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ANTISYMMETRY AND PRESENTATIONS OF  $G_{32}$  LAYER SYMMETRY GROUPS

( Received 26.04.1988 )

The presentations and structures of layer symmetry groups  $G_{32}$  are defined by the use of antisymmetry groups of ornaments  $G'_2$ .

## 1. INTRODUCTION

The idea of antisymmetry, introduced with the representations of the  $G_{321}$  band and  $G_{32}$  layer symmetry groups using Weber black-white diagrams, i.e. by the  $G'_2$  antisymmetric friezes and  $G'_2$  ornaments, is realised in the works of H. Heesch and A. V. Shubnikov. The established relation between the  $G'_{r\dots}$  antisymmetry groups and  $G_{(r+1)r\dots}$  symmetry groups of the  $(r+1)$ -dimensional space was applied in the early studies of antisymmetry by H. Heesch [1, 2] for the derivation of four-dimensional  $G_{430}$  symmetry groups and for the approximate valuation of the  $G_{43}$  symmetry group number (less than 2000).

In the course of development of the antisymmetry theory, the mentioned relations have been widely used by A. V. Shubnikov, A. M. Zamorzaev, A. F. Palistrant and numerous other authors. After the multiple antisymmetry theory, introduced by A. M. Zamorzaev [3], the mentioned relation, which makes possible the dimensional changes, is generalized in the works of A. M. Zamorzaev and A. F. Palistrant [3, 4, 5].

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AMS Subject Classification (1980): Primary 20H15

The principle of the relation between the  $G_{r\dots}^1$  multiple antisymmetry groups and subperiodic symmetry groups of the  $(r+1)$ -dimensional space is expressed by the recursive relations connecting the  $G_{r\dots}^1$  and  $G_{(r+1)r\dots}^{1-1}$  (multiple) antisymmetry categories.

By gathering the original results and works of numerous authors, H.S.M. Coxeter and W.O.J. Moser in their monograph [6] have given the most complete info on discrete group presentations, where the presentations and structures of the  $G_2$  ornamental symmetry groups [6, Chapter 4] are discussed.

In this paper the presentations and structures of the  $G_{r\dots}$  symmetry groups are the basis of presentation and structure defining of the  $G_{(r+1)r\dots}$  symmetry groups. The suggested procedure makes possible the generalization, i.e. the application of the  $G_{r\dots}^1$  multiple antisymmetry groups, connecting the  $G_{r\dots}$  symmetry groups of the  $r$ -dimensional space with subperiodic symmetry groups of the  $(r+1)$ -dimensional space. The first of the given possibilities is the object of this study, while the second, generalization, exceeds its domain.

In Section 2 are given the theoretical assumptions, making possible the use of presentations and structures of the  $G_{r\dots}$  symmetry and  $G'_{r\dots}$  antisymmetry groups for defining the presentations and structures of the  $G_{(r+1)r\dots}$  symmetry groups. Section 3 contains illustrations of this universal procedure and its practical results: the defining of the presentations and structures of the 80  $G_{32}$  layer symmetry groups obtained by using the presentations and structures of the 17  $G_2$  ornamental symmetry groups and  $G'_2$  antisymmetry groups. The final results are represented in the catalogue, in Section 4.

## 2. THE UNIVERSAL METHOD OF PRESENTATION AND STRUCTURE DEFINING OF $G_{(r+1)r\dots}$ SYMMETRY GROUPS BY USE OF $G_{r\dots}$ SYMMETRY AND $G'_{r\dots}$ ANTISSYMMETRY GROUPS

Every symmetry group  $G$  is given by the presentation:

$$\{s_1, s_2, \dots, s_m\} \quad g_k(s_1, s_2, \dots, s_m) = E \quad k=1, 2, \dots, s \quad (1)$$

Group  $G$  and all groups derived from  $G$  by introducing the  $e_1$  antiidentity transformation which satisfies the relations:

$$e_1^2 = E \quad e_1 S_i = S_i e_1 \quad i=1, 2, \dots, m \quad (2)$$

are antisymmetry groups.

Antisymmetry groups are divided into three types: the generating (i.e. symmetry groups), senior and junior groups [3,4]. If G is the generating group, then the senior or antisymmetry group derived from G is of the  $Gx\{e_1\}$  form. All junior antisymmetry groups  $G'_j$  ( $j \in N$ ), derived from G are isomorphous with the G group [3].

Let the symmetry group G be given by the presentation (1) and the relations (2) are satisfied. The  $e_1$  antiidentity transformation can be identified with the (hyper) plane reflection [7]:

$$R_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad \dim R_1 = (r+1) \times (r+1)$$

If group G is the symmetry group of the r-dimensional space, the member of the  $G_{r\dots}$  category, in accordance with [3,4,5], the senior antisymmetry group of the  $Gx\{e_1\}$  form defines the corresponding symmetry group of the  $(r+1)$ -dimensional space, the member of the  $G_{(r+1)r\dots}$  category. These two groups have the following presentations respectively:

$$\begin{array}{ccc} \{S_1, S_2, \dots, S_m\} \times \{e_1\} & \cong & \{S_1, S_2, \dots, S_m\} \times \{R_1\} \\ g_k(S_1, S_2, \dots, S_m) = E & & g_k(S_1, S_2, \dots, S_m) = E \quad k=1, 2, \dots, s \\ e_1^2 = E \quad e_1 S_i = S_i e_1 & & R_1^2 = E \quad R_1 S_i = S_i R_1 \quad i=1, 2, \dots, m \end{array}$$

and the structure  $GxD_1$ .

The junior antisymmetry group  $G'_j$  ( $j \in N$ ), which is isomorphous with the generating G group, as well as the corresponding symmetry group of the  $(r+1)$ -dimensional space, the member of the  $G_{(r+1)r\dots}$  category, have the presentation:

$$\{S'_1, S'_2, \dots, S'_m\} \quad g_k(S'_1, S'_2, \dots, S'_m) = E \quad k=1, 2, \dots, s$$

and the same structure as group G. The generator set  $\{S'_1, S'_2, \dots, S'_m\}$  consists of (anti)generators  $S'_i$  ( $S'_i = S_i$  or  $S'_i = e_1 S_i$ ,  $i=1, 2, \dots, m$ ) or corresponding symmetries  $S'_i$  of the  $(r+1)$ -dimensional space ( $S'_i = S_i$  or  $S'_i = R_1 S_i$ ,  $i=1, 2, \dots, m$ ) respectively, obtained by the identification  $e_1 = R_1$ .

By using the minimal generator set of the G group, we can obtain presentations of subperiodic  $G_{(r+1)r\dots}$  symmetry groups with minimal generator sets.

If the number and structures of non-isomorphous groups of the  $G_{r\dots}$  category are known, we can state the number of non-isomorphous  $G_{(r+1)\dots}$  category groups.

### 3. PRESENTATIONS AND STRUCTURES OF $G_{32}$ LAYER SYMMETRY GROUPS

In order to define the presentations and structures of the 80  $G_{32}$  layer symmetry groups, the mentioned theoretical suppositions have been used. The 17  $G_2$  ornamental symmetry groups, given with one or several different presentations, comprising the structure symbols [6, Chapter 4] and the 46  $G'_2$  ornamental junior antisymmetry groups, obtained by antisymmetric characteristic method [8] are the basis of these defining.

The final result is a catalogue of the 80  $G_{32}$  layer symmetry groups with presentations and structure symbols. Besides these data, the catalogue also gives complete information on the derivation process and relations existing between the  $G'_2$  ornamental antisymmetry groups and  $G_{32}$  layer symmetry groups.

The concept of the catalogue can be illustrated with the example of the layer symmetry groups, derived from the III) pm ornamental symmetry group. The III) pm ornamental symmetry group is given by two different presentations [6, Chapter 4]:

$$8) \quad \{X, Y, R\} \quad XY=YX \quad R^2=E \quad RXR=X^{-1} \quad RYR=Y \\ 8') \quad \{Y, R, R'\} \quad R^2=R'^2=E \quad RY=YR \quad R'Y=YR'$$

and has the structure  $D_\infty \times C_\infty$ .

Regarding the III) pm  $G_2$  ornamental symmetry group as the generating symmetry group, the 8.Pm11 [3, 9] $G_{32}$  layer symmetry group of the  $D_\infty \times C_\infty$  structure, with the same two presentations:

$$8) \quad pm=\{X, Y, R\} \quad 8.Pm11 \quad D_\infty \times C_\infty \\ 8') \quad pm=\{Y, R, R'\}$$

is obtained.

The next 24.Pm2m layer symmetry group of the  $D_\infty \times C_\infty \times D_1$  structure is obtained from the  $pm \times \{e_1\}$  senior ornamental antisymmetry group by identification of the  $e_1$  antiidentity transformation with the  $R_1$  plane reflection in the invariant plane of the ornament pm. The result are two different presentations of the 24.Pm2m layer sym-

metry group:

$$9) \quad pm \times D_1 = \{X, Y, R\} \times \{R_1\}$$

$$9') \quad pm \times D_1 = \{Y, R, R'\} \times \{R_1\}$$

$$24. Pm2m \quad D_\infty \times C_\infty \times D_1$$

In accordance with the direct group product definition [6] beside the 8) or 8') relations, defining the pm ornamental symmetry group, in the presentations 9), 9') the relations  $X, Y, R \rightleftharpoons R_1$  or  $Y, R, R' \rightleftharpoons R_1$  are satisfied respectively.

The following five 10)-14)  $G_{32}'$  layer symmetry groups are derived from the corresponding junior antisymmetry groups, generated from the pm by applying the antisymmetric characteristic method [8]. One or several generator sets and the group/subgroup symbol of the corresponding  $G_2'$  junior antisymmetry group [10] are given in the first line of the catalogue. The second line contains the related generator set(s) and the crystallographic symbol [3, 9] of the layer symmetry group resulting from the identification of the  $e_1$  with  $R_1$ . Since junior antisymmetry groups are isomorphous with the generating symmetry group, the same applies for the related layer symmetry groups. It means that they posses generators, given in the same order, which satisfy the same defining relations as the generators of the generating  $G_2$  ornamental symmetry group, as well as the same structure as the generating group.

In the catalogue, the translations are denoted by X, Y, Z (except for the 40"), 41"), 42"), 43"), 44") layer symmetry groups, where the S denotes a translation), the reflections by R, the half-turns by T, the glide reflections by P, Q, O, the central inversion by I, the rotations by S (except for the 40"), 41"), 42"), 43"), 44") layer symmetry groups), the  $2_1$  rotational reflection by V, W and rotational reflections by  $\bar{S}$ .

When denoting the  $G_2'$  ornamental antisymmetry groups, the group/subgroup symbols [10] give the complete information, except for the 11), 13) antisymmetry groups, where by additional mark m1 or 1m [10] is denoted the symmetric frieze subgroup of the considered antisymmetric ornament.

With regard to relation of isomorphism, since the  $G_2$  category consists of the 17 non-isomorphous ornamental groups [6, Chapter 4], the  $G_{32}$  category consists of the 34 [3] non-isomorphous layer symmetry groups. The presentations and structures of the 80 layer symmetry groups can be used for the next dimensional change, in order to obtain

by the same procedure the presentations and structures of the  $G_{432}$  four-dimensional superperiodic groups, applying the presentations and structures of the  $G_{32}$  layer symmetry groups and  $G'_{32}$  antisymmetry groups [9] etc. The suggested method, applied on the different well known  $G_{r\dots}$  symmetry group categories can be a very efficient instrument for the analysis of the multi-dimensional groups [5].

#### 4. CATALOGUE OF PRESENTATIONS AND STRUCTURES OF $G_{32}$ LAYER SYMMETRY GROUPS

I) p1	1)	$p1=\{X, Y\}$	1.P1	$C_\infty^2$
	1')	$p1=\{X, Y, Z\}$		
	2)	$p1xD_1=\{X, Y\}x\{R_1\}$	7.P11m	$C_\infty^2 \times D_1$
	2')	$p1xD_1=\{X, Y, Z\}x\{R_1\}$		
	3)	$\{X, e_1 Y\}$	$p1/p1$	
		$\{X, P\}$	10.P11b	$C_\infty^2$
	3')	$\{X, e_1 Y, e_1 Z\}$		
		$\{X, P, Q\}$		
II) p2	4)	$p2=\{X, Y, T\}$	3.P112	
	4')	$p2=\{T_1, T_2, T_3\}$		
	4'')	$p2=\{T_1, T_2, T_3, T_4\}$		
	5)	$p2xD_1=\{X, Y, T\}x\{R_1\}$	12.P11 $\frac{2}{m}$	
	5')	$p2xD_1=\{T_1, T_2, T_3\}x\{R_1\}$		
	5'')	$p2xD_1=\{T_1, T_2, T_3, T_4\}x\{R_1\}$		
	6)	$\{X, Y, e_1 T\}$	$p2/p1$	
		$\{X, Y, I\}$	2.P1	
	6')	$\{e_1 T_1, e_1 T_2, e_1 T_3\}$		
		$\{I_1, I_2, I_3\}$		
	6'')	$\{e_1 T_1, e_1 T_2, e_1 T_3, e_1 T_4\}$		
		$\{I_1, I_2, I_3, I_4\}$		
	7)	$\{X, e_1 Y, T\}=\{e_1 X, e_1 Y, T\}=\{X, e_1 Y, e_1 T\}=\{e_1 X, e_1 Y, e_1 T\}$	$p2/p2$	
		$\{X, Q, T\}=\{P, Q, T\}=\{X, Q, I\}=\{P, Q, I\}$		16.P11 $\frac{2}{b}$

	7')	$\{e_1 T_1, T_2, T_3\} = \{e_1 T_1, e_1 T_2, T_3\}$		
		$\{I_1, T_2, T_3\} = \{I_1, I_2, T_3\}$		
	7'')	$\{e_1 T_1, e_1 T_2, T_3, T_4\}$		
		$\{I_1, I_2, T_3, T_4\}$		
III) pm	8)	$pm = \{X, Y, R\}$	8.Pm11	$D_\infty \times C_\infty$
	8'')	$pm = \{Y, R, R'\}$		
	9)	$pm \times D_1 = \{X, Y, R\} \times \{R_1\}$	24.Pm2m	$D_\infty \times C_\infty \times D_1$
	9'')	$pm \times D_1 = \{Y, R, R'\} \times \{R_1\}$		
	10)	$\{X, Y, e_1 R\}$	pm/pl	
		$\{X, Y, T\}$	4.P121	$D_\infty \times C_\infty$
	10'')	$\{Y, e_1 R, e_1 R'\}$		
		$\{Y, T, T'\}$		
	11)	$\{X, e_1 Y, R\}$	pm/pm(m1)	
		$\{X, Q, R\}$	25.Pm2 <sub>1</sub> b	$D_\infty \times C_\infty$
	11'')	$\{e_1 Y, R, R'\}$		
		$\{Q, R, R'\}$		
	12)	$\{X, e_1 Y, e_1 R\}$	pm/pg	
		$\{X, Q, T\}$	26.Pb2b	$D_\infty \times C_\infty$
	12'')	$\{e_1 Y, e_1 R, e_1 R'\}$		
		$\{Q, T, T'\}$		
	13)	$\{e_1 X, Y, R\} = \{e_1 X, Y, e_1 R\}$	pm/pm(1m)	
		$\{P, Y, R\} = \{P, Y, T\}$		
	13'')	$\{Y, R, e_1 R'\}$		
		$\{Y, R, T'\}$		
	14)	$\{e_1 X, e_1 Y, R\} = \{e_1 X, e_1 Y, e_1 R\}$	pm/cm	
		$\{P, Q, R\} = \{P, Q, T\}$	36.Cm2a	$D_\infty \times C_\infty$
	14'')	$\{e_1 Y, e_1 R, R'\}$		
		$\{Q, T, R'\}$		
IV) pg	15)	$pg = \{X, Y, P\}$	9.Pb11	$\langle 2, 2, \infty \rangle$
	15'')	$pg = \{P, Q\}$		
	16)	$pg \times D_1 = \{X, Y, P\} \times \{R_1\}$	29.Pb2 <sub>1</sub> m	$\langle 2, 2, \infty \rangle \times D_1$
	16'')	$pg \times D_1 = \{P, Q\} \times \{R_1\}$		

	17)	$\{X, Y, e_1 P\}$	pg/p1
		$\{X, Y, V\}$	$5.P12_1^1 \quad \langle 2, 2, \infty \rangle$
	17')	$\{e_1 P, e_1 Q\}$	
		$\{V, W\}$	
	18)	$\{e_1 X, Y, P\} = \{e_1 X, Y, e_1 P\}$	pg/pg
		$\{P_1, Y, P\} = \{P_1, Y, V\}$	$30.Pb2_1 a \quad \langle 2, 2, \infty \rangle$
	18')	$\{e_1 P, Q\}$	
		$\{V, Q\}$	
V) p4g	19)	$p4g = \{R_1, R_2, R_3, R_4, S\}$	$56.P4bm \quad [4^+, 4]$
	19')	$p4g = \{R, S\}$	
	20)	$p4gx D_1 = \{R_1, R_2, R_3, R_4, S\} \times \{R'\}$	$63.P\frac{4}{m}bm \quad [4^+, 4]_{xD_1}$
	20')	$p4gx D_1 = \{R, S\} \times \{R'\}$	
	21)	$\{R_1, R_2, R_3, R_4, e_1 S\}$	pg/cmm
		$\{R_1, R_2, R_3, R_4, \bar{S}\}$	$58.P\bar{4}2_1 m \quad [4^+, 4]$
	21')	$\{R, e_1 S\}$	
		$\{R, \bar{S}\}$	
	22)	$\{e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4, S\}$	pg/pg
		$\{T_1, T_2, T_3, T_4, S\}$	$54.P42_1 2 \quad [4^+, 4]$
	22')	$\{e_1 R, S\}$	
		$\{T, S\}$	
	23)	$\{e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4, e_1 S\}$	pg/pgg
		$\{T_1, T_2, T_3, T_4, \bar{S}\}$	$60.P\bar{4}b2 \quad [4^+, 4]$
	23')	$\{e_1 R, e_1 S\}$	
		$\{T, \bar{S}\}$	
VI) p3	24)	$p3 = \{X, Y, Z, S_1\}$	$65.P3 \quad \Delta^+$
	24')	$p3 = \{S_1, S_2\}$	
	25)	$p3xD_1 = \{X, Y, Z, S_1\} \times \{R_1\}$	$74.P\bar{6} \quad \Delta^+_{xD_1}$
	25')	$p3xD_1 = \{S_1, S_2\} \times \{R_1\}$	
VII) p31m	26)	$p31m = \{S_1, S_2, R\}$	$70.P31m \quad [3^+, 6]$
	26')	$p31m = \{R, S\}$	
	27)	$p31mx D_1 = \{S_1, S_2, R\} \times \{R_1\}$	$79.P\bar{6}2m \quad [3^+, 6]_{xD_1}$
	27')	$p31mx D_1 = \{R, S\} \times \{R_1\}$	

	28) $\{S_1, S_2, e_1 R\}$ $\{S_1, S_2, T\}$	p31m/p3 68.P 321	$[3^+, 6]$
	28') $\{e_1 R, S\}$ $\{T, S\}$		
VIII) p3m1	29) $p3m1 = \{S_1, S_2, R\}$	69.P 3m1	$\Delta$
	29') $p3m1 = \{R_1, R_2, R_3\}$		
	30) $p3m1xD_1 = \{S_1, S_2, R\} \times \{R'\}$	78.P $\bar{6}m2$	$\Delta \times D_1$
	30') $p3m1xD_1 = \{R_1, R_2, R_3\} \times \{R'\}$		
	31) $\{S_1, S_2, e_1 R\}$ $\{S_1, S_2, T\}$	p3m1/p3 67.P 312	$\Delta$
	31') $\{e_1 R_1, e_1 R_2, e_1 R_3\}$ $\{T_1, T_2, T_3\}$		
IX) p6	32) $p6 = \{S_1, S_2, T\}$	73.P 6	$[6, 3]^+$
	32') $p6 = \{S, T\}$		
	33) $p6xD_1 = \{S_1, S_2, T\} \times \{R_1\}$	75.P $\frac{6}{m}$	$[6, 3]^+ \times D_1$
	33') $p6xD_1 = \{S, T\} \times \{R_1\}$		
	34) $\{S_1, S_2, e_1 T\}$ $\{S_1, S_2, I\}$	p6/p3 66.P $\bar{3}$	$[6, 3]^+ \times D_1$
	34') $\{S, e_1 T\}$ $\{S, I\}$		
X) p6m	35) $p6m = \{R, R_1, R_2, R_3\}$	77.P 6m	$[6, 3]$
	35') $p6m = \{R, R_1, R_2\}$		
	36) $p6mxD_1 = \{R, R_1, R_2, R_3\} \times \{R'\}$	80.P $\frac{6}{m}$ mm	$[6, 3] \times D_1$
	36') $p6mxD_1 = \{R, R_1, R_2\} \times \{R'\}$		
	37) $\{e_1 R, R_1, R_2, R_3\}$ $\{T, R_1, R_2, R_3\}$	p6m/p3m1 72.P $\bar{3}m1$	$[6, 3]$
	37') $\{e_1 R, R_1, R_2\}$ $\{T, R_1, R_2\}$		
	38) $\{R, e_1 R_1, e_1 R_2, e_1 R_3\}$ $\{R, T_1, T_2, T_3\}$	p6m/p31m 71.P $\bar{3}1m$	$[6, 3]$
	38') $\{R, e_1 R_1, e_1 R_2\}$ $\{R, T_1, T_2\}$		

	39) $\{e_1 R, e_1 R_1, e_1 R_2, e_1 R_3\}$ $\{T, T_1, T_2, T_3\}$	p6m/p6	
	39') $\{e_1 R, e_1 R_1, e_1 R_2\}$ $\{T, T_1, T_2\}$	76.P 622	[6,3]
XI) cm	40) cm = {P, Q, R}	11.Cm11	
	40') cm = {P, R}		
	40'') cm = {R, S}		
	41) cmxD <sub>1</sub> = {P, Q, R} x {R <sub>1</sub> }	35.Cm2m	
	41') cmxD <sub>1</sub> = {P, R} x {R <sub>1</sub> }		
	41'') cmxD <sub>1</sub> = {R, S} x {R <sub>1</sub> }		
	42) {P, Q, e <sub>1</sub> R} {P, Q, T}	cm/pg	
	42') {P, e <sub>1</sub> R} {P, T}	31.Pb2n	
	42'') {e <sub>1</sub> S, e <sub>1</sub> R} {P <sub>1</sub> , T}		
	43) {e <sub>1</sub> P, e <sub>1</sub> Q, R} {V, W, R}	cm/pm	
	43') {e <sub>1</sub> P, R} {V, R}	32.Pm2 <sub>1</sub> n	
	43'') {e <sub>1</sub> S, R} {P <sub>1</sub> , R}		
	44) {e <sub>1</sub> P, e <sub>1</sub> Q, e <sub>1</sub> R} {V, W, T}	cm/pl	
	44') {e <sub>1</sub> P, e <sub>1</sub> R} {V, T}	6.C121	
	44'') {S, e <sub>1</sub> R} {S, T}		
XII) pmm	45) pmm = {R, R', R <sub>2</sub> , Y}	23.Pmm2	D <sub>∞</sub> xD <sub>∞</sub>
	45') pmm = {R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> }		
	46) pmmxD <sub>1</sub> = {R, R', R <sub>2</sub> , Y} x {R''}	37.Pmmm	D <sub>∞</sub> xD <sub>∞</sub> xD <sub>1</sub>
	46') pmmxD <sub>1</sub> = {R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> } x {R''}		

- 47)  $\{e_1 R, R', R_2, Y\} = \{R, R', R_2, e_1 Y\} = \{R, R', e_1 R_2, e_1 Y\}$  pmm/pmm  
 $\{T, R', R_2, Y\} = \{R, R', R_2, Q\} = \{R, R', T_2, Q\}$  41.P mmb  $D_{\infty} \times D_{\infty}$   
 47')  $\{e_1 R_1, R_2, R_3, R_4\}$   
 $\{T_1, R_2, R_3, R_4\}$   
 48)  $\{R, R', e_1 R_2, Y\} = \{e_1 R, e_1 R', R_2, Y\}$  pmm/pm  
 $\{R, R', T_2, Y\} = \{T, T', R_2, Y\}$   $13.P \frac{2}{m} 11$   $D_{\infty} \times D_{\infty}$   
 48')  $\{e_1 R_1, R_2, e_1 R_3, R_4\}$   
 $\{T_1, R_2, T_3, R_4\}$   
 49)  $\{e_1 R, R', e_1 R_2, Y\} = \{e_1 R, e_1 R', R_2, e_1 Y\} = \{e_1 R, e_1 R', e_1 R_2, e_1 Y\}$ ,  
 $\{T, R', T_2, Y\} = \{T, T', R_2, Q\} = \{T, T', T_2, Q\}$   
 pmm/pmg  
 38.Pbmb  $D_{\infty} \times D_{\infty}$   
 49')  $\{e_1 R_1, e_1 R_2, e_1 R_3, R_4\}$   
 $\{T_1, T_2, T_3, R_4\}$   
 50)  $\{e_1 R, R', R_2, e_1 Y\} = \{e_1 R, R', e_1 R_2, e_1 Y\}$  pmm/cmm  
 $\{T, R', R_2, Q\} = \{T, R', T_2, Q\}$  48.Cmma  $D_{\infty} \times D_{\infty}$   
 50')  $\{e_1 R_1, e_1 R_2, R_3, R_4\}$   
 $\{T_1, T_2, R_3, R_4\}$   
 51)  $\{e_1 R, e_1 R', e_1 R_2, Y\}$  pmm/p2  
 $\{T, T', T_2, Y\}$   $19.P \frac{2}{2} 22$   $D_{\infty} \times D_{\infty}$   
 51')  $\{e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4\}$   
 $\{T_1, T_2, T_3, T_4\}$   
 XIII) pmg 52) pmg = {P, Q, R} 28.Pbm2  
 52') pmg = {T<sub>1</sub>, T<sub>2</sub>, R}  
 53) pmgx D<sub>1</sub> = {P, Q, R} x {R<sub>1</sub>} 40.P bmm  
 53') pmgx D<sub>1</sub> = {T<sub>1</sub>, T<sub>2</sub>, R} x {R<sub>1</sub>}  
 54) {e<sub>1</sub>P, Q, R} pmg/pmg  
 $\{V, Q, R\}$  45.Pbma  
 54') {e<sub>1</sub>T<sub>1</sub>, T<sub>2</sub>, R}  
 $\{I_1, T_2, R\}$   
 55) {P, Q, e<sub>1</sub>R} pmg/pg  
 $\{P, Q, T\}$   $17.P \frac{2}{b} 11$

	55') {e <sub>1</sub> T <sub>1</sub> , e <sub>1</sub> T <sub>2</sub> , e <sub>1</sub> R }	
	{I <sub>1</sub> , I <sub>2</sub> , T}	
56)	{e <sub>1</sub> P, e <sub>1</sub> Q, R}	pmg/pm
	{V, W, R}	14.P1 <sub>m</sub> <sup>21</sup>
56')	{e <sub>1</sub> T <sub>1</sub> , e <sub>1</sub> T <sub>2</sub> , R}	
	{I <sub>1</sub> , I <sub>2</sub> , R}	
57)	{e <sub>1</sub> P, Q, e <sub>1</sub> R}	pmg/pgg
	{V, Q, T}	43.Pbaa
57')	{e <sub>1</sub> T <sub>1</sub> , T <sub>2</sub> , e <sub>1</sub> R }	
	{I <sub>1</sub> , T <sub>2</sub> , T}	
58)	{e <sub>1</sub> P, e <sub>1</sub> Q, e <sub>1</sub> R}	pmg/p2
	{V, W, T}	21.P22 <sub>1</sub> 2
58')	{T <sub>1</sub> , T <sub>2</sub> , e <sub>1</sub> R}	
	{T <sub>1</sub> , T <sub>2</sub> , T}	
XIV) pgg	59) pgg={P, Q, T}	33.Pba2 (oo,oo 2,2)
	59') pgg={P, O}	
60)	pggx <sub>D<sub>1</sub></sub> ={P, Q, T}x{R <sub>1</sub> }	44.Pbam (oo,oo 2,2)x <sub>D<sub>1</sub></sub>
60')	pggx <sub>D<sub>1</sub></sub> ={P, O}x{R <sub>1</sub> }	
61)	{P, Q, e <sub>1</sub> T}	pgg/pg
	{P, Q, I}	18.P <sub>b</sub> <sup>21</sup> 11 (oo,oo 2,2)
61')	{e <sub>1</sub> P, O}	
	{V, O}	
62)	{e <sub>1</sub> P, e <sub>1</sub> Q, T}	pgg/p2
	{V, W, T}	20.P2 <sub>1</sub> <sup>2</sup> <sub>1</sub> 2 (oo,oo 2,2)
62')	{e <sub>1</sub> P, e <sub>1</sub> O}	
	{V, V <sub>1</sub> }	
XV) cmm	63) cmm={R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> , T}	34.Cmm2
	63') cmm={R <sub>1</sub> , R <sub>2</sub> , T}	
64)	cmmx <sub>D<sub>1</sub></sub> ={R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> , T}x{R'}	47.Cmmm
64')	cmmx <sub>D<sub>1</sub></sub> ={R <sub>1</sub> , R <sub>2</sub> , T}x{R'}	
65)	{R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> , e <sub>1</sub> T}	cmm/pmm
	{R <sub>1</sub> , R <sub>2</sub> , R <sub>3</sub> , R <sub>4</sub> , I}	46.Pmmn

	65')	$\{R_1, R_2, e_1 T\}$ $\{R_1, R_2, I\}$		
	66)	$\{e_1 R_1, R_2, e_1 R_3, R_4, T\}$ $\{T_1, R_2, T_3, R_4, T\}$	cmm/pmg	
	66')	$\{e_1 R_1, R_2, T\}$ $\{T_1, R_2, T\}$	42.Pmbn	
	67)	$\{e_1 R_1, R_2, e_1 R_3, R_4, e_1 T\}$ $\{T_1, R_2, T_3, R_4, I\}$	cmm/cm	
	67')	$\{e_1 R_1, R_2, e_1 T\}$ $\{T_1, R_2, I\}$	$15.C_m^2 11$	
	68)	$\{e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4, T\}$ $\{T_1, T_2, T_3, T_4, T\}$	cmm/p2	
	68')	$\{e_1 R_1, e_1 R_2, T\}$ $\{T_1, T_2, T\}$	22.C222	
	69)	$\{e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4, e_1 T\}$ $\{T_1, T_2, T_3, T_4, I\}$	cmm/pgg	
	69')	$\{e_1 R_1, e_1 R_2, e_1 T\}$ $\{T_1, T_2, I\}$	39.Pban	
XVI) p4	70)	$p4=\{T_1, T_2, T_3, T_4, S\}$	49.P4	$[4, 4]^+$
	70')	$p4=\{S, T\}$		
	71)	$p4xD_1=\{T_1, T_2, T_3, T_4, S\}x\{R_1\}$	$51.P_m^{\frac{4}{m}}$	$[4, 4]^+xD_1$
	71')	$p4xD_1=\{S, T\}x\{R_1\}$		
	72)	$\{T_1, T_2, T_3, T_4, e_1 S\}$ $\{T_1, T_2, T_3, T_4, \bar{S}\}$	p4/p2	
	72')	$\{e_1 S, T\}$ $\{\bar{S}, T\}$	50.P4	$[4, 4]^+$
	73)	$\{e_1 T_1, e_1 T_2, e_1 T_3, e_1 T_4, S\}$ $\{I_1, I_2, I_3, I_4, S\}$	$p4/p4$ $52.P_n^{\frac{4}{n}}$	$[4, 4]^+$
	73')	$\{S, e_1 T\}$ $\{S, I\}$		
XVII) p4m	74)	$p4m=\{R, R_1, R_2, R_3, R_4\}$	55.P4mm	$[4, 4]$
	74')	$p4mm=\{R, R_1, R_2\}$		

75)	$p4mx D_1 = \{R, R_1, R_2, R_3, R_4\} \times \{R'\}$	$61.P\frac{4}{m} mm$	$[4,4] \times D_1$
75')	$p4mx D_1 = \{R, R_1, R_2\} \times \{R'\}$		
76)	$\{e_1 R, R_1, R_2, R_3, R_4\}$ $\{T, R_1, R_2, R_3, R_4\}$	$p4m/pmm$	
76')	$\{e_1 R, R_1, R_2\}$ $\{T, R_1, R_2\}$	$59.P\bar{4}m2$	$[4,4]$
77)	$\{R, e_1 R_1, R_2, R_3, e_1 R_4\}$ $\{R, T_1, R_2, R_3, T_4\}$	$p4m/p4m$	
77')	$\{R, e_1 R_1, R_2\}$ $\{R, T_1, R_2\}$	$64.P\frac{4}{n} mm$	$[4,4]$
78)	$\{e_1 R, e_1 R_1, R_2, R_3, e_1 R_4\}$ $\{T, T_1, R_2, R_3, T_4\}$	$p4m/p4g$	
78')	$\{e_1 R, e_1 R_1, R_2\}$ $\{T, T_1, R_2\}$	$62.P\frac{4}{n} bm$	$[4,4]$
79)	$\{R, e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4\}$ $\{R, T_1, T_2, T_3, T_4\}$	$p4m/cmm$	
79')	$\{R, e_1 R_1, e_1 R_2\}$ $\{R, T_1, T_2\}$	$57.P\bar{4}2m$	$[4,4]$
80)	$\{e_1 R, e_1 R_1, e_1 R_2, e_1 R_3, e_1 R_4\}$ $\{T, T_1, T_2, T_3, T_4\}$	$p4m/p4$	
80')	$e_1 R, e_1 R_1, e_1 R_2$ $T, T_1, T_2$	$53.P422$	$[4,4]$

#### REFERENCES

- [1] HEESCH, H.: Zur Strukturtheorie der ebenen Symmetriegruppen. Z.Kristallogr. 71 (1929), 95-102
- [2] HEESCH, H.: Zur systematischen Strukturtheorie IV. Über die Symmetrien zweiter Art in Kontinuen und Semidiskontinuen. Z.Kristallogr. 73 (1930), 399-407
- [3] ZAMORZAEV, A.M.: Teoriya prostoi i kratnoi antisymmetrii. Shtiintsa, Kishinev 1976
- [4] ZAMORZAEV, A.M. & PALISTRANT, A.F.: Antisymmetry, its generaliza-

- tions and geometrical applications. Z. Kristallogr. 151 (1980), 231-248
- [5] ZAMORZAEV, A. M.; KARPOVA, Y.S.; LUNGU, A.P. & PALISTRANT, A.F.: P-simetriya i eye dalneishee razvitiye. Shtiintsa, Kishinev 1986
  - [6] COXETER, H. S. M. & MOSER, W.O.J.: Generators and Relations for Discrete Groups. Springer-Verlag, 4th. ed., Berlin-Heidelberg-New York 1980
  - [7] SHUBNIKOV, A.V.; BELOV, N.V. et al. : Colored Symmetry. Pergamon Press, New York 1964
  - [8] JABLJAN, S.V.: A new method of generating plane groups of simple and multiple antisymmetry. Acta Crystallogr. A42 (1986), 209-212
  - [9] JABLJAN, S.V.: Groups of simple and multiple antisymmetry of layers. Z.Kristallogr. 176 (1986), 283-290
  - [10] COXETER, H. S. M.: The seventeen black and white frieze types. C.R.Math. Rep.Acad.Sci.Canada, Vol.VII ,No. 5, 1985

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#### ANTISIMETRIJA I PREZENTACIJE GRUPA SIMETRIJE SLOJEVA $G_{32}$

Prezentacije i strukture grupa simetrije slojeva  $G_{32}$  izučene su pomoću grupa antisimetrije ornamenata  $G_2'$ .

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