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Weighted and Voronovskaja type approximation by q-Szász-Kantorovich operators involving Appell polynomials

Md. Nasiruzzaman[a], **Khursheed J. Ansari**[b], **M. Mursaleen**[c]

[a]*Department of Mathematics, Faculty of Science, University of Tabuk, PO Box-4279, Tabuk-71491, Saudi Arabia*
[b]*Department of Mathematics, College of Science, King Khalid University, 61413, Abha, Saudi Arabia*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*

Abstract. In this article, we concentrate on the Szász-Jakimovski-Leviatan operators imposed by Appell polynomials using q-calculus. We analyze the classical Szász-Jakimovski-Leviatan-Kantorovich and derive the approximation results connected to the non-negative parameters $\varsigma \in [\frac{1}{2}, \infty)$ in q-analogue. In order to combining with the earlier investigation by utilizing the Korovkin's theorem we study the local as well as global approximation theorems in terms of uniform modulus of continuity of order one and two. We calculate the rate of convergence by using of Lipschitz-maximal functions. Moreover, the Voronovskaja-type approximation theorem is also calculated here.

1. Introduction and Preliminaries

Due to the rapid development of the Appell polynomials [3] defined in 1880, an advance technique of Appell polynomials have been attempted by the mathematician Jakimovski and Leviatan in 1969 [8] by the identity

$$P(w)e^{wx} = \sum_{k=0}^{\infty} \beta_k(x) w^k, \tag{1}$$

which have been received more significant attention with the expression $\beta_k(x) = \sum_{j=0}^{k} \alpha_i \frac{x^{r-j}}{(r-j)!}$ $(r \in \mathbb{N})$ and $P(w) = \sum_{k=0}^{\infty} \alpha_k w^k$, $P(1) \neq 0$ and Szász-Jakimovski-Leviatan-operators given by Wood, [29]

$$\mathcal{W}_r(f; x) = \frac{1}{P(1)e(rx)} \sum_{k=0}^{\infty} P_k(rx) f\left(\frac{k}{r}\right), \tag{2}$$

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Corresponding author: Khursheed J. Ansari
Email addresses: nasir3489@gmail.com (Md. Nasiruzzaman), ansari.jkhursheed@gmail.com (Khursheed J. Ansari), mursaleenm@gmail.com (M. Mursaleen)

where $P(1) \neq 0$, $P(a) = \sum_{k=0}^{\infty} b_k a^s$, $P_k(x) = \sum_{j=0}^{k} b_j \frac{x^{r-j}}{(r-j)!}$ $(r \in \mathbb{N})$ and $\frac{b_j}{P(1)} \geq 0$. Further for more precisely, if take $P(1) = 1$ in (2) then the classical Szász operators [28] obtained. Later on, the great achievement in manufacturing techniques of Appell polynomials , which because of being provided with an improved performance of Appell polynomials and introduced by Al-Salam (see [2, 11]) in q-calculus by initiating the generating functions $P_q(t) = \sum\limits_{n=0}^{\infty} P_{r,q} \frac{t^n}{[n]_q!}$, $P_q(1) \neq 0$. Al-Salam, proposed a model of the family of q-Appell polynomials by

$$P_{r,q}(x) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q P_{r-k,q} x^k, \ (r \in \mathbb{N})$$

and q-differential, $D_{q,x}\Big(P_{r,q}(x)\Big) = [r]_q P_{r-1,q}(x)$, $r = 1, 2, \ldots$, where $P_{0,q}(x)$ is a non zero constant. Furthermore, $D_{q,x}\Big(P_{1,q}(x)\Big) = [1]_q P_{0,q}(x) = P_{0,q}$ and $P_q(t) e_q(tx) = \sum\limits_{r=0}^{\infty} P_{r,q}(x) \frac{t^r}{[r]_q!}$, $\quad 0 < q < 1$. These types of approaches were successfully identified the great achievement and high efficiency corresponding with the earlier classical Appell polynomials [3]. To understand the better information regrading the above mathematical polynomials we review the basics of q-calculus. For each non-negative integer r, the q-integer is defined as

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \end{cases} \quad \text{for } r \in \mathbb{N} \text{ and } [0]_q = 0.$$

For $|q| < 1$, the q-factorial $[r]_q!$ is defined by

$$[r]_q! = \begin{cases} 1 & (r = 0) \\ \prod\limits_{k=1}^{r} [k]_q & (r \in \mathbb{N}). \end{cases} \tag{3}$$

In the standard approach the exponential functions for q-calculus:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}. \tag{4}$$

For $0 < |q| < 1$, the q-Jackson integral from 0 to $u \in \mathbb{R}$, given by [7, 9]

$$\int_0^u f(x) \mathrm{d}_q x = u(1-q) \sum_{i=0}^{\infty} f(uq^i) q^i,$$

and the q-Jackson integral on a general interval $[u, v]$ given by

$$\int_u^v f(x) \mathrm{d}_q x = \int_0^v f(x) \mathrm{d}_q x - \int_0^u f(x) \mathrm{d}_q x.$$

For more concerned about q-calculus, especially, their notations and formula we prefer to see [7, 9, 10]). In 1950, Szász gave a comprehensive investigation of positive linear operators for the set of all continuous function f on $[0, \infty)$, and rapidly there are various model of Szász operators studied by mathematician (see [20, 25]). The Szász Jakimovski and Leviatan types model are also typically represented by the researcher for example, Mursaleen et al., studied the model of q-analogue of Jakimovski-Levitian operators [17] and the model of Stacu Jakimovski-Levitian-Durmeyer operators [18], Alotaibi et al., studied the model of q-Jakimovski-Leviatan-Beta operators [1], Nasiruzzaman et al., studied by including the Dunkl generalization, the model of Szász-Jakimovski-Leviatan-Beta [23], Szász-Jakimovski-Leviatan [21] and Szász-Jakimovski-Leviatan-Kantorovich [22]. For the set of all continuous $f \in C_\Phi[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\Phi)\}$ and all $x \in [0,\infty)$, with $\Phi > r, \quad r \in \mathbb{N}, \ P(1) \neq 0, \ \varsigma \geq 0$, the Szász-Jakimovski-Leviatan-Beta operators [23] given by,

$$\mathcal{B}^*_{r,\varsigma}(f;x) = \frac{1}{P(1)e_\varsigma(rx)} \sum_{k=0}^{\infty} P_k(rx) f\left(\frac{k+2\varsigma\theta_k}{r}\right), \tag{5}$$

and the Szász-Jakimovski-Leviatan-Kantorovich operators [22] given by

$$\mathcal{K}^*_{r,\varsigma}(f;x) = \frac{r}{P(1)e_\varsigma(rx)} \sum_{k=0}^{\infty} P_k(rx) \int_{\frac{k+2\varsigma\theta_k}{r}}^{\frac{k+1+2\varsigma\theta_k}{r}} f(t)\mathrm{d}t. \tag{6}$$

In this work our more concentrates on the recent investigations [21, 22] and by initiating the q-analogue by impose the presence of Dunkl parameter ς for the interval $[\frac{1}{2},\infty)$ we study the approximation properties of Szász-Jakimovski-Leviatan-Kantorovich operators given by [22]. By Combining with the results of [21, 22] we get the our new results in q-analogue are more effective rather than the earlier. We utilizing the Korovkin's theorem and study the local as well as global approximation theorems in terms of uniform modulus of continuity of order one and two. We calculate the rate of convergence by using of Lipschitz-maximal functions and also obtain the Voronovskaja-type approximation theorems. For more related concepts we present here to see the published article [14–16].

2. The q-variant of Kantorovich operators involving the Appell polynomials

In this section, our main aim is to construct the q-variant of recent investigation of Kantorovich positive linear operators induced by the Appel polynomial, which were explained in [22]. We suppose all classes of continuous and nondecreasing function f on $[0,\infty)$ denoted by $C_\Phi[0,\infty)$. For all $0 < q < 1$ and $f \in C_\Phi[0,\infty)$ if we take $C_\Phi[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\Phi)\}$ whenever $t \to \infty$, and $x \in [0,\infty) \subset \mathbb{R}, \quad \Phi > r$ with $r \in \mathbb{N}, \ P_q(1) \neq 0$ and $\varsigma \in [\frac{1}{2},\infty)$, then our new operators are defined by:

$$\mathcal{L}^*_{r,q}(f;x) = \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_q x)} \sum_{r=0}^{\infty} \frac{P_{r,q}([r]_q x)}{[r]_q!} \int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[r+2\varsigma\theta_r+1]_q}{[r]_q}} f(t)\mathrm{d}_q t, \tag{7}$$

where for all $r = 0,1,2,3,\cdots$ the θ_r given as

$$\theta_r = \begin{cases} 0 & \text{if } r \in \{0,2,4,6,\cdots\}, \\ 1 & \text{if } r \in \{1,3,5,7,\cdots\}, \end{cases} \tag{8}$$

and the q-structure of Dunkl exponential and their recursion we know

$$e_{\varsigma,q}(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\gamma_{\varsigma,q}(\nu)}, \quad x \in [0,\infty), \quad E_{\varsigma,q}(x) = \sum_{\nu=0}^{\infty} \frac{q^{\frac{\nu(\nu-1)}{2}} x^\nu}{\gamma_{\varsigma,q}(\nu)}, \quad x \in [0,\infty) \tag{9}$$

$$\gamma_{\varsigma,q}(\nu+1) = \left(\frac{1-q^{2\varsigma\theta_{\nu+1}+\nu+1}}{1-q}\right)\gamma_{\varsigma,q}(\nu), \quad \nu \in \mathbb{N}\cup\{0\}; \quad \gamma_{\varsigma,q}(0) = 1. \tag{10}$$

In the studies of equality (7), if we take $q = 1$, then we direct go through the results of published article by [22]. In addition, if we put $P_q(1) = 1$ and apply the extensive properties of Appell polynomials, then we directly go through the published article by [19]. Thus we say that, in this article our results are more extensive compared to [19, 22] and have the importance for further research area.

Lemma 2.1. *For all $q \in (0,1)$ let we consider the polynomial*

$$P_q(t)e_{\varsigma,q}(t\mu) = \sum_{r=0}^{\infty} P_{r,q}(\mu)\frac{t^r}{[r]_q!}. \tag{11}$$

Then for $t = 1$ and $\mu = [r]_q x$, we get following identities:

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{P_{r,q}([r]_q x)}{[r]_q!} &= P_q(1)e_{\varsigma,q}([r]_q x),\\
\sum_{r=0}^{\infty} [r]_q \frac{P_{r,q}([r]_q x)}{[r]_q!} &= \left([r]_q P_q(1)x + P'_q(1)\right) e_{\varsigma,q}([r]_q x),\\
\sum_{r=0}^{\infty} [r]_q^2 \frac{P_{r,q}([r]_q x)}{[r]_q!} &= \left([r]_q^2 P_q(1)x^2 + 2[r]_q P'_q(1)x + P''_q(1)\right) e_{\varsigma,q}([r]_q x),\\
\sum_{r=0}^{\infty} [r]_q^3 \frac{P_{r,q}([r]_q x)}{[r]_q!} &= \left([r]_q^3 P_q(1)x^3 + 3[r]_q^2 P'_q(1)x^2 + 3[r]_q P''_q(1)x + P'''_q(1)\right) e_{\varsigma,q}([r]_q x)\\
\sum_{r=0}^{\infty} [r]_q^4 \frac{P_{r,q}([r]_q x)}{[r]_q!} &= \Big([r]_q^4 P_q(1)x^4 + 4[r]_q^3 P'_q(1)x^3 + 6[r]_q^2 P''_q(1)x^2\\
&\quad +4[r]_q P'''_q(1)x + P'^{v}_q(1)\Big) e_{\varsigma,q}([r]_q x).
\end{aligned}$$

Lemma 2.2. *Let $\varsigma \geq \frac{r}{2}$ and $0 < q < 1$, for $r \in \mathbb{N}$, then we get that*

$$[2\varsigma + r]_q = q^{2\varsigma}[r]_q + 1 + q[2\varsigma - 1]_q, \quad [2\varsigma + r + 1]_q = 1 + q[2\varsigma + r]_q;$$

$$[2\varsigma + r]_q^2 = q^{4\varsigma}[r]_q^2 + 2q^{2\varsigma}\left(1 + q[2\varsigma - 1]_q\right)[r]_q + 1 + 2q[2\varsigma - 1]_q + q^2[2\varsigma - 1]_q^2;$$

$$\begin{aligned}
[2\varsigma + r]_q^3 &= q^{6\varsigma}[r]_q^3 + q^{4\varsigma}\left(3 + 3q[2\varsigma - 1]_q\right)[r]_q^2 + q^{2\varsigma}\left(3 + 6q[2\varsigma - 1]_q + 3q^2[2\varsigma - 1]_q^2\right)[r]_q\\
&+ 1 + 2q[2\varsigma - 1]_q + 3q^2[2\varsigma - 1]_q^2 + q^3[2\varsigma - 1]_q^3;
\end{aligned}$$

$$\begin{aligned}
[2\varsigma + r]_q^4 &= q^{8\varsigma}[r]_q^4 + q^{6\varsigma}\left(4 + 4q[2\varsigma - 1]_q\right)[r]_q^3 + q^{4\varsigma}\left(6 + 12q[2\varsigma - 1]_q + 6q^2[2\varsigma - 1]_q^2\right)[r]_q^2\\
&+ q^{2\varsigma}\left(4 + 11q[2\varsigma - 1]_q + 12q^2[2\varsigma - 1]_q^2 + 4q^3[2\varsigma - 1]_q^3\right)[r]_q\\
&+ 1 + 3q[2\varsigma - 1]_q + 5q^2[2\varsigma - 1]_q^2 + 4q^3[2\varsigma - 1]_q^3 + q^4[2\varsigma - 1]_q^4.
\end{aligned}$$

Lemma 2.3. *Let $q \in (0,1)$ and we use the equality $[1 + r + 2\varsigma\theta_r]_q = 1 + q[r + 2\varsigma\theta_r]_q$, then we get that*

$$\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[1+r+2\varsigma\theta_r]_q}{[r]_q}} d_q t = \frac{[1 + r + 2\varsigma\theta_r]_q}{[r]_q} - \frac{[r + 2\varsigma\theta_r]_q}{[r]_q} = \frac{1}{[r]_q},$$

$$\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[1+r+2\varsigma\theta_r]_q}{[r]_q}} t\, d_q t = \frac{1}{[2]_q[r]_q^2}\left(1 + 2q[r + 2\varsigma\theta_r]_q\right),$$

$$\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[1+r+2\varsigma\theta_r]_q}{[r]_q}} t^2 d_q t = \frac{1}{[3]_q[r]_q^3}\left(1 + 3q[r + 2\varsigma\theta_r]_q + 3q^2[r + 2\varsigma\theta_r]_q^2\right),$$

$$\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[1+r+2\varsigma\theta_r]_q}{[r]_q}} t^3 d_q t = \frac{1}{[4]_q[r]_q^4}\left(1 + 4q[r + 2\varsigma\theta_r]_q + 6q^2[r + 2\varsigma\theta_r]_q^2 + 4q^3[r + 2\varsigma\theta_r]_q^3\right),$$

$$\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[1+r+2\varsigma\theta_r]_q}{[r]_q}} t^4 d_q t = \frac{1}{[5]_q[r]_q^5}\left(1 + 5q[r + 2\varsigma\theta_r]_q + 10q^2[r + 2\varsigma\theta_r]_q^2 + 10q^3[r + 2\varsigma\theta_r]_q^3 + 5q^4[r + 2\varsigma\theta_r]_q^4\right).$$

Lemma 2.4. *For the test function $f(t)=t^j$, if $j=0,1,2,3,4$, the operators $\mathcal{L}^*_{r,q}(\,\cdot\,;\,\cdot\,)$ have the following equalities:*

$$
\begin{aligned}
(1)\quad \mathcal{L}^*_{r,q}(1;x) &= 1,\\
(2)\quad \mathcal{L}^*_{r,q}(t;x) &= \frac{1+2q+2q^2[2\varsigma-1]_q}{[2]_q[r]_q}+\frac{2q^{1+2\varsigma}}{[2]_q}\left(x+\frac{P'_q(1)}{[r]_qP_q(1)}\right),\\
(3)\quad \mathcal{L}^*_{r,q}(t^2;x) &= \frac{1}{[3]_q[r]_q^2}\left(1+3q+3q^2(1+[2\varsigma-1]_q)+6q^3[2\varsigma-1]_q+3q^4[2\varsigma-1]_q^2\right)\\
&\quad+\frac{3q^{1+2\varsigma}}{[3]_q[r]_q}(1+2q+2q^2[2\varsigma-1]_q)\left(x+\frac{P'_q(1)}{[r]_qP_q(1)}\right)\\
&\quad+\frac{3q^{2(1+\varsigma)}}{[3]_q}\left(x^2+\frac{2P'_q(1)}{[r]_qP_q(1)}x+\frac{P''_q(1)}{[r]_q^2P_q(1)}\right),\\
(4)\quad \mathcal{L}^*_{r,q}(t^3;x) &= \frac{1}{[4]_q[r]_q^3}\left(1+4q+6q^2+4q^3+4q^2(1+3q+2q^2)[2\varsigma-1]_q\right.\\
&\quad\left.+6q^4(1+2q)[2\varsigma-1]_q^2\right)\\
&\quad+\frac{1}{[4]_q}\left(4q^{1+2\varsigma}\left(1+3q+3q^2+3q^2(1+2q)[2\varsigma-1]_q\right)+3q^2[2\varsigma-1]_q^2\right)\\
&\quad\times\left(\frac{1}{[r]_q^2}x+\frac{P'_q(1)}{[r]_q^3P_q(1)}\right)\\
&\quad+\frac{1}{[4]_q}\left(6q^{2+4\varsigma}\left(1+2q+2q^2[2\varsigma-1]_q\right)\right)\left(\frac{1}{[r]_q}x^2+\frac{2P'_q(1)}{[r]_q^2P_q(1)}+\frac{P''_q(1)}{[r]_q^3P_q(1)}\right)\\
&\quad+\frac{4q^{3+6\varsigma}}{[4]_q}\left(x^3+\frac{3P'_q(1)}{[r]_qP_q(1)}x^2+\frac{3P''_q(1)}{[r]_q^2P_q(1)}x+\frac{P'''_q(1)}{[r]_q^3P_q(1)}\right),
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad \mathcal{L}^*_{r,q}(t^4;x) &= \frac{1}{[5]_q[r]_q^4}\left(1+5q+10q^2+10q^3+5q^4+5q^2(1+4q+4q^2+3q^3)[2\varsigma-1]_q\right.\\
&\quad\left.+5q^4(2+6q+5q^2)[2\varsigma-1]_q^2+10q^6(1+2q)[2\varsigma-1]_q^3+5q^8[2\varsigma-1]_q^4\right)\\
&\quad+\frac{5q^{1+2\varsigma}}{[5]_q}\left(1+4q+6q^2+4q^3+q^2(4+12q+11q^2)[2\varsigma-1]_q\right.\\
&\quad\left.+6q^4(1+2q)[2\varsigma-1]_q^2+4q^6[2\varsigma-1]_q^3\right)\left(\frac{1}{[r]_q^3}x+\frac{P'_q(1)}{[r]_q^4P_q(1)}\right)\\
&\quad+\frac{10q^{2+4\varsigma}}{[5]_q}\left(1+3q+3q^2+3q^2(1+2q)[2\varsigma-1]_q+3q^4[2\varsigma-1]_q^2\right)\\
&\quad\times\left(\frac{1}{[r]_q^2}x^2+\frac{2P'_q(1)}{[r]_q^3P_q(1)}x+\frac{P''_q(1)}{[r]_q^4P_q(1)}\right)\\
&\quad+\frac{10q^{3+6\varsigma}}{[5]_q}\left(1+2q+2q^2[2\varsigma-1]_q\right)\\
&\quad\times\left(\frac{1}{[r]_q}x^3+\frac{3P'_q(1)}{[r]_q^2P_q(1)}x^2+\frac{3P''_q(1)}{[r]_q^3P_q(1)}x+\frac{P'''_q(1)}{[r]_q^4P_q(1)}\right)\\
&\quad+\frac{5q^{4+8\varsigma}}{[5]_q}\left(x^4+\frac{4P'_q(1)}{[r]_qP_q(1)}x^3+\frac{6P''_q(1)}{[r]_q^2P_q(1)}x^2+\frac{4P'''_q(1)}{[r]_q^3P_q(1)}x+\frac{P'^{v}_q(1)}{[r]_q^4P_q(1)}\right).
\end{aligned}
$$

Proof. For the prove of results we taking into account the results from Lemmas 2.1, 2.2 and 2.3. Let $f(t) = 1$, then

$$\begin{aligned}\mathcal{L}^*_{r,q}(1;x) &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[r+2\varsigma\theta_r+1]_q}{[r]_q}} \mathrm{d}_qt\\ &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\frac{1}{[r]_q}\\ &= 1.\end{aligned}$$

If $f(t) = t$, then

$$\begin{aligned}\mathcal{L}^*_{r,q}(t;x) &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[r+2\varsigma\theta_r+1]_q}{[r]_q}} t\mathrm{d}_qt\\ &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\frac{1}{[2]_q[r]_q^2}\left(1+2q[r+2\varsigma\theta_r]_q\right)\\ &= \frac{1}{[2]_q[r]_qP_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\left(1+2q+2q^2[2\varsigma-1]_q+2q^{1+2\varsigma}[r]_q\right)\\ &= \frac{1+2q+2q^2[2\varsigma-1]_q}{[2]_q[r]_q}+\frac{2q^{1+2\varsigma}}{[2]_q[r]_qP_q(1)e_{\varsigma,q}([r]_qx)}\left([r]_qP_q(1)x+P'_q(1)\right)e_{\varsigma,q}([r]_qx)\\ &= \frac{1+2q+2q^2[2\varsigma-1]_q}{[2]_q[r]_q}+\frac{2q^{1+2\varsigma}}{[2]_q}\left(x+\frac{P'_q(1)}{[r]_qP_q(1)}\right).\end{aligned}$$

And for $f(t) = t^2$,

$$\begin{aligned}\mathcal{L}^*_{r,q}(t^2;x) &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\int_{\frac{q[r+2\varsigma\theta_r]_q}{[r]_q}}^{\frac{[r+2\varsigma\theta_r+1]_q}{[r]_q}} t^2\mathrm{d}_qt\\ &= \frac{[r]_q}{P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\frac{1}{[3]_q[r]_q^3}\left(1+3q[r+2\varsigma\theta_r]_q+3q^2[r+2\varsigma\theta_r]_q^2\right)\\ &= \frac{1}{[3]_q[r]_q^2P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}\frac{P_{r,q}([r]_qx)}{[r]_q!}\\ &\quad\times\left(1+3q+3q^2(1+[2\varsigma-1]_q)+6q^3[2\varsigma-1]_q+3q^4[2\varsigma-1]_q^2\right)\\ &\quad+\frac{1}{[3]_q[r]_q^2P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}[r]_q\frac{P_{r,q}([r]_qx)}{[r]_q!}3q^{1+2\varsigma}\left(1+2q+2q^2[2\varsigma-1]_q\right)\\ &\quad+\frac{3q^{2(1+\varsigma)}}{[3]_q[r]_q^2P_q(1)e_{\varsigma,q}([r]_qx)}\sum_{r=0}^{\infty}[r]_q^2\frac{P_{r,q}([r]_qx)}{[r]_q!}\\ &= \frac{1}{[3]_q[r]_q^2}\left(1+3q+3q^2(1+[2\varsigma-1]_q)+6q^3[2\varsigma-1]_q+3q^4[2\varsigma-1]_q^2\right)\\ &\quad+\frac{3q^{1+2\varsigma}}{[3]_q[r]_q}(1+2q+2q^2[2\varsigma-1]_q)\left(x+\frac{P'_q(1)}{[r]_qP_q(1)}\right)\\ &\quad+\frac{3q^{2(1+\varsigma)}}{[3]_q}\left(x^2+\frac{2P'_q(1)}{[r]_qP_q(1)}x+\frac{P''_q(1)}{[r]_q^2P_q(1)}\right).\end{aligned}$$

Similarly in the view of Lemmas 2.1, 2.2 and 2.3 we easily get the results for $f(t) = t^3$ and $f(t) = t^4$. □

Lemma 2.5. *Operators $\mathcal{L}^*_{r,q}$ have the following properties for the central moments $(t-x)^\eta$ if $\eta = 1, 2, 4$:*

$$(1)\quad \mathcal{L}^*_{r,q}((t-x);x) = \left(\frac{2q^{1+2\varsigma}}{[2]_q} - 1\right)x + \frac{1}{[2]_q[r]_q}\left(1 + 2q + 2q^2[2\varsigma-1]_q + 2q^{1+2\varsigma}\frac{P'_q(1)}{P_q(1)}\right),$$

$$\begin{aligned}(2)\quad \mathcal{L}^*_{r,q}\left((t-x)^2;x\right) = & \left(\frac{3q^{2(1+\varsigma)}}{[3]_q} - \frac{4q^{1+2\varsigma}}{[2]_q} + 1\right)x^2 \\ & + \frac{1}{[3]_q[r]_q}\left(3q^{1+2\varsigma}\left(1 + 2q + 2q^2[2\varsigma-1]_q + 2q\frac{P'_q(1)}{P_q(1)}\right)\right)x \\ & - \frac{2}{[2]_q[r]_q}\left(1 + 2q + 2q^2[2\varsigma-1]_q - 2q^{1+2\varsigma}\frac{P'_q(1)}{P_q(1)}\right)x \\ & + \frac{1}{[3]_q[r]_q^2}\left(1 + 3q + 3q^2 + 3q^2[2\varsigma-1]_q + 6q^3[2\varsigma-1]_q + 3q^4[2\varsigma-1]_q^2\right. \\ & \left. + 3q^{1+2\varsigma}(1 + 2q + 2q^2[2\varsigma-1]_q)\frac{P'_q(1)}{P_q(1)} + 3q^{2(1+\varsigma)}\frac{P''_q(1)}{P_q(1)}\right),\end{aligned}$$

$$(3)\quad \mathcal{L}^*_{r,q}\left((t-x)^4;x\right) = x^4 + O\left(\frac{1}{[r]_q^3}\right)x^3 + O\left(\frac{1}{[r]_q^2}\right)x^2 + O\left(\frac{1}{[r]_q}\right)x + O\left(\frac{1}{[r]_q^4}\right).$$

3. Approximations in weighted space

Here we want to give the approximations in weighted spaces and for this purpose, we apply the analogous of P.P. Korovkin's theorem. We go along with the well-known Gadžiev [5] results and look back on weighted spaces and to obtain the uniform approximations we proceed with the assumptions $\Phi(x) = 1 + \Phi^2(x)$ and $x \to \Phi(x)$ such kind of continuous and strictly increasing functions, where $\lim_{x\to\infty}\Phi(x) = \infty$. In addition, we suppose all bounded functions class be $B_\Phi[0,\infty)$ and satisfying the analogues:

$$B_\Phi[0,\infty) = \{g : |g(x)| \le M_g\Phi(x)\}, \tag{12}$$

where the equipped norm on g is defined by

$$\|g\|_\Phi = \sup_{x\in[0,\infty)} \frac{|g(x)|}{\Phi(x)}. \tag{13}$$

More precisely, we take $C_\Phi[0,\infty) = B_\Phi[0,\infty) \cap C[0,\infty)$, and $C_\Phi[0,\infty)$ is the subset of $C[0,\infty)$. Additionally, take into consideration from [5] the sequence of positive linear operators $\{J_r\}_{r\ge1}$ acting from $C_\Phi[0,\infty)$ to $B_\Phi[0,\infty)$ satisfying

$$|J_r(\Phi;x)| \le M\Phi(x),$$

where $M > 0$, and constant, and for all $m \in \mathbb{N}$,

$$C^m_\Phi[0,\infty) = \{g \in C_\Phi[0,\infty) : \lim_{x\to\infty}\frac{g(x)}{\Phi(x)} = c, \text{exists and finite}\}. \tag{14}$$

Theorem 3.1. *For all $f \in \mathcal{U}_f \bigcap C_\Phi[0,\infty)$, and $\mathcal{U}_f = \left\{f : \text{ such that } \frac{f(x)}{\Phi(x)} \text{ is convergent when } x \to \infty\right\}$, the operators $\mathcal{L}^*_{r,q}$ satisfying*

$$\mathcal{L}^*_{r,q}(f;x) \Longrightarrow f,$$

where $\Longrightarrow$ stands for uniform convergence for all compact subset of $[0,\infty)$.

Proof. For the prove of results asserted by Theorem 3.1, we take into account the Lemma 2.4, and use the analogues of Korovkin's Theorem by [12], then simply enough to verify for all $\kappa = 0, 1, 2$

$$\mathcal{L}^*_{r,q}(t^\kappa; x) \to x^\kappa, \text{ is uniformly on compact subset of } [0, \infty),$$

It is obvious, to see from Lemma 2.4, for all $\kappa = 0, 1, 2$ the $\lim_{r\to\infty} \mathcal{L}^*_{r,q}(1; x) = 1$, $\lim_{r\to\infty} \mathcal{L}^*_{r,q}(t; x) = x$ and $\lim_{r\to\infty} \mathcal{L}^*_{r,q}(t^2; x) = x^2$ which gives the prove Theorem 3.1. □

Theorem 3.2. *[5, 6] Operators $\{K_r\}_{r\geq 1}$ acting from $C_\Phi[0, \infty)$ to $B_\Phi[0, \infty)$ and verifies that $\lim_{r\to\infty} \|K_r(t^\kappa) - x^\kappa\|_\Phi = 0$. Then for every $f \in C^m_\Phi[0, \infty)$, $m \in \mathbb{N}$ it satisfying*

$$\lim_{r\to\infty} \|K_r(f) - f\|_\Phi = 0.$$

Theorem 3.3. *Let the operators $\mathcal{L}^*_{r,q}$ acting from $C_\Phi[0, \infty)$ to $B_\Phi[0, \infty)$ and satisfies $\lim_{r\to\infty} \|\mathcal{L}^*_{r,q}(t^\kappa) - x^\kappa\|_\Phi = 0$. Then for any $g \in C^m_\Phi[0, \infty)$, $m \in \mathbb{N}$ it follows that*

$$\lim_{r\to\infty} \|\mathcal{L}^*_{r,q}(g) - g\|_\Phi = 0.$$

Proof. We use Theorem 3.2 and well-known Korovkin's theorem enough to show

$$\lim_{r\to\infty} \|\mathcal{L}^*_{r,q}(t^\kappa) - x^\kappa\|_\Phi = 0, \quad \kappa = 0, 1, 2.$$

From the Lemma 2.4, we easily get that

$$\|\mathcal{L}^*_{r,q}(1) - 1\|_\Phi = \sup_{x\in[0,\infty)} \frac{|\mathcal{L}^*_{r,q}(1; x) - 1|}{\Phi(x)} = 0.$$

Take $\kappa = 1$, then

$$\begin{aligned}
\|\mathcal{L}^*_{r,q}(t) - x\|_\Phi &= \sup_{x\in[0,\infty)} \frac{\left|\mathcal{L}^*_{r,q}(t; x) - x\right|}{\Phi(x)} \\
&= \sup_{x\in[0,\infty)} \frac{x}{\Phi(x)} \left|\frac{2q^{1+2\varsigma}}{[2]_q} - 1\right| \\
&\quad + \sup_{x\in[0,\infty)} \frac{1}{\Phi(x)} \left|\frac{1}{[2]_q[r]_q}\left(1 + 2q + 2q^2[2\varsigma - 1]_q + 2q^{1+2\varsigma}\frac{P'_q(1)}{P_q(1)}\right)\right|.
\end{aligned}$$

Take $r \to \infty$, then $\|\mathcal{L}^*_{r,q}(t) - x\|_\Phi \to 0$. For $\kappa = 2$, we get

$$\begin{aligned}
\|\mathcal{L}^*_{r,q}(t^2) - x^2\|_\Phi &= \sup_{x\in[0,\infty)} \frac{\left|\mathcal{L}^*_{r,q}(t^2; x) - x^2\right|}{\Phi(x)} \\
&= \sup_{x\in[0,\infty)} \frac{x^2}{\Phi(x)} \left|\frac{3q^{2(1+\varsigma)}}{[3]_q} - 1\right| \\
&\quad + \sup_{x\in[0,\infty)} \frac{x}{\Phi(x)} \left|\frac{1}{[3]_q[r]_q}\left(3q^{1+2\varsigma}\left(1 + 2q + 2q^2[2\varsigma - 1]_q + 2q\frac{P'_q(1)}{P_q(1)}\right)\right)\right| \\
&\quad + \sup_{x\in[0,\infty)} \frac{1}{\Phi(x)} \left|\frac{1}{[3]_q[r]^2_q}\left(1 + 3q + 3q^2 + 3q^2[2\varsigma - 1]_q + 6q^3[2\varsigma - 1]_q + 3q^4[2\varsigma - 1]^2_q\right)\right|.
\end{aligned}$$

Thus if as $r \to \infty$, then we have $\|\mathcal{L}^*_{r,q}(t^2) - x^2\|_\Phi \to 0$. □

Theorem 3.4. *Let $\ell \in C_{\Phi}^{m}[0,\infty)$, $m \in \mathbb{N}$, then for any $\lambda \in [0,\infty)$ then operators $\mathcal{L}_{r,q}^{*}$ satisfies the equality*

$$\lim_{r\to\infty} \sup_{z\in[0,\infty)} \frac{|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} = 0.$$

Proof. Taking into account the inequality $|\ell(x)| \leq \|\ell\|_{\Phi}\Phi(x)$ then for any positive number x_0 obvious that

$$\begin{aligned}\lim_{r\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} &\leq \sup_{x\leq x_0} \frac{|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} + \sup_{x\geq x_0} \frac{|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} \\ &\leq \|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)\|_{C[0,x_0]} \\ &\quad + \|\ell\|_{\Phi} \sup_{x\geq x_0} \frac{|\mathcal{L}_{r,q}^{*}(1+t^2;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} + \sup_{x\geq x_0} \frac{|\ell(x)|}{(\Phi(x))^{1+\lambda}} \\ &= \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3, \text{(suppose)}.\end{aligned}$$

Thus

$$\mathcal{K}_3 = \sup_{x\geq x_0} \frac{|\ell(x)|}{(\Phi(x))^{1+\lambda}} \leq \sup_{x\geq x_0} \frac{\|\ell\|_{\Phi}\Phi(x)}{(\Phi(x))^{1+\lambda}} \leq \frac{\|\ell\|_{\Phi}}{(\Phi(x_0))^{\lambda}}. \tag{15}$$

From Lemma 2.4, it follows that

$$\lim_{r\to\infty} \sup_{x\geq x_0} \frac{\mathcal{L}_{r,q}^{*}(1+t^2;x)}{\Phi(x)} = 1.$$

Now a given $\epsilon^* > 0$, exists a positive $r_1 \in \mathbb{N}$ with $r \geq r_1$ which verifying the inequality

$$\sup_{x\geq x_0} \frac{\mathcal{L}_{r,q}^{*}(1+t^2;x)}{\Phi(x)} \leq \frac{(\Phi(x_0))^{\lambda}}{\|\ell\|_{\Phi}} \frac{\epsilon^*}{3} + 1.$$

For $r \geq r_1$

$$\mathcal{K}_2 = \|\ell\|_{\Phi} \sup_{x\geq x_0} \frac{\mathcal{L}_{r,q}^{*}(1+t^2;x)}{(\Phi(x))^{1+\lambda}} \leq \frac{\|\ell\|_{\Phi}}{(\Phi(x_0))^{\lambda}} + \frac{\epsilon^*}{3}. \tag{16}$$

We take in account of (15) and (16), we see

$$\mathcal{K}_2+\mathcal{K}_3 \leq 2\frac{\|\ell\|_{\Phi}}{(\Phi(x_0))^{\lambda}} + \frac{\epsilon^*}{3}.$$

Choose x_0 very large, such that $\frac{\|\ell\|_{\Phi}}{(\Phi(x_0))^{\lambda}} \leq \frac{\epsilon^*}{6}$, then get

$$\mathcal{K}_2+\mathcal{K}_3 \leq \frac{2\epsilon^*}{3}, \quad \text{for } r \geq r_1. \tag{17}$$

Similarly, for $r \geq r_2$ we have

$$\mathcal{K}_1 = \|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)\|_{C[0,x_0]} \leq \frac{\epsilon^*}{3}. \tag{18}$$

Lastly, we take $r_3 = \max(r_1,r_2)$ and take (17) as well as (18), thus we see

$$\sup_{x\in[0,\infty)} \frac{|\mathcal{L}_{r,q}^{*}(\ell;x)-\ell(x)|}{(\Phi(x))^{1+\lambda}} < \epsilon^*,$$

which completes the proof. □

Definition 3.5. *For all uniformly continuous function $f \in \tilde{C}[0,\infty)$, the usual modulus of continuity is given by:*

$$\omega^*(f;\delta^*) = \sup_{|\chi_1-\chi_2|\leq\delta^*} |f(\chi_1) - f(\chi_2)|, \qquad \chi_1, \chi_2 \in [0,\infty),$$

$$|f(\chi_1) - f(\chi_2)| \leq \left(1 + \frac{|\chi_1 - \chi_2|}{(\delta^*)^2}\right)\omega^*(f;\delta^*). \tag{19}$$

Theorem 3.6. *[26] If the sequences of positive linear operators $\{P\}_{r\geq 1}$ acting from $[a,b] \to C[c,d]$ such that $[c,d] \subseteq [a,b]$, then*

1. *for all $\varphi \in C[a,b]$ and $x \in [c,d]$, it satisfies that*
$$\begin{aligned}|P_r(\varphi;x) - \varphi(x)| \quad &\leq \quad |\varphi(x)||P_r(1;x) - 1| \\ &+ \left\{P_r(1;x) + \frac{1}{\delta^*}\sqrt{P_r((t-x)^2;x)}\sqrt{P_r(1;x)}\right\}\omega^*(\varphi;\delta^*),\end{aligned}$$
2. *for all $\varphi' \in C[a,b]$ and $x \in [c,d]$, one has*
$$\begin{aligned}|P_r(\varphi;x) - \varphi(x)| \quad &\leq \quad |\varphi(x)||P_r(1;x) - 1| + |\varphi'(x)||P_r(t-x;x)| \\ &+ P_r((t-x)^2;x)\left\{\sqrt{P_r(1;x)} + \frac{1}{\delta^*}\sqrt{P_r((t-x)^2;x)}\right\}\omega^*(\varphi';\delta^*).\end{aligned}$$

Theorem 3.7. *Take $f \in C_\Phi[0,\infty)$ and $x \in [0,\infty) \subset \mathbb{R}$, then operators $\mathcal{L}^*_{r,q}$ have the property:*

$$|\mathcal{L}^*_{r,q_r}(f;x) - f(x)| \quad \leq \quad 2\omega^*\left(f;\sqrt{\delta^*_{r,q_r}(x)}\right),$$

where $q = q_r$ with $0 < q_r < 1$ and $\delta^* = \sqrt{\delta^*_{r,q_r}(x)} = \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}$.

Proof. We apply the expression of Theorem 3.6 and Lemma 2.4, we easily get here:

$$\begin{aligned}|\mathcal{L}^*_{r,q}(f;x) - f(x)| \quad &\leq \quad +\frac{1}{\delta^*}\sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}\sqrt{\mathcal{L}^*_{r,q_r}(1;x)}\Big\}\omega^*(f;\delta^*) \\ &+|f(x)||\mathcal{L}^*_{r,q_r}(1;x) - 1| + \Big\{\mathcal{L}^*_{r,q_r}(1;x),\end{aligned}$$

choose, $\delta^* = \sqrt{\delta^*_{r,q_r}(x)} = \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}$, gives our result. □

Theorem 3.8. *Take the positive sequence $q = q_r$ with $0 < q_r < 1$, then for all $\psi \in C'_\Phi[0,\infty)$, we have the property*

$$|\mathcal{L}^*_{r,q_r}(\psi;x) - \psi(x)| \quad \leq \quad \left|\Omega_{r,q_r}(x)\right||\psi'(x)| + 2\delta^*_{r,q_r}(x)\omega^*\left(\psi';\sqrt{\delta^*_{r,q_r}(x)}\right),$$

where $\delta^* = \sqrt{\delta^*_{r,q_r}(x)} = \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}$ and

$$\Omega_{r,q_r}(x) = \left(\frac{2q_r^{1+2\varsigma}}{[2]_{q_r}} - 1\right)x + \frac{1}{[2]_{q_r}[r]_{q_r}}\left(1 + 2q_r + 2q_r^2[2\varsigma - 1]_{q_r} + 2q_r^{1+2\varsigma}\frac{P'_{q_r}(1)}{P_{q_r}(1)}\right).$$

Proof. In the view of the Lemmas 2.4, 2.5 and Theorem 3.6, obvious to get that

$$\begin{aligned}|\mathcal{L}^*_{r,q_r}(\psi;x) - \psi(x)| \quad &\leq \quad |\mathcal{L}^*_{r,q_r}(1;x) - 1||\psi(x)| + |\psi'(x)||\mathcal{L}^*_{r,q_r}(t-x;x)| \\ &+\mathcal{L}^*_{r,q_r}((t-x)^2;x)\left\{\sqrt{\mathcal{L}^*_{r,q_r}(1;x)} + \frac{1}{\delta^*}\sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}\right\}\omega^*(\psi';\delta^*),\end{aligned}$$

choose, $\delta^* = \sqrt{\delta^*_{r,q_r}(x)} = \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^2;x)}$, then our result is proved. □

For prove the another theorem in weighted modulus of continuity we use the results from [4] and for an arbitrary $\varphi \in C^m_\Phi[0,\infty)$, $m \in \mathbb{N} \cup \{0\}$ one has

$$\omega^*(\varphi;\delta^*) = \sup_{x\in[0,\infty),|\mu|\leq\delta^*} \frac{|\varphi(x+\mu)-\varphi(x)|}{(1+\mu^2)(1+x^2)}, \tag{20}$$

and weighed modulus of continuity satisfying the equality $\lim_{\delta^*\to 0}\omega^*(\varphi;\delta^*) = 0$, and

$$|\varphi(t)-\varphi(x)| \leq 2\left(1+\frac{|t-x|}{\delta^*}\right)(1+(\delta^*)^2)(1+x^2)(1+(t-x)^2)\omega^*(\varphi;\delta^*), \tag{21}$$

where $t, x \in [0,\infty)$.

Theorem 3.9. *Suppose $\varphi \in C^m_\Phi[0,\infty)$, then for any $x \in [0,\infty)$ and $q = q_r$ $0 < q_r < 1$ operators $\mathcal{L}^*_{r,q}$ satisfy the inequality*

$$\sup_{x\in[0,A_{r,q_r}(\varsigma))} \frac{\left|\mathcal{L}^*_{r,q_r}(\varphi;x)-\varphi(x)\right|}{1+x^2} \leq C\left(1+A_{r,q_r}(\varsigma)\right)\Omega\left(\varphi;\sqrt{A_{r,q_r}(\varsigma)}\right),$$

where $C = 2(2+C_1+\sqrt{C_2}) > 0$, for $C_1 > 0$ and $C_2 > 0$, and are constants while, $A_{r,q_r}(\varsigma)$ depends on parameter ς.

Proof. In the view of the inequality (20), (21) and on the positive linear operators $\mathcal{L}^*_{r,q_r}$ we apply the Cauchy-Schwarz inequality, then

$$\begin{aligned}&\left|\mathcal{L}^*_{r,q_r}(\varphi;x)-\varphi(x)\right| \\ &\leq 2(1+(\delta^*)^2)(1+x^2)\omega^*(\varphi;\delta^*)\left\{1+\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)+\mathcal{L}^*_{r,q_r}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta^*};x\right)\right\}.\end{aligned} \tag{22}$$

On simplify we have

$$\begin{aligned}&\mathcal{L}^*_{r,q_r}\left(\frac{|t-x|}{\delta^*(1+(t-x)^2)};x\right) \\ &\quad= \mathcal{L}^*_{r,q_r}\left((t-x)^2\frac{|t-x|}{\delta^*};x\right)+\frac{1}{\delta^*}\mathcal{L}^*_{r,q_r}(|t-x|;x) \\ &\quad\leq \left(\mathcal{L}^*_{r,q_r}\left((t-x)^4;x\right)\right)^{\frac{1}{2}}\left\{\mathcal{L}^*_{r,q_r}\left(\frac{(t-x)^2}{(\delta^*)^2};x\right)\right\}^{\frac{1}{2}}+\frac{1}{\delta^*}\left(\mathcal{L}^*_{r,q_r}(t-x)^2;x\right)^{\frac{1}{2}} \\ &\quad= \frac{1}{\delta^*}\left(\mathcal{L}^*_{r,q_r}(t-x)^2;x\right)^{\frac{1}{2}}\left\{1+\sqrt{\mathcal{L}^*_{r,q_r}((t-x)^4;x)}\right\}.\end{aligned} \tag{23}$$

From Lemma 2.5, we easily see that

$$\mathcal{L}^*_{r,q_r}((t-x)^2;x) \leq A_{r,q_r}(\varsigma)(x^2+x+1) \leq C_1(x+1)^2 \text{ as } r\to\infty, \tag{24}$$

$$\mathcal{L}^*_{r,q_r}((t-x)^4;x) \leq B_{r,q_r}(\varsigma)(x^4+x^3+x^2+x+1) \leq C_2(x+1)^4 \text{ as } r\to\infty, \tag{25}$$

where $C_1 > 0$ and $C_2 > 0$ and are constants.

$$\begin{aligned} A_{r,q_r}(\varsigma) &= \max\left\{\left\{\frac{3q_r^{2(1+\varsigma)}}{[3]_{q_r}} - \frac{4q_r^{1+2\varsigma}}{[2]_{q_r}} + 1\right\},\right. \\ &\left\{\frac{3q_r^{1+2\varsigma}}{[3]_{q_r}[r]_{q_r}}\left(1 + 2q_r + 2q_r^2[2\varsigma-1]_{q_r} + 2q\frac{P'_{q_r}(1)}{P_{q_r}(1)}\right)\right\}, \\ &\left\{\frac{2}{[2]_{q_r}[r]_{q_r}}\left(1 + 2q_r + 2q_r^2[2\varsigma-1]_{q_r} - 2q_r^{1+2\varsigma}\frac{P'_{q_r}(1)}{P_{q_r}(1)}\right)\right\}, \\ &\left\{\frac{1}{[3]_{q_r}[r]_{q_r}^2}\left(1 + 3q_r + 3q_r^2 + 3q_r^2[2\varsigma-1]_{q_r} + 6q_r^3[2\varsigma-1]_{q_r} + 3q_r^4[2\varsigma-1]_{q_r}^2\right)\right\}, \\ &\left.\left\{\frac{1}{[3]_{q_r}[r]_{q_r}^2}\left(3q_r^{1+2\varsigma}(1 + 2q_r + 2q_r^2[2\varsigma-1]_{q_r})\frac{P'_{q_r}(1)}{P_{q_r}(1)} + 3q_r^{2(1+\varsigma)}\frac{P''_{q_r}(1)}{P_{q_r}(1)}\right)\right\}\right\}, \end{aligned}$$

$$B_{r,q_r}(\varsigma) = \max\left\{a_{r,q_r}, b_{r,q_r}, c_{r,q_r}, d_{r,q_r}\right\},$$

where $a_{r,q_r} = O\left(\frac{1}{[r]_{q_r}^4}\right)$, $b_{r,q_r} = O\left(\frac{1}{[r]_{q_r}^3}\right)$, $c_{r,q_r} = O\left(\frac{1}{[r]_{q_r}^2}\right)$, $d_{r,q_r} = O\left(\frac{1}{[r]_{q_r}}\right)$.
Thus from inequality (22), we get

$$\begin{aligned} \left|\mathcal{L}^*_{r,q_r}(\varphi;x) - \varphi(x)\right| &\leq 2(1+(\delta^*)^2)(1+x^2)\omega^*(\varphi;\delta^*)\left[1 + \mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)\right. \\ &\left. + \frac{1}{\delta^*}\left(\mathcal{L}^*_{r,q_r}(t-x)^2;x\right)^{\frac{1}{2}}\left\{1 + \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^4;x)}\right\}\right] \\ &\leq 2(1+(\delta^*)^2)(1+x^2)\omega^*(\varphi;\delta^*)\left[1 + C_1(x+1)^2\right. \\ &\left. + \frac{1}{\delta^*}\sqrt{A_{r,q_r}(\varsigma)}(x+1)\left\{1 + \sqrt{C_2}(x+1)^2\right\}\right]. \end{aligned}$$

Choose $\delta^* = \sqrt{A_{r,q_r}(\varsigma)}$ and if take the supremum $x \in [0, A_{r,q_r}(\varsigma))$, then easily get our complete result. □

4. Direct approximation results of $\mathcal{L}^*_{r,q}$

In the present part we follow the space of K-functional and Lipschitz spaces and then give the approximation properties of our new operators (7)

Definition 4.1. *For every $\delta^* > 0$, and $f \in C[0,\infty)$, the properties of K-functional defined by:*

$$\mathcal{K}_\Psi(f;\delta^*) = \inf\left\{\left(\|f-\Psi\|_{C_\Phi[0,\infty)} + \delta^*\|\Psi''\|_{C_\Phi[0,\infty)}\right) : \Psi, \Psi' \in C^2_\Phi[0,\infty)\right\}, \tag{26}$$

$$C^k_\Phi[0,\infty) = \{f : f \in C_\Phi[0,\infty), k \in \mathbb{N}; \quad \text{such that } \lim_{x\to\infty}\frac{f(x)}{\Phi(x)} = c_f < \infty\}. \tag{27}$$

For an absolute and positive constant C one proved that

$$\mathcal{K}_\Psi(f;\delta^*) \leq C\{\min(1,\delta^*)\|f\|_{C_\Phi[0,\infty)} + \omega^*_2(f;\sqrt{\delta^*})\},$$

where $\omega^*_2(f;\delta^*)$ denotes the second order modulus of continuity given by

$$\omega^*_2(f;\delta^*) = \sup_{0<\mu<\delta^*}\sup_{x\in[0,\infty)}|f(x+2\mu) - 2f(x+\mu) + f(x)|. \tag{28}$$

While, the first order modulus of continuity is given by

$$\omega^*(f;\delta^*) = \sup_{0<\mu<\delta^*}\sup_{x\in[0,\infty)}|f(x+\mu) - f(x)|. \tag{29}$$

Theorem 4.2. *Let* $\Upsilon_{r,q}(x) = \frac{2q^{1+2\varsigma}}{[2]_q}x + \frac{1}{[2]_q[r]_q}\left(1 + 2q + 2q^2[2\varsigma - 1]_q + 2q^{1+2\varsigma}\frac{P'_q(1)}{P_q(1)}\right)$*, then for all* $\Psi \in C^2_\Phi[0,\infty)$*, we define an auxiliary operator* $\mathcal{K}^*_{r,q}$ *such that*

$$\mathcal{K}^*_{r,q}(\Psi;x) = \mathcal{L}^*_{r,q}(\Psi;x) + \Psi(x) - \Psi\left(\Upsilon_{r,q}(x)\right). \tag{30}$$

If we take $q = q_r,\ 0 < q_r < 1$*, then, for any* $\Psi \in C^2_\Phi[0,\infty)$ *operators* (30) *satisfying the property*

$$\left|\mathcal{K}^*_{r,q_r}(\Psi;x) - \Psi(x)\right| \left\{\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right) + \left(\mathcal{B}_{r,q_r}(x)\right)^2\right\} \parallel \Psi'' \parallel,$$

where $\delta^*_{r,q_r}(x)$ *is defined by Theorem 3.7 and*

$$\mathcal{B}_{r,q_r}(x) = \left(\frac{2q_r^{1+2\varsigma}}{[2]_{q_r}} - 1\right)x + \frac{1}{[2]_{q_r}[r]_{q_r}}\left(1 + 2q_r + 2q_r^2[2\varsigma-1]_{q_r} + 2q_r^{1+2\varsigma}\frac{P'_{q_r}(1)}{P_{q_r}(1)}\right).$$

Proof. For any $\Psi \in C^2_\Phi[0,\infty)$, it is easy to verify that $\mathcal{K}^*_{r,q_r}(1;x) = 1$ and

$$\mathcal{K}^*_{r,q_r}(t;x) = \mathcal{L}^*_{r,q_r}(t;x) + x - \left(\Upsilon_{r,q_r}(x)\right) = x.$$

We have

$$\parallel \mathcal{L}^*_{r,q_r}(\Psi;x) \parallel \leq \parallel \Psi \parallel,$$

and

$$\mid \mathcal{K}^*_{r,q_r}(\Psi;x) \mid \leq \mid \mathcal{L}^*_{r,q_r}(\Psi;x) \mid + \mid \Psi(x) \mid + \left|\Psi\left(\Upsilon_{r,q_r}(x)\right)\right| \leq 3 \parallel \psi \parallel. \tag{31}$$

Take $\Psi \in C^2_\Phi[0,\infty)$, then from Taylor series expansion we get that

$$\Psi(t) = \Psi(x) + (t-x)\Psi'(x) + \int_x^t (t-\varphi)\Psi''(\varphi)\mathrm{d}\varphi.$$

Therefore, on operating $\mathcal{K}^*_{r,q_r}$, we get

$$\begin{aligned}
\mathcal{K}^*_{r,q_r}(\Psi;x) - \Psi(x) &= \mathcal{K}^*_{r,q_r}\left(\int_x^t (t-\varphi)\Psi''(\varphi)\mathrm{d}\varphi;x\right) + \mathcal{K}^*_{r,q_r}(t-x;x)\Psi'(x) \\
&= \mathcal{K}^*_{r,q_r}\left(\int_x^t (t-\varphi)\Psi''(\varphi)\mathrm{d}\varphi;x\right) \\
&= \mathcal{L}^*_{r,q_r}\left(\int_x^t (t-\varphi)\varphi''(\varphi)\mathrm{d}\varphi;x\right) + \int_x^x (x-\varphi)\varphi''(\varphi)\mathrm{d}\varphi;x \\
&\quad - \int_x^{\Upsilon_{r,q_r}(x)} \left(\Upsilon_{r,q_r}(x) - \varphi\right)\Psi''(\varphi)\mathrm{d}\varphi;
\end{aligned}$$

$$\begin{aligned}
\mid \mathcal{K}^*_{r,q_r}(\Psi;x) - \Psi(x) \mid &\leq \left|\mathcal{L}^*_{r,q_r}\left(\int_x^t (t-\varphi)\Psi''(\varphi)\mathrm{d}\varphi;x\right)\right| \\
&\quad + \left|\int_x^{\Upsilon_{r,q_r}(x)} \left(\Upsilon_{r,q_r}(x) - \varphi\right)\Psi''(\varphi)\mathrm{d}\varphi\right|.
\end{aligned}$$

We know the inequality

$$\left|\int_x^t (t-\varphi)\Psi''(\varphi)\mathrm{d}\varphi\right| \leq (t-x)^2 \parallel \Psi'' \parallel$$

and

$$\left|\int_x^{\Upsilon_{r,q_r}(x)}\left(\Upsilon_{r,q_r}(x)-\varphi\right)\Psi''(\varphi)\mathrm{d}\varphi\right| \leq \left(\mathcal{B}_{r,q_r}(x)\right)^2 \parallel \Psi'' \parallel,$$

where

$$\mathcal{B}_{r,q_r}(x)=\left(\frac{2q_r^{1+2\varsigma}}{[2]_{q_r}}-1\right)x+\frac{1}{[2]_{q_r}[r]_{q_r}}\left(1+2q_r+2q_r^2[2\varsigma-1]_{q_r}+2q_r^{1+2\varsigma}\frac{P'_{q_r}(1)}{P_{q_r}(1)}\right)$$

Thus we get

$$|\,\mathcal{K}^*_{r,q_r}(\Psi;x)-\Psi(x)\,|\leq\left\{\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)+\left(\mathcal{B}_{r,q_r}(x)\right)^2\right\}\parallel \Psi''\parallel.$$

This gives the complete proof. □

Theorem 4.3. *Suppose $q=q_r$, $0<q_r<1$ and $\Psi\in C^2_\Phi[0,\infty)$, then for each $f\in C_\Phi[0,\infty)$ operators $\mathcal{L}^*_{r,q_r}$ have the property by*

$$\begin{aligned}\left|\mathcal{L}^*_{r,q_r}(f;x)-f(x)\right| \leq\ & \mathcal{M}\left[\omega_2^*\left\{f;\frac{1}{2}\sqrt{\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2}\right\}\right.\\ &\left.+\min\left\{1;\frac{1}{4}\left(\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2\right)\right\}\|f\|_{C_\Phi[0,\infty)}\right]\\ &+\omega^*\left(f;\left|\mathcal{B}_{r,q_r}(x)\right|\right),\end{aligned}$$

where $\mathcal{B}_{r,q_r}(x)$ defined by Theorem 4.2 and $\delta^*_{r,q_r}(x)$ is given in Theorem 3.7.

Proof. We take into account the Theorem 4.2 and suppose $\Psi\in C^2_\Phi[0,\infty)$, then for each given $f\in C_\Phi[0,\infty)$ we get that

$$\begin{aligned}|\,\mathcal{L}^*_{r,q_r}(f;x)-f(x)\,| =\ & \left|\mathcal{K}^*_{r,q_r}(f;x)-f(x)+f\left(\Upsilon_{r,q_r}(x)\right)-f(x)\right|\\ \leq\ & \left|\mathcal{K}^*_{r,q_r}(f-\Psi;x)\right|+\left|\mathcal{K}^*_{r,q_r}(\Phi;x)-\Psi(x)\right|\\ &+\left|\Psi(x)-f(x)\right|+\left|f\left(\Upsilon_{r,q_r}(x)\right)-f(x)\right|\\ \leq\ & 4\parallel f-\Psi\parallel+\omega^*\left(f;\left|\mathcal{B}_{r,q_r}(x)\right|\right)\\ &+\left\{\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2\right\}\parallel\Psi''\parallel.\end{aligned}$$

Apply the Infimum value for all $\Psi\in C^2_\Phi[0,\infty)$ and use of (26), it is obvious to get that

$$\begin{aligned}\left|\mathcal{L}^*_{r,q_r}(f;x)-f(x)\right| \leq\ & 4K_\Psi\left\{f;\frac{1}{4}\left(\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2\right)\right\}\\ &+\omega^*\left(f;\left|\mathcal{B}_{r,q_r}(x)\right|\right)\\ \leq\ & \mathcal{M}\left[\omega_2^*\left\{f;\frac{1}{2}\sqrt{\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2}\right\}\right.\\ &\left.+\min\left\{1;\frac{1}{4}\left(\delta^*_{r,q_r}(x)+\left(\mathcal{B}_{r,q_r}(x)\right)^2\right)\right\}\|f\|_{C_\Phi[0,\infty)}\right]\\ &+\omega^*\left(f;\left|\mathcal{B}_{r,q_r}(x)\right|\right).\end{aligned}$$

We get the prove here. □

The present part has tried to present the approximation results in local direct estimates for the operators $\mathcal{L}^*_{r,q}$ by (7). The study of the characteristic of Lipschitz-maximal function containing the parameters $\chi_1,\ \chi_2 > 0$ and $\xi \in (0,1]$. From [24], we easily recall the basic results of Lipschitz-maximal function such that

$$Lip^{\xi}_{\mathcal{M}} = \left\{ f \in C_{\Phi}[0,\infty) : |f(t) - f(x)| \leq \mathcal{M}\frac{|t-x|^{\xi}}{(\chi_1 x^2 + \chi_2 x + t)^{\frac{\xi}{2}}};\ x,\ t \in [0,\infty) \right\},$$

where $\mathcal{M}$ is a positive constant .

Theorem 4.4. *Suppose $\ell \in Lip^{\xi}_{\mathcal{M}}$ verifying the equality (**??**), then for $q = q_r,\ 0 < q_r < 1$ operators $\mathcal{L}^*_{r,q_r}$ hold the property*

$$|\mathcal{L}^*_{r,q_r}(\ell;x) - \ell(x)| \leq \mathcal{M}\left(\frac{\delta^*_{r,q_r}(x)}{(\chi_1 x^2 + \chi_2 x)}\right)^{\frac{\xi}{2}},$$

where $\delta^*_{r,q_r}(x)$ is obtained by Theorem 3.7.

Proof. Let $\ell \in Lip^{\xi}_{\mathcal{M}}$ for any $0 < \xi \leq 1$, then first we want to prove the results will be true for $\xi = 1$, and the we show it for $0 < \xi \leq 1$. If $\chi_1,\ \chi_2 \geq 0$, then obvious to get that $(\chi_1 x^2 + \chi_2 x + t)^{-1/2} \leq (\chi_1 x^2 + \chi_2 x)^{-1/2}$. Thus in order to apply the well-known results of Cauchy-Schwarz inequality, obvious to write here

$$\begin{aligned}
|\mathcal{L}^*_{r,q_r}(\ell;x) - \ell(x)| &\leq |\mathcal{L}^*_{r,q_r}(|\ell(t) - \ell(x)|;x)| + \ell(x)\,|\mathcal{L}^*_{r,q_r}(1;x) - 1| \\
&\leq \mathcal{L}^*_{r,q_r}\left(\frac{|t-x|}{(\chi_1 x^2 + \chi_2 x + t)^{\frac{1}{2}}};x\right) \\
&\leq \mathcal{M}(\chi_1 x^2 + \chi_2 x)^{-1/2}\mathcal{L}^*_{r,q_r}(|t-x|;x) \\
&\leq \mathcal{M}\mathcal{L}^*_{r,q}((t-x)^2(\chi_1 x^2 + \chi_2 x)^{-1/2};x)\Big|^{1/2}.
\end{aligned}$$

Thus due to above detailed mathematical explanation our statement is valid for $\xi = 1$. Now we establish the validation of statement for $0 < \xi < 1$. In addition to prove it we apply the monotonicity property to $\mathcal{L}^*_{r,q_r}$ and use Hölder's inequality

$$\begin{aligned}
\left|\mathcal{L}^*_{r,q_r}(\ell;x) - \ell(x)\right| &\leq \mathcal{L}^*_{r,q_r}(|\ell(t) - \ell(x)|;x) \\
&\leq \left(\mathcal{L}^*_{r,q_r}\left(|\ell(t) - \ell(x)|^{\frac{2}{\xi}};x\right)\right)^{\frac{\xi}{2}}\left(\mathcal{L}^*_{r,q_r}(1;x)\right)^{\frac{2-\xi}{2}} \\
&\leq \mathcal{M}\left\{\frac{\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)}{t + \chi_1 x^2 + \chi_2 x}\right\}^{\frac{\xi}{2}} \\
&\leq \mathcal{M}(\chi_1 x^2 + \chi_2 x)^{-\xi/2}\left\{\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)\right\}^{\frac{\xi}{2}} \\
&\leq \mathcal{M}(\chi_1 x^2 + \chi_2 x)^{-\xi/2}\left(\mathcal{L}^*_{r,q_r}(t-x)^2;x\right)^{\frac{\xi}{2}} \\
&= \mathcal{M}\left(\frac{\delta^*_{r,q_r}(x)}{(\chi_1 x^2 + \chi_2 x)}\right)^{\frac{\xi}{2}},
\end{aligned}$$

which gives the proof. □

Here we try to present the anther local direct approximation results for the operators $\mathcal{L}^*_{r,q}$. Let the function $\chi \in C_{\Phi}[0,\infty),\ 0 < \xi \leq 1$ and $t, x \in [0,\infty)$, from [13] we recall that

$$\omega^*_{\xi}(\chi;x) = \sup_{t \neq x,\ t \in [0,\infty)} \frac{|\ \chi(t) - \chi(x)\ |}{|\ t - x\ |^{\xi}}. \tag{32}$$

Theorem 4.5. *For all $\chi \in C_{\Phi}[0,\infty)$ and $x \in [0,\infty)$, if $q = q_r,\ 0 < q_r < 1$, then*

$$\left|\mathcal{L}^*_{r,q_r}(\chi;x) - \chi(x)\right| \leq \left(\delta^*_{r,q_r}(x)\right)^{\frac{\xi}{2}} \omega^*_{\xi}(\chi;x),$$

where $\omega^*_{\xi}(\chi;x)$ is given by and $\delta^*_{r,q_r}(x)$ is defined by Theorem 3.7.

Proof. To prove it we use Hölder inequality, thus we get

$$\begin{aligned}
\left|\mathcal{L}^*_{r,q_r}(\chi;x) - \chi(x)\right| &\leq \mathcal{L}^*_{r,q_r}(|\chi(t) - \chi(x)|\,;x) \\
&\leq \omega^*_{\xi}(\chi;x) \mid \mathcal{L}^*_{r,q_r}\left(|t-x|^{\xi}\,;x\right) \\
&\leq \omega^*_{\xi}(\chi;x)\left(\mathcal{L}^*_{r,q_r}(1;x)\right)^{\frac{2-\xi}{2}}\left(\mathcal{L}^*_{r,q_r}(|t-x|^2\,;x)\right)^{\frac{\xi}{2}} \\
&= \omega^*_{\xi}(\chi;x)\left(\mathcal{L}^*_{r,q_r}\left((t-x)^2;x\right)\right)^{\frac{\xi}{2}}.
\end{aligned}$$

This gives the complete proof here. □

5. Voronovskaja-type approximation results

In this portion, we obtain the quantitative approximation by use of Voronovskaja-type theorem for our operators $\mathcal{L}^*_{r,q}(f;x)$ defined by equality (7):

Theorem 5.1. *For all $x \in [0,\infty) \subset \mathbb{R}$ and $\varphi \in C_{\Phi}[0,\infty)$, operators $\mathcal{L}^*_{r,q}$ satisfying that*

$$\lim_{r\to\infty}\{\mathcal{L}^*_{r,q_r}(\varphi;x) - \varphi(x)\} = \left(\frac{2q_r^{1+2\varsigma}}{[2]_{q_r}} - 1\right)x\varphi'(x) + \left(\frac{3q_r^{2(1+\varsigma)}}{[3]_{q_r}} - \frac{4q_r^{1+2\varsigma}}{[2]_{q_r}} + 1\right)x^2\frac{\varphi''(x)}{2},$$

where $\varphi'(x)$ and $\varphi''(x)$ belongs to $C_{\Phi}[0,\infty)$, and $q = q_r,\ 0 < q_r < 1$.

Proof. For any $\varphi(x) \in C_{\Phi}[0,\infty)$, the Taylor's series expansion give us

$$\varphi(t) = \varphi(x) + \varphi'(x)(t-x) + \varphi''(x)\frac{(t-x)^2}{2} + (t-x)^2\, S_x(t), \tag{33}$$

where $S_x(t)$ denoted for remainder term containing in $[0,\infty)$ and satisfying $S_x(t) \to 0$ when $t \to x$. On operating $\mathcal{L}^*_{r,q_r}$ to (33), and by the consequence of Cauchy-Schwarz inequality, we see

$$\begin{aligned}
\mathcal{L}^*_{r,q_r}(\varphi;x) - \varphi(x) &= \varphi'(x)\mathcal{L}^*_{r,q_r}(t-x;x) + \frac{\varphi''(x)}{2}\mathcal{L}^*_{r,q_r}((t-x)^2;x) + \mathcal{L}^*_{r,q_r}((t-x)^2 S_x(t);x) \\
&\leq \varphi'(x)\mathcal{L}^*_{r,q_r}(t-x;x) + \frac{\varphi''(x)}{2}\mathcal{L}^*_{r,q_r}((t-x)^2;x) \\
&\quad + \sqrt{\mathcal{L}^*_{r,q_r}((t-x)^4;z)}\sqrt{\mathcal{L}^*_{r,q_r}(S_x^2(t);x)}.
\end{aligned}$$

Since we have $\lim_{r\to\infty}\mathcal{L}^*_{r,q_r}(S_x(t);x) = 0$, therefore

$$\lim_{r\to\infty}\{\mathcal{L}^*_{r,q_r}((t-x)^2 S_x(t);x)\} = 0.$$

Therefore, we have

$$\begin{aligned}
\lim_{r\to\infty}\{\mathcal{L}^*_{r,q_r}(\varphi;x) - \varphi(x)\} &= \varphi'(x)\lim_{r\to\infty}\left\{\mathcal{L}^*_{r,q_r}(t-x;x) + \frac{\varphi''(x)}{2}\mathcal{L}^*_{r,q_r}((t-x)^2;x)\right. \\
&\quad \left. + \mathcal{L}^*_{r,q_r}((t-x)^2 S_x(t);x)\right\}.
\end{aligned}$$

Which gives the desired prove. □

Inspired by Theorem 5.1 we directly conclude the below corollary.

Corollary 5.2. *For all $\varphi \in C[0,\infty)$, it satisfies that*

$$\lim_{r\to\infty}\left[\mathcal{L}^{*}_{r,q}(\varphi;x)-\varphi(x)-\left(\frac{2q^{1+2\varsigma}}{[2]_q}-1\right)x\varphi'(x)+\left(\frac{3q^{2(1+\varsigma)}}{[3]_q}-\frac{4q^{1+2\varsigma}}{[2]_q}+1\right)x^2\frac{\varphi''(x)}{2}\right]=0.$$

6. Conclusion

In our study we investigate the approximation in q-variant of recent studied by [21, 22]. We directly apply the basic definitions and properties of q-calculus and introduced the q-variant of [22] the Kantorovich Szász-Mirakjan-operators involving the Appell induced by the parametric variant of Dunkl generalizations. These types of approximation results are better generalized version of published article rather than the earlier study demonstrations investigated by [19, 21, 22, 27].Finally, we have discuss the global approximation, local direct approximations, Lipschitz-type approximations, rate of convergence and Voronovskaja-type approximation theorems of our new operators.

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Conflicts of Interest
The author declares no conflict of interest.

Availability of data and material
Not applicable.

Authors' contributions
Author's declare that they have equally contributed the manuscript.

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