Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 2(1988), 3-7

Stojan Bogdanović

# GENERALIZED U-SEMIGROUPS

(Received 10.5.1988.)

Abstract. In this paper we consider semigroups with the following condition
$$(\forall x,y)(\exists m)\ (xy)^m \in \langle x\rangle \cup \langle y\rangle .$$

In conection with a lettice of subsemigroups of some semigroup the imporatant position is captured by U-semigroups. A semigroup $S$ is a U-semigroup if the union of every two subsemigroups of $S$ is a subsemigroup of $S$, equivalently $xy \in \langle x\rangle \cup \langle y\rangle$ for all $x,y \in S$. These semigroups have been considered more a time, predominantly in special cases. More detaildly about these we can find in the book of M.Petrich, [4].

In this paper we consider a generalization of U-semigroups.

By $Z^+$ we denote the set of all positive integers. By $Reg(S)$ $(Gr(S)$, $E(S))$ we denote the set of all regular (completely regular, idempotent) elements of a semigroup $S$.

For non defined notions and notations we refer to [3] and [4].

AMS Subject Classification (1980): Primary 20M

DEFINITION 1. $S$ *is a* GU-*semigroup* (*generalized* U-*semigroup*) *if for every* $x,y\in S$ *there exists* $m\in Z^+$ *such that*

$$(1) \qquad (xy)^m \in \langle x\rangle \cup \langle y\rangle \ .$$

LEMMA 1. *Every subsemigroup and every homomorphic image of a* GU-*semigroup is a* GU-*semigroup*.

A *singular band* is a semigroup which is either a left or a right zero semigroup.

A semigroup $S = \bigcup_{\alpha\in Y} S_\alpha$ is an *ordinal sum of semigroups* $S_\alpha$, $\alpha\in Y$, if $Y$ is a chain, $S_\alpha \cap S_\beta = \emptyset$ if $\alpha\neq\beta$ and for any $a\in S_\alpha$, $b\in S_\beta$, $\alpha<\beta$ implies $ab = ba = b$, [4].

LEMMA 2.[5]. $S$ *is an ordinal sum of singular bands if and only if*

$$(\forall x,y\in S)\ xy\in\{x,y\}.$$

LEMMA 3. *Let* $S$ *be a* GU-*semigroup and let* $E(S)\neq\emptyset$. *Then*

$$(\forall e,f\in E(S))\ ef\in\{e,f\}.$$

A semigroup $S$ is *power joined* if for every $x,y\in S$ there exist $m,n\in Z^+$ such that $x^m = y^n$, [4].

LEMMA 4. *A power joined semigroup is a* GU-*semigroup*.

COROLLARY 1. *A nil-semigroup is a* GU-*semigroup*.

COROLLARY 2. *A periodic group is a* GU-*semigroup*.

LEMMA 5. *Let* $G$ *be a group with the identity element* $e$. *Then* $G$ *is a* GU-*semigroup if and only if* $G$ *is periodic*.

*Proof.* Let $G$ be a GU-semigroup. Then for any $x\in G$ there exist $m,p,q\in Z^+$ such that $e = (xx^{-1})^m = x^p$ or $e = (xx^{-1})^m = (x^{-1})^q$. So $G$ is periodic.

The converse follows by Corollary 2.

In any semigroup $S$, define a relation $\mathcal{K}$ by :

$$a\,\mathcal{K}\,b \iff (\exists m,n\in Z^+)\ a^m = b^n .$$

It is immediate that $\mathcal{K}$ is an equivalence relation.

THEOREM 1. $S$ *is a left zero band of power joined semigroups if and only if*

$$(2) \qquad (\forall x,y\in S)(\exists m\in Z^+)\ (xy)^m \in \langle x\rangle .$$

*Proof.* Let $S$ be a left zero band $Y$ of power joined semigroups

$S_\alpha$, $\alpha \in Y$. For $x \in S_\alpha$, $y \in S_\beta$ we have $xy \in S_{\alpha\beta} = S_\alpha$, whence $(xy)^m = x^n$ for some $m,n \in Z^+$. Thus the condition (2) holds.

Conversely, let $S$ satisfy (2). Let $x \mathcal{K} y$. Then $(xy)^m = x^n$, for some $m,n \in Z^+$. So each $\mathcal{K}$-class is a power joined subsemigroup of $S$. We shall show that $\mathcal{K}$ is a congruence on $S$. Suppose $x \mathcal{K} y$ and $z \in S$. Then $x^m = y^n$ for some $m,n \in Z^+$, and by (2) we have that

$$(xz)^k = x^p \quad , \quad (yz)^r = y^q$$

for some $k,p,r,q \in Z^+$. Hence

$$(xz)^{mkq} = x^{pmq} = y^{pnq} = (yz)^{pnr} \quad .$$

Thus $xz \mathcal{K} yz$. On the other hand $(zx)^s = z^t$ and $(zy)^i = z^j$ for some $s,t,i,j \in Z^+$. So $(zx)^{sj} = z^{tj} = (zy)^{it}$, i.e. $zx \mathcal{K} zy$. Therefore, $\mathcal{K}$ is a congruence and since $a \mathcal{K} a^2$ for every $a \in S$, we have that $S$ is a band of power joined semigroups. It is clear that $S$ is a left zero band of power joined semigroups.

COROLLARY 3. $S$ _is a periodic left group if and only if_ $S$ _is a left zero band of periodic groups._

LEMMA 6. $S$ _is a completely simple_ GU-_semigroup if and only if_ $S$ _is a left or a right periodic group._

_Proof._ Let $S$ be a completely simple GU-semigroup. Then by Lemmas 1. and 5. $S$ is periodic. By Theorem 7.1. [1] every subsemigroup of $S$ is simple. So $E(S)$ is simple and by Lemmas 2. and 3. we have that $E(S)$ is a left or a right zero band. By Theorem IV 3.9. [3] we have that $S$ is a periodic left or right group.

Conversely, let $S$ be a periodic left (right) group. Then by Corollary 3. $S$ is a left (right) zero band of periodic groups. By Theorem 1. $S$ is a GU-semigroup.

DEFINITION 2. _A chain_ $Y$ _of semigroups_ $S = \bigcup_{\alpha \in Y} S_\alpha$ _is a_ GU-_chain of semigroups if for every_ $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$ _there exists_ $m \in Z^+$ _such that_ $(xy)^m \in \langle x \rangle \cup \langle y \rangle$.

THEOREM 2. $S$ _is a regular_ GU-_semigroup if and only if_ $S$ _is a_ GU-_chain of periodic left and right groups._

_Proof._ Let $S$ be a regular GU-semigroup. Let $a = axa$. Then by Lemma 3. we have that $ax = axxa$ or $xa = axxa$. So $a = axa = ax^2aa = ax^2a^2$ or $a = a^2x^2a$. Thus $S$ is a completely regular semigroup. By Lemmas 1. and 5. we have that $S$ is periodic. Since $S$ is a semilattice of completely simple semigroups we have by Lemma 6. that $S$ is a semilattice of periodic left and right groups. By Lemmas 2. and 3. and by

Theorem 5.2.[2] we have that $S$ is a chain of periodic left and right groups. It is clear that $S$ is a GU-chain of periodic left and right groups.

The converse follows immediately.

THEOREM 3. *$S$ is a GU-semigroup if and only if $S$ is a union $Y$ of power joined semigroups $S_\alpha$, $\alpha \in Y$ and for every $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$ there exists $m \in Z^+$ such that $(xy)^m \in \langle x \rangle \cup \langle y \rangle$.*

*Proof.* Let $S$ be a GU-semigroup. Let $x \mathcal{H} y$. Then $(xy)^m = x^n$ for some $m,n \in Z^+$. So each $\mathcal{H}$-class is a power joined subsemigroup of $S$. Therefore, $S$ is a union $Y$ of power joined semigroups and it is clear that for every $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$, $\alpha,\beta \in Y$ there exists $m \in Z^+$ such that $(xy)^m \in \langle x \rangle \cup \langle y \rangle$.

The converse follows by the hypothesis and by Lemma 4.

A semigroup $S$ is *$\pi$-regular* if for every $a \in S$ there exists $m \in Z^+$ such that $a^m \in a^m S a^m$.

THEOREM 4. *The following conditions are equivalent on a semigroup $S$:*

(i) *$S$ is a $\pi$-regular GU-semigroup,*

(ii) *$S$ is a GU-chain $Y$ of semigroups $S_\alpha$, where $S_\alpha$, $\alpha \in Y$ is an ideal extension of a periodic left or right group $K_\alpha$ by a nil-semigroup and for every $x \in S_\alpha - K_\alpha$, $y \in K_\alpha$ there exist $m,n \in Z^+$ such that*

$$(xy)^m ,\ (yx)^n \in \langle x \rangle \cup \langle y \rangle , \tag{3}$$

(iii) *$S$ is a union $Y$ of ideal extensions $S_\alpha$, $\alpha \in Y$ of periodic groups by nil-semigroups and for every $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$ there exists $m \in Z^+$ such that $(xy)^m \in \langle x \rangle \cup \langle y \rangle$.*

*Proof.* (i) $\Longrightarrow$ (ii). Let $S$ be a $\pi$-regular GU-semigroup. Let $a = axa$. Then by Lemma 3. $(ax)(xa) \in \{ax, xa\}$. So $a = ax^2a^2$ or $a = a^2x^2a$. Thus $Reg(S) = Gr(S)$. Hence $S$ is a GV-semigroup. By Theorem 5.2.[2] we have that $S$ is a chain $Y$ of nil-extensions of rectangular groups $S_\alpha$, $\alpha \in Y$. By Lemma 6. $S_\alpha$, $\alpha \in Y$ is an ideal extension of a periodic left or right group $K_\alpha$. It is clear that (3) holds and that $S$ is a GU-chain $Y$ of semigroups $S_\alpha$, $\alpha \in Y$.

(ii) $\Longrightarrow$ (i). This implication follows by Lemma 5. and Corollary1. and by (3).

(i) $\Longrightarrow$ (iii). By equivalence (i) $\Longleftrightarrow$ (ii) we have that $S$ is periodic. Then by Theorem 3. we have the assertion.

(iii) $\Longrightarrow$ (i). This implication follows immediately.

## R E F E R E N C E S


1. S.Bogdanović and S.Gilezan,Semigroups with completely simple kernel, Zbornik radova PMF Novi Sad 12(1982),429-445.
2. S.Bogdanović,Semigroups of Galbiati-Veronesi,Proc.of the conference "Algebra and Logic",Zagreb, 1984,9-20.
3. M.Petrich,Introduction to semigroups,Merill,Ohio,1973.
4. M.Petrich,Lectures in semigroups,Akad.-Verlag,Berlin,1977.
5. L.Rédei,Algebra I , Pergamon Press Oxford, 1967, pp. 81.


## GENERALISANE U-POLUGRUPE

U vezi sa izučavanjem mreže podpolugrupa neke polugrupe važnu ulogu igraju U-polugrupe.Polugrupa $S$ je U-polugrupa ako je unija svake dve podpolugrupe iz $S$ podpolugrupa od $S$ ,što je ekvivalentno sa $xy \in \langle x\rangle \cup \langle y\rangle$ za svaki $x,y \in S$ . Ovde se razmatra jedno uopštenje U-polugrupe,tj. razmatraju se polugrupe u kojima važi zakon:

$$(\forall x,y)(\exists m)\ (xy)^m \in \langle x\rangle \cup \langle y\rangle .$$


Matematički institut
11000 Beograd
Knez Mihailova 35
Yugoslavia