However, we expect that this model construction should provide some interesting models in certain special cases,eg. when applied to various order relations.

Let $\mathcal{M}$ be a measure over k. We consider $\prod_{\mathcal{M}} (k, <) = (K, \prec)$. Clearly, $\prec$ is a partial order with 0 and has some nice layers; we define

$$F_\alpha = F_\alpha^{\mathcal{M}} = \prod_{\mathcal{M}} (\alpha, <) - \bigcup_{\beta < \alpha} \prod_{\mathcal{M}} \beta \quad .$$

Then $K = \bigcup_{\alpha \leq k} F_\alpha$ is a union of disjoint layers. If $\mathcal{M}$ is an ultrafilter then F_α's are intervals, singletons for α a successor and thicker or empty intervals for α a limit ordinal. If $\mathcal{M}$ is an arbitrary measure discretness is preserved: between $f_{\mathcal{M}}$ and $(f+1)_{\mathcal{M}}$ there are no other classes. Thus $(f+1)_{\mathcal{M}}$ is an immediate successor of $f_{\mathcal{M}}$. For α a limit, F_α contains some nice chains. For all $f_{\mathcal{M}} \in K$, $\{g_{\mathcal{M}} : g_{\mathcal{M}} \prec f_{\mathcal{M}}\} \cong \prod_{\mathcal{M}} (f(i), <)$. Important in RK order are normality conditions for ultrafilters and it should be interesting to determine to what extent can the theory of P points be transfered to this context. On the other hand, we hope that understanding of the measure products structure should give some important facts on the measure position in

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