Filomat 37:1 (2023), 97–106
https://doi.org/10.2298/FIL2301097X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A system of matrix equations over the commutative quaternion ring

Lv-Ming Xie[a], Qing-Wen Wang[a]

[a]*Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China*

Abstract. In this paper, we propose a necessary and sufficient condition for the solvability to a system of matrix equations over the commutative quaternion ring, and establish an expression of its general solution when it is solvable. We also present an algorithm for finding an approximate solution to the system when it is inconsistent. Finally, we give an example to illustrate the main results of this paper.

1. Introduction

As well known that the Hamilton quaternion discovered in 1843 [1] was not only applied in mathematics, but also penetrated into mechanics, quantum physics, signal and color image processing, etc (e.g. [3–7]). However, the multiplication of Hamilton quaternions is not commutative which causes many difficulties in studying problems. In 1892, Segre [15] proposed another kind of quaternions satisfying the commutative property of multiplication. The set of all commutative quaternions is a ring which contains zero-divisor and isotropic elements. The collection of all commutative quaternions is a four-dimensional space over the real number field. The commutative quaternions have been widely used in signal and image processing (e.g. [8, 10]). In [2], Kosal et al. gave the complex matrix representations of a commutative quaternion and a commutative quaternion matrix, respectively. After this, Kosal et al.[12] presented a universal similarity factorization equality to give the real matrix representations of a commutative quaternion and a commutative quaternion matrix, respectively. On this basis, they gave an expression of the general solution to the commutative quaternion matrix equation $AX = B$ when it was solvable. Moreover, Kosal et al.[16] also investigated the so called the Kalman-Yakubovich-conjugate matrix equations by the real representations of commutative quaternion matrices.

We know that Sylvester-type matrix equations are widely used in system science and control theory. Wang et al. [17, 18] considered some systems of one-sided coupled Sylvester-type quaternion matrix equations. In 2019, Wang et al.[9] also found the solvable conditions and an expression of the general solution to the following two-sided coupled Sylvester-type matrix equations over the quaternion algebra,

$$\begin{cases} A_1X = C_1, A_2Y = C_2, A_3Z = C_3, \\ XB_1 = D_1, YB_2 = D_2, ZB_3 = D_3, \\ A_4XB_4 + C_4YD_4 = P, A_5ZB_5 + C_5YD_5 = Q, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification*. 15A09; 15A24, 15B33.
Keywords. (Sylvester-type matrix equations; Commutative quaternion; General solution; Approximate solution)
Received: 07 February 2021; Accepted: 03 March 2021
Communicated by Dragana Cvetković Ilić
Corresponding author: Qing-Wen Wang
Research supported by the National Natural Science Foundation of China [grant number 11971294].
Email addresses: xlm1817@163.com (Lv-Ming Xie), wqw@t.shu.edu.cn, wqw369@yahoo.com (Qing-Wen Wang)

where X, Y, Z are unknown matrices and the other matrices are given with appropriate orders. To our best knowledge, so far there has been little information on the exact solution and the approximate solution to the system (1) over the commutative quaternion ring. Motivated by this mentioned above, we in this paper consider the solvability conditions, the general solution, and the approximate solution of the system (1) over the commutative quaternion ring.

This paper is organized as follows. In Section 2, we discuss the structure of the operator vec(RYU) over the commutative quaternion ring. In Section 3, using the different method from one in [9], we present a necessary and sufficient condition for the solvability to (1) and an expression of the general solution to (1) when it is solvable. We also present an algorithm for finding an approximate solution to the system (1) when it is inconsistent. In Section 4, we provide an algorithm and a numerical example to illustrate the main results of this paper. Finally, we conclude this paper by giving some remarks in Section 5.

Throughout this paper, we denote the real number field, the complex number field, the commutative quaternion ring by $\mathbb{R}, \mathbb{C}, \mathbb{Q}_c$, respectively. We denote the set of all $m \times n$ matrices over $\mathbb{Q}_c$ ($\mathbb{C}$ or $\mathbb{R}$) by $\mathbb{Q}_c^{m\times n}$ ($\mathbb{C}^{m\times n}$ or $\mathbb{R}^{m\times n}$). For $A \in \mathbb{C}^{m\times n}$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real part and the imaginary part of A, respectively. We denote the addition of the main diagonal elements of a square matrix A by $\mathrm{tr}(A)$. We denote $A^T \in \mathbb{Q}_c^{n\times m}$ as the transpose of A. Let $A \otimes B = (a_{ij}B)$ be the Kronecker product of A and B. The Moore-Penrose inverse of $A \in \mathbb{R}^{m\times n}$, denoted by $A^\dagger$, is a unique matrix X satisfying the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX, \quad (XA)^T = XA.$$

2. Preliminary

In this section, we first recall the complex representation of the commutative quaternions, some properties associated with the commutative quaternions, and the structure of the operator vec(RYU) over $\mathbb{Q}_c$. The following notation used in this paper is as in [11].

The set of all commutative quaternions is denoted by

$$\mathbb{Q}_c = \{a = a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\}, \tag{2}$$

where i, j, k satisfy

$$i^2 = k^2 = -1, j^2 = 1, ijk = -1, ij = ji = k, jk = kj = i, ki = ik = -j.$$

Let $a = a_0 + a_1 i + a_2 j + a_3 k, b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{Q}_c, \mu \in \mathbb{R}$. Then we easily have that

$$\begin{aligned} ab = ba =&(a_0b_0 - a_1b_1 + a_2b_2 - a_3b_3) + (a_1b_0 + a_0b_1 + a_3b_2 + a_2b_3)i \\ &+ (a_0b_2 + a_2b_0 - a_1b_3 - a_3b_1)j + (a_3b_0 + a_0b_3 + a_1b_2 + a_2b_1)k, \\ a + b =& (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k, \\ \mu a =& \mu(a_0 + a_1 i + a_2 j + a_3 k) = \mu a_0 + \mu a_1 i + \mu a_2 j + \mu a_3 k. \end{aligned}$$

Clearly, the commutative law holds in $\mathbb{Q}_c$. For any given $a \in \mathbb{Q}_c$, there are three different types of conjugates with

$$\begin{aligned} a^{(1)} &= a_0 - a_1 i + a_2 j - a_3 k, \\ a^{(2)} &= a_0 + a_1 i - a_2 j - a_3 k, \\ a^{(3)} &= a_0 - a_1 i - a_2 j + a_3 k. \end{aligned}$$

Definition 2.1. [2] *If $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{Q}_c$, then the induced norm over $a \in \mathbb{Q}_c$ is defined as*

$$\begin{aligned} \|a\|^4 &= aa^{(1)}a^{(2)}a^{(3)} \\ &= \left[(a_0 + a_2)^2 + (a_1 + a_3)^2\right]\left[(a_0 - a_2)^2 + (a_1 - a_3)^2\right] \geq 0. \end{aligned} \tag{3}$$

2.1. The complex representation over $\mathbb{Q}_c$

Theorem 2.2. [2] *Every commutative quaternion can be represented by a 2×2 complex matrix.*

From Theorem 2.2, we know that

$$g:\mathbb{Q}_c \longrightarrow M := \left\{\begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} : b_1, b_2 \in \mathbb{C}\right\}$$
$$a = b_1 + b_2 j \longmapsto g(a) = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}$$

is isomorphism. We call $g(a)$ is a complex representation matrix for commutative quaternion a. It is easy to verify that the following statements are true.

Proposition 2.3. *Let $a, b \in \mathbb{Q}_c$, and $\lambda \in \mathbb{R}$. Then*

1. $a = b \Leftrightarrow g(a) = g(b)$,
2. $g(a+b) = g(a) + g(b)$,
3. $g(ab) = g(a)g(b)$,
4. $g(\lambda a) = \lambda g(a)$,
5. $g(a)^T = g(a)$ *and* $\mathrm{tr}(g(a)) = a + a^{(2)}$.

If $A \in \mathbb{Q}_c^{m\times n}$, then three different types of conjugates of A are given by

$$A^{(1)} = \left(a_{ij}^{(1)}\right), A^{(2)} = \left(a_{ij}^{(2)}\right) \text{ and } A^{(3)} = \left(a_{ij}^{(3)}\right).$$

According to the $i^{th}(i = 1, 2, 3)$ conjugate of A, $A^{*_i} = (A^{(i)})^T \in \mathbb{Q}_c^{n\times m}$ is said to be the $i^{th}(i = 1, 2, 3)$ conjugate transpose of A. For any $a \in \mathbb{Q}_c$ and $b_1, b_2 \in \mathbb{C}$, it is easy to know that a can be written as $a = b_1 + b_2 j$. Similarly, if $A \in \mathbb{Q}_c^{m\times n}$, $A_1, A_2 \in \mathbb{C}^{m\times n}$, then A can be written as $A = A_1 + A_2 j \in \mathbb{Q}_c^{m\times n}$.

Definition 2.4. [2] *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$ and $A = A_1 + A_2 j \in \mathbb{Q}_c^{m\times n}$ be given. We define the complex representation of A as the following*

$$G(A) := \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}.$$

Theorem 2.5. [2] *Let $C, D \in \mathbb{Q}_c^{n\times n}$. Then*

1. $G(I_n) = I_{2n}$,
2. $G(C + D) = G(C) + G(D)$,
3. $G(CD) = G(C)G(D)$,
4. $G(C^{-1}) = (G(C))^{-1}$, *if C^{-1} exists,*
5. $G(C^{*_1}) = (G(C))^*$.

Remark 2.6. *In general, $G(C^{*_2}) \neq (G(C))^*$, $G(C^{*_3}) \neq (G(C))^*$ where $(G(C))^*$ is the conjugate transpose of $G(C)$.*

2.2. The structure of the operator vec(RYU)

For any $C = C_1 + C_2 j \in \mathbb{Q}_c^{m\times n}$, $C_1, C_2 \in \mathbb{C}^{m\times n}$, we have

$$C_1 + C_2 j = C \cong \Psi_C = [C_1, C_2],$$

where the symbol $\cong$ denotes an identification. Define

$$\hat{C}_1 = \begin{bmatrix} \mathrm{Re}\,(C_1) \\ \mathrm{Im}\,(C_1) \end{bmatrix}, \hat{C} = \begin{bmatrix} \mathrm{Re}\,(C_1) \\ \mathrm{Im}\,(C_1) \\ \mathrm{Re}\,(C_2) \\ \mathrm{Im}\,(C_2) \end{bmatrix}.$$

We have that

$$\mathrm{vec}(\hat{C}_1) = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,(C_1)) \\ \mathrm{vec}(\mathrm{Im}\,(C_1)) \end{bmatrix}, \mathrm{vec}(\hat{C}) = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,(C_1)) \\ \mathrm{vec}(\mathrm{Im}\,(C_1)) \\ \mathrm{vec}(\mathrm{Re}\,(C_2)) \\ \mathrm{vec}(\mathrm{Im}\,(C_2)) \end{bmatrix}.$$

For any given $C_1 \in \mathbb{C}^{m\times n}$, its Frobenius norm is defined as

$$\|C_1\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left\|c_{ij}\right\|^2}, \|c_{ij}\|^2 = (\mathrm{Re}\, c_{ij})^2 + (\mathrm{Im}\, c_{ij})^2.$$

For any given $C = C_1 + C_2 j \in \mathbb{Q}_c^{m\times n}$, we define

$$\|\hat{C}\| = \sqrt{\|\mathrm{Re}\, C_1\|^2 + \|\mathrm{Im}\, C_1\|^2 + \|\mathrm{Re}\, C_2\|^2 + \|\mathrm{Im}\, C_2\|^2}.$$

Obviously, $\|\Psi_C\| = \|\hat{C}\| = \|\mathrm{vec}(\hat{C})\|$. Next, we present some properties related to Ψ_C as follows.

Theorem 2.7. *Let $C = C_1 + C_2 j \in \mathbb{Q}_c^{m\times n}, D = D_1 + D_2 j \in \mathbb{Q}_c^{m\times n}$, where $C_1, C_2, D_1, D_2 \in \mathbb{C}^{m\times n}$. Then*
1. *$C = D$ if and only if $\Psi_C = \Psi_D$,*
2. *$\Psi_{C+D} = \Psi_C + \Psi_D$ and $\Psi_{lC} = l\Psi_C, l \in \mathbb{R}$,*
3. *$\Psi_{CD} = \Psi_C G(D)$.*

Proof. Clearly, (1) and (2) hold, we only need to prove (3). By calculating, we have

$$CD = (C_1 + C_2 j)(D_1 + D_2 j) = (C_1D_1 + C_2D_2) + (C_1D_2 + C_2D_1)\, j.$$

Thus

$$\begin{aligned} \Psi_{CD} &= [C_1D_1 + C_2D_2, C_1D_2 + C_2D_1] \\ &= [C_1, C_2]\begin{bmatrix} D_1 & D_2 \\ D_2 & D_1 \end{bmatrix} \\ &= \Psi_C G(D). \end{aligned}$$

□

Based on this theorem, we have the following results.

Theorem 2.8. *Suppose that $R = R_1 + R_2 j \in \mathbb{Q}_c^{m\times n}, Y = Y_1 + Y_2 j \in \mathbb{Q}_c^{n\times s}$ and $U = U_1 + U_2 j \in \mathbb{Q}_c^{s\times t}$, where $R_1, R_2 \in \mathbb{C}^{m\times n}, Y_1, Y_2 \in \mathbb{C}^{n\times s}$ and $U_1, U_2 \in \mathbb{C}^{s\times t}$, then*

$$\mathrm{vec}\,(\Psi_{RYU}) = \left[G(U)^T \otimes R_1, G(Uj)^T \otimes R_2\right]\begin{bmatrix} \mathrm{vec}\,(\Psi_Y) \\ \mathrm{vec}\,(\Psi_Y) \end{bmatrix}. \tag{4}$$

Proof. By Theorem 2.7, it follows that

$$\begin{aligned} \Psi_{RYU} =&\Psi_R G(YU) \\ =&\Psi_R G(Y)G(U) \\ =&[R_1, R_2]\begin{bmatrix} Y_1 & Y_2 \\ Y_2 & Y_1 \end{bmatrix}\begin{bmatrix} U_1 & U_2 \\ U_2 & U_1 \end{bmatrix} \\ =&[R_1Y_1U_1 + R_2Y_2U_1 + R_1Y_2U_2 + R_2Y_1U_2 \\ &R_1Y_1U_2 + R_2Y_2U_2 + R_1Y_2U_1 + R_2Y_1U_1]. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname{vec}(\Psi_{RYU}) \\
&= \begin{bmatrix} \left(U_1^T \otimes R_1\right)\operatorname{vec}(Y_1) + \left(U_1^T \otimes R_2\right)\operatorname{vec}(Y_2) + \left(U_2{}^T \otimes R_1\right)\operatorname{vec}(Y_2) + \left(U_2{}^T \otimes R_2\right)\operatorname{vec}(Y_1) \\ \left(U_2^T \otimes R_1\right)\operatorname{vec}(Y_1) + \left(U_2^T \otimes R_2\right)\operatorname{vec}(Y_2) + \left(U_1{}^T \otimes R_1\right)\operatorname{vec}(Y_2) + \left(U_1{}^T \otimes R_2\right)\operatorname{vec}(Y_1) \end{bmatrix} \\
&= \left[\begin{bmatrix} U_1 & U_2 \\ U_2 & U_1 \end{bmatrix}^T \otimes R_1, \begin{bmatrix} U_2 & U_1 \\ U_1 & U_2 \end{bmatrix}^T \otimes R_2\right]\begin{bmatrix} \operatorname{vec}(\Psi_Y) \\ \operatorname{vec}(\Psi_Y) \end{bmatrix} \\
&= \left[G(U)^T \otimes R_1, G(Uj)^T \otimes R_2\right]\begin{bmatrix} \operatorname{vec}(\Psi_Y) \\ \operatorname{vec}(\Psi_Y) \end{bmatrix}.
\end{aligned}$$

□

Note that the above results are important for solving a system of constrained two-sided coupled Sylvester-type matrix equations over the commutative quaternion ring.

Lemma 2.9. *For $B = B_1 + B_2 j \in \mathbb{Q}_c^{n\times s}, B_1, B_2 \in \mathbb{C}^{n\times s}$. Let*

$$K_s = \begin{bmatrix} I_{ns} & iI_{ns} & 0 & 0 \\ 0 & 0 & I_{ns} & iI_{ns} \\ I_{ns} & iI_{ns} & 0 & 0 \\ 0 & 0 & I_{ns} & iI_{ns} \end{bmatrix}. \tag{5}$$

Then

$$\begin{bmatrix} \operatorname{vec}(\Psi_B) \\ \operatorname{vec}(\Psi_B) \end{bmatrix} = K_s \operatorname{vec}(\hat{B}). \tag{6}$$

Proof. If $B = B_1 + B_2 j \in \mathbb{Q}_c^{n\times s}$, then it follows that

$$\begin{aligned}
\begin{bmatrix} \operatorname{vec}(\Psi_B) \\ \operatorname{vec}(\Psi_B) \end{bmatrix} &= \begin{bmatrix} \operatorname{vec}(B_1) \\ \operatorname{vec}(B_2) \\ \operatorname{vec}(B_1) \\ \operatorname{vec}(B_2) \end{bmatrix} \\
&= \begin{bmatrix} I_{ns} & iI_{ns} & 0 & 0 \\ 0 & 0 & I_{ns} & iI_{ns} \\ I_{ns} & iI_{ns} & 0 & 0 \\ 0 & 0 & I_{ns} & iI_{ns} \end{bmatrix}\begin{bmatrix} \operatorname{vec}(\operatorname{Re}(B_1)) \\ \operatorname{vec}(\operatorname{Im}(B_1)) \\ \operatorname{vec}(\operatorname{Re}(B_2)) \\ \operatorname{vec}(\operatorname{Im}(B_2)) \end{bmatrix} \\
&= K_s \operatorname{vec}(\hat{B}).
\end{aligned} \tag{7}$$

□

By Theorem 2.8 and Lemma 2.9, we can obtain the following corollary.

Corollary 2.10. *If $R = R_1 + R_2 j \in \mathbb{Q}_c^{m\times n}, Y = Y_1 + Y_2 j \in \mathbb{Q}_c^{n\times s}$ and $U = U_1 + U_2 j \in \mathbb{Q}_c^{s\times t}$, then*

$$\operatorname{vec}(\Psi_{RYU}) = \left(G(U)^T \otimes R_1, G(jU)^T \otimes R_2\right) K_s \operatorname{vec}(\hat{Y}). \tag{8}$$

Lemma 2.11. [13] *The matrix equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$, has a solution $x \in \mathbb{R}^n$ if and only if*

$$AA^{\dagger}b = b. \tag{9}$$

In this case, it has the general solution given by

$$x = A^{\dagger}b + \left(I_n - A^{\dagger}A\right)y, \tag{10}$$

where $y \in \mathbb{R}^n$ is an arbitrary matrix, and it has a unique solution $A^{\dagger}b$ if $\operatorname{rank}(A) = n$.

3. The Solution of (1)

From the previous discussion, we now turn our attention to solving the system of commutative quaternion matrix equations (1). For convenience, we define some useful notations which will be used in the sequel. Let $A_1 = A_{11}+A_{12}j, A_2 = A_{21}+A_{22}j, A_3 = A_{31}+A_{32}j \in \mathbb{Q}_c^{m\times n}, C_1, C_2, C_3 \in \mathbb{Q}_c^{m\times n}, B_1, B_2, B_3, D_1, D_2, D_3 \in \mathbb{Q}_c^{n\times k}, A_4 = A_{41} + A_{42}j, A_5 = A_{51} + A_{52}j, C_4 = C_{41} + C_{42}j, C_5 = C_{51} + C_{52}j \in \mathbb{Q}_c^{s\times n}, B_4, B_5, D_4, D_5 \in \mathbb{Q}_c^{n\times t}$ and $P, Q \in \mathbb{Q}_c^{s\times t}$. Set

$$L = \begin{bmatrix} G(I)^T \otimes A_{11} & G(jI)^T \otimes A_{12} \\ 0 & 0 \\ 0 & 0 \\ G(B_1)^T \otimes I & 0 \\ 0 & 0 \\ 0 & 0 \\ G(B_4)^T \otimes A_{41} & G(jB_4)^T \otimes A_{42} \\ 0 & 0 \end{bmatrix} K_n, \quad M = \begin{bmatrix} 0 & 0 \\ G(I)^T \otimes A_{21} & G(jI)^T \otimes A_{22} \\ 0 & 0 \\ 0 & 0 \\ G(B_2)^T \otimes I & 0 \\ 0 & 0 \\ G(D_4)^T \otimes C_{41} & G(jD_4)^T \otimes C_{42} \\ G(D_5)^T \otimes C_{51} & G(jD_5)^T \otimes C_{52} \end{bmatrix} K_n,$$

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ G(I)^T \otimes A_{31} & G(jI)^T \otimes A_{32} \\ 0 & 0 \\ 0 & 0 \\ G(B_3)^T \otimes I & 0 \\ 0 & 0 \\ G(B_5)^T \otimes A_{51} & G(jB_5)^T \otimes A_{52} \end{bmatrix} K_n, E = \begin{bmatrix} \mathrm{vec}(\Psi_{C_1}) \\ \mathrm{vec}(\Psi_{C_2}) \\ \mathrm{vec}(\Psi_{C_3}) \\ \mathrm{vec}(\Psi_{D_1}) \\ \mathrm{vec}(\Psi_{D_2}) \\ \mathrm{vec}(\Psi_{D_3}) \\ \mathrm{vec}(\Psi_P) \\ \mathrm{vec}(\Psi_Q) \end{bmatrix},$$

$$\begin{aligned} &L_1 = \mathrm{Re}\, L, \quad L_2 = \mathrm{Im}\, L, \quad M_1 = \mathrm{Re}\, M, \quad M_2 = \mathrm{Im}\, M, \quad N_1 = \mathrm{Re}\, N, \quad N_2 = \mathrm{Im}\, N, \\ &V_1 = [L_1, M_1, N_1], V_2 = [L_2, M_2, N_2], \end{aligned} \tag{11}$$

and

$$E_1 = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}\,\Psi_{C_1}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_{C_2}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_{C_3}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_{D_1}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_{D_2}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_{D_3}) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_P) \\ \mathrm{vec}(\mathrm{Re}\,\Psi_Q) \end{bmatrix}, E_2 = \begin{bmatrix} \mathrm{vec}(\mathrm{Im}\,\Psi_{C_1}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_{C_2}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_{C_3}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_{D_1}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_{D_2}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_{D_3}) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_P) \\ \mathrm{vec}(\mathrm{Im}\,\Psi_Q) \end{bmatrix}, \varepsilon = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{12}$$

Now we can give the expression of general solution to the system (1).

Theorem 3.1. *For $A_1, A_2, A_3, C_1, C_2, C_3 \in \mathbb{Q}_c^{m\times n}, B_1, B_2, B_3, D_1, D_2, D_3 \in \mathbb{Q}_c^{n\times k}, A_4, A_5, C_4, C_5 \in \mathbb{Q}_c^{s\times n}, B_4, B_5, D_4, D_5 \in \mathbb{Q}_c^{n\times t}$ and $P, Q \in \mathbb{Q}_c^{s\times t}$. Let V_1, V_2, ε be defined in* (11) *and* (12)*. Then the system* (1) *has a solution $X, Y, Z \in \mathbb{Q}_c^{n\times n}$ if and only if*

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^\dagger \varepsilon = \varepsilon. \tag{13}$$

In that case, the set of general solution can be expressed as

$$\Gamma = \left\{ [X, Y, Z] \middle| \begin{bmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \\ \mathrm{vec}(\hat{Z}) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^\dagger \varepsilon + \left[I_{12n^2} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^\dagger \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right] y \right\}, \tag{14}$$

where y is an arbitrary vector with appropriate order. Moreover, the system of commutative quaternion matrix equations (1) *has a unique solution $[X, Y, Z] \in \Gamma$ if and only if*

$$\operatorname{rank}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 12n^2. \tag{15}$$

In this case, we have

$$\Gamma = \left\{ [X, Y, Z] \middle| \begin{bmatrix} \operatorname{vec}(\hat{X}) \\ \operatorname{vec}(\hat{Y}) \\ \operatorname{vec}(\hat{Z}) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \varepsilon \right\}. \tag{16}$$

Proof. By Corollary 2.10 and Theorem 2.7, it follows that

$$\begin{aligned}
(1) \Longleftrightarrow & \begin{cases} \Psi_{A_1X} = \Psi_{C_1}, \quad \Psi_{A_2Y} = \Psi_{C_2}, \quad \Psi_{A_3Z} = \Psi_{C_3}, \\ \Psi_{XB_1} = \Psi_{D_1}, \quad \Psi_{YB_2} = \Psi_{D_2}, \quad \Psi_{ZB_3} = \Psi_{D_3}, \\ \Psi_{A_4XB_4} + \Psi_{C_4YD_4} = \Psi_P, \quad \Psi_{A_5ZB_5} + \Psi_{C_5YD_5} = \Psi_Q, \end{cases} \\
\Longleftrightarrow & L \operatorname{vec}(\hat{X}) + M \operatorname{vec}(\hat{Y}) + N \operatorname{vec}(\hat{Z}) = E, \\
\Longleftrightarrow & (\operatorname{Re} L + i \operatorname{Im} L) \operatorname{vec}(\hat{X}) + (\operatorname{Re} M + i \operatorname{Im} M) \operatorname{vec}(\hat{Y}) + (\operatorname{Re} N + i \operatorname{Im} N) \operatorname{vec}(\hat{Z}) \\
& = E_1 + iE_2, \\
\Longleftrightarrow & \begin{bmatrix} \operatorname{Re} L & \operatorname{Re} M & \operatorname{Re} N \\ \operatorname{Im} L & \operatorname{Im} M & \operatorname{Im} N \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\hat{X}) \\ \operatorname{vec}(\hat{Y}) \\ \operatorname{vec}(\hat{Z}) \end{bmatrix} = \varepsilon, \\
\Longleftrightarrow & \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\hat{X}) \\ \operatorname{vec}(\hat{Y}) \\ \operatorname{vec}(\hat{Z}) \end{bmatrix} = \varepsilon.
\end{aligned} \tag{17}$$

By Lemma 2.11, we can see that the system (1) has a solution $[X, Y, Z] \in \Gamma$ if and only if (13) holds. Consequently,

$$\begin{bmatrix} \operatorname{vec}(\hat{X}) \\ \operatorname{vec}(\hat{Y}) \\ \operatorname{vec}(\hat{Z}) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \varepsilon + \left[I_{12n^2} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right] y,$$

which implies (14) holds. Furthermore, if (13) holds, then the system (1) has a unique solution $[X, Y, Z] \in \Gamma$ if and only if

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = I_{12n^2},$$

that is to say, (15) holds and (16) is trivial. □

Next, we consider the Moore-Penrose generalized inverse of the column block matrix. Let

$$\begin{aligned}
q &= 6mn + 6kn + 4st, \\
S &= \left(I_{12n^2} - V_1^{\dagger} V_1\right) V_2^T, \\
W &= \left(I_q + \left(I_q - S^{\dagger} S\right) V_2 V_1^{\dagger} V_1^{\dagger T} V_2^T \left(I_q - S^{\dagger} S\right)\right)^{-1}, \\
J &= S^{\dagger} + \left(I_q - S^{\dagger} S\right) W V_2 V_1^{\dagger} V_1^{\dagger T} \left(I_{12n^2} - V_2^T S^{\dagger}\right), \\
\Theta_1 &= I_q - V_1 V_1^{\dagger} + V_1^{\dagger T} V_2^T W \left(I_q - S^{\dagger} S\right) V_2 V_1^{\dagger}, \\
\Theta_2 &= -V_1^{\dagger T} V_2^T \left(I_q - S^{\dagger} S\right) W, \\
\Theta_3 &= \left(I_q - S^{\dagger} S\right) W.
\end{aligned}$$

From the results in [14], we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} V_1^{\dagger} - J^T V_2 V_1^{\dagger} & J^T \end{bmatrix}, \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = V_1^{\dagger} V_1 + SS^{\dagger}. \tag{18}$$

$$I_{2q} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix}. \tag{19}$$

Corollary 3.2. *The system of commutative quaternion matrix equations* (1) *has a solution* $[X, Y, Z]$ *if and only if*

$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \varepsilon = 0. \tag{20}$$

In this case, the set of general solution of system (1) *can be expressed as*

$$\Gamma = \left\{ [X, Y, Z] \left| \begin{bmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \\ \mathrm{vec}(\hat{Z}) \end{bmatrix} = [V_1^{\dagger} - J^T V_2 V_1^{\dagger} \quad J^T]\varepsilon + \left(I_{12n^2} - V_1^{\dagger} V_1 - SS^{\dagger} \right) y \right. \right\}, \tag{21}$$

where $X, Y, Z \in \mathbb{Q}_c^{n \times n}$ *and* y *is an arbitrary vector with appropriate order. Furthermore, if* (20) *holds, then the system* (1) *has a unique solution* $[X, Y, Z] \in \Gamma$ *if and only if* (15) *holds. In this case,*

$$\Gamma = \left\{ [X, Y, Z] \left| \begin{bmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \\ \mathrm{vec}(\hat{Z}) \end{bmatrix} = \begin{bmatrix} V_1^{\dagger} - J^T V_2 V_1^{\dagger} & J^T \end{bmatrix} \varepsilon \right. \right\}. \tag{22}$$

Corollary 3.3. *Let the condition be satisfied in Corollary* 3.2. *Then the optimization problem*

$$\min_{[X,Y,Z] \in \Gamma} \left(\|\Psi_X\|^2 + \|\Psi_Y\|^2 + \|\Psi_Z\|^2 \right)$$

has a unique minimizer $[X_w, Y_w, Z_w]$ *which satisfies*

$$\begin{bmatrix} \mathrm{vec}(\hat{X_w}) \\ \mathrm{vec}(\hat{Y_w}) \\ \mathrm{vec}(\hat{Z_w}) \end{bmatrix} = [V_1^{\dagger} - J^T V_2 V_1^{\dagger} \quad J^T]\varepsilon. \tag{23}$$

Proof. From (21), we can see that the solution set Γ is a nonempty closed convex set. Hence,

$$\begin{aligned} \min_{[X,Y,Z] \in \Gamma} \left(\|\Psi_X\|^2 + \|\Psi_Y\|^2 + \|\Psi_Z\|^2 \right) &= \min_{[X,Y,Z] \in \Gamma} \left(\|\hat{X}\|^2 + \|\hat{Y}\|^2 + \|\hat{Z}\|^2 \right) \\ &= \min_{[X,Y,Z] \in \Gamma} \left(\|\mathrm{vec}(\hat{X})\|^2 + \|\mathrm{vec}(\hat{Y})\|^2 + \|\mathrm{vec}(\hat{Z})\|^2 \right) \\ &= \min_{[X,Y,Z] \in \Gamma} \left\| \begin{bmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \\ \mathrm{vec}(\hat{Z}) \end{bmatrix} \right\|^2. \end{aligned}$$

By Corollary 3.2, we have $\begin{bmatrix} \mathrm{vec}(\hat{X_w}) \\ \mathrm{vec}(\hat{Y_w}) \\ \mathrm{vec}(\hat{Z_w}) \end{bmatrix}$ is in the form (23). □

4. Numerical exemplification

Next, we give a numerical algorithm and a numerical example to solve the system (1).

Algorithm 1

1. Input the matrix: $A_1 = A_{11} + A_{12}j, A_2 = A_{21} + A_{22}j, A_3 = A_{31} + A_{32}j \in \mathbb{Q}_c^{m\times n}, C_1, C_2, C_3 \in \mathbb{Q}_c^{m\times n}, B_1, B_2, B_3, D_1, D_2, D_3 \in \mathbb{Q}_c^{n\times k}, A_4 = A_{41} + A_{42}j, A_5 = A_{51} + A_{52}j, C_4 = C_{41} + C_{42}j, C_5 = C_{51} + C_{52}j \in \mathbb{Q}_c^{s\times n}, B_4, B_5, D_4, D_5 \in \mathbb{Q}_c^{n\times t}$ and $P, Q \in \mathbb{Q}_c^{s\times t}$.
2. Compute $V_1, V_2, S, W, J, \Theta_1, \Theta_2, \Theta_3$, and ε.
3. If both (15) and (20) hold, then calculate $[X_w, Y_w, Z_w] \in \Gamma$ according to (22).
4. If (20) hold, then calculate $[X_w, Y_w, Z_w] \in \Gamma$ according to (21). Otherwise, go to next step.
5. Calculate $[X_w, Y_w, Z_w] \in \Gamma$ according to (23).

If the system (1) is consistent, then we have that

$$Q_1 = \left\| \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} \varepsilon - \varepsilon \right\|, \quad Q_2 = \left\| \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \varepsilon \right\|$$

and

$$Q_3 = \left\| I - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^{\dagger} - \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \right\|$$

are small.

Example 4.1. Given the commutative quaternion matrices:

$$A_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, C_1 = \begin{bmatrix} i - j + k & -1 + i + k \\ 1 + 0.5i - j & 0.5k \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} -2 + 0.5i + k & -j + k \\ 0 & -0.25i \end{bmatrix}, C_3 = \begin{bmatrix} 1 + i + 0.5k & 0.25j \\ 1 + i + 0.5k & 0.25j \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 1 + j + k & 1 + i + j \\ 0.5 - i + k & 0.5j \end{bmatrix}, D_2 = \begin{bmatrix} -0.5 - 2i - j & -j - k \\ 0 & -0.25i \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} i & i \\ 0 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} j & i \\ i & k \end{bmatrix}, D_4 = \begin{bmatrix} 1 & j \\ i & k \end{bmatrix},$$
$$D_3 = \begin{bmatrix} -1 + i & -1 + i \\ -0.5j & -0.5j \end{bmatrix}, B_5 = \begin{bmatrix} k & j \\ i & 0 \end{bmatrix}, C_5 = \begin{bmatrix} j & 0 \\ 0 & i \end{bmatrix}, D_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0 & 1 + i \\ 0 & -0.25 \end{bmatrix}, P = \begin{bmatrix} 0.25 + 4i + j + 2k & 3i - 1.75j + 4k \\ 0.25 + 3i + 0.5j + 0.5k & 1 + 2i - 0.75j + 2k \end{bmatrix},$$
$$A_1 = A_2, \quad B_1 = -B_2 = A_4, \quad A_3 = C_4, \quad A_5 = 0.$$

Taking

$$\tilde{X} = \begin{bmatrix} 1 + j + k & 1 + i + j \\ 0.5 - i + k & 0.5j \end{bmatrix}, \tilde{Y} = \begin{bmatrix} 0.5 + 2i + j & j + k \\ 0 & 0.25i \end{bmatrix}, \tilde{Z} = \begin{bmatrix} 1 + i & 0.25j \\ 0.5k & 0 \end{bmatrix}.$$

Let

$$\Psi_{C_1} = \Psi_{A_1} G(\tilde{X}), \quad \Psi_{C_2} = \Psi_{A_2} G(\tilde{Y}), \quad \Psi_{C_3} = \Psi_{A_3} G(\tilde{Z}),$$
$$\Psi_{D_1} = \Psi_{\tilde{X}} G(B_1), \quad \Psi_{D_2} = \Psi_{\tilde{Y}} G(B_2), \quad \Psi_{D_3} = \Psi_{\tilde{Z}} G(B_3),$$
$$\Psi_P = \Psi_{A_4} G(\tilde{X}) G(B_4) + \Psi_{C_4} G(\tilde{Y}) G(D_4), \quad \Psi_Q = \Psi_{A_5} G(\tilde{Z}) G(B_5) + \Psi_{C_5} G(\tilde{Y}) G(D_5).$$

From Matlab and Algorithm 1, we obtain

$$\operatorname{rank} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 44 < 12n^2 = 48, \quad Q_2 = 1.3172 \times 10^{-14}.$$

Therefore, we can easily see that the system (1) is consistent. Besides, we can also compute $Q_1 = 1.4357 \times 10^{-14}, Q_3 = 2.7365 \times 10^{-14}$. Consequently, the system (1) has infinite solutions $[X, Y, Z] \in \Gamma$ and an approximate solution $[X_w, Y_w, Z_w] \in \Gamma$ and we can get $\|\Psi_{[\tilde{X},\tilde{Y},\tilde{Z}]} - \Psi_{[X_w,Y_w,Z_w]}\| = 9.8306 \times 10^{-15}$.

5. Conclusion

In this paper, we have proposed a necessary and sufficient condition for the solvability of the system (1) by the complex representation of commutative quaternion matrices. In this case, we establish an expression of the general solution of the system (1). If the system (1) is inconsistent, then we develop an algorithm to obtain its approximate solution. Moreover, we also provide an example to illustrate our results.

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