Filomat 37:1 (2023), 37–41
https://doi.org/10.2298/FIL2301037A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Jordan $*$-derivations on standard operator algebras

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Abstract. Let $\mathcal{H}$ be a real or complex Hilbert space with $dim(\mathcal{H}) > 1$, $B(\mathcal{H})$ be algebra of all bounded linear operators on H and $A(\mathcal{H}) \subseteq B(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. If $D : A(\mathcal{H}) \to B(\mathcal{H})$ is a linear mapping satisfying $D(A^{n+1}) = \sum_{i=0}^{n} A^i D(A)(A^*)^{n-i}$ for all $A \in A(\mathcal{H})$, then D is a Jordan $*$-derivation on $A(\mathcal{H})$. Later, we discuss some algebraic identities on semiprime rings.

1. Introduction

Throughout R will represent an associative ring. A ring R is said to be n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. Let A be an algebra over a real or complex field and B be a subalgebra of A. A linear mapping $D : B \to A$ is called linear derivation if $D(b_1b_2) = D(b_1)b_2 + b_1D(b_2)$ for all $b_1, b_2 \in B$. In the case of ring, an additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. A derivation d is said to be inner if there exists a fixed element $a \in R$ such that $d(x) = ax - xa$ for all x in R. Every derivation is a Jordan derivation but the converse need not be true in general. A classical result of Herstein [4, Theorem 3.3] states that every Jordan derivation on a prime ring of characteristic different from two is a derivation. Bresar and Vukman [2] reword this result. Further, Cusack [3] rescript the same idea for semiprime ring. In fact author proved that every Jordan derivation on a 2-torsion free semiprime ring is a derivation (see also [3] for an alternate proof). Following [1], an additive mapping $d : R \to R$ is called Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. One can easily proved that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation. Bresar [1] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. An additive mapping $d : R \to R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$.

Let $\mathcal{H}$ be a real or complex Hilbert space with $dim(\mathcal{H}) > 1$ and $B(\mathcal{H})$ denotes algebra of all bounded linear operators on $\mathcal{H}$. Throughout the present paper, $F(\mathcal{H})$ stands for the subalgebra of bounded finite ranks operators. Note that $F(\mathcal{H})$ forms a $*$-closed ideal in $B(\mathcal{H})$. An algebra $A(\mathcal{H})$ which is contained in

2020 *Mathematics Subject Classification*. 16W25, 16N60, 16W10, 13N15, 46K15.
Keywords. Semiprime ring; Standard operator algebras; Hilbert space; Jordan $*$-derivation; Involution.
Received: 28 October 2020; Accepted: 20 September 2021
Communicated by Dragan S. Djordjević
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$B(\mathcal{H})$ is called standard operator algebra if $F(\mathcal{H})$ is proper subset of $A(\mathcal{H})$. Hahn-Banach theorem says that any standard operator algebra is prime. In 2005, Vukman et al. [7] proved that D will be a derivation if it satisfies the condition $D(x^n) = \sum_{j=1}^{n} x^{n-j}D(x)x^{j-1}$ for all $x \in R$ with some suitable torsion restrictions. Further, they have extended this result on standard operator algebras and semisimple H* algebras. In view of the above theorems, it is legitimate to think whether a linear mapping $D : A(\mathcal{H}) \to B(\mathcal{H})$ satisfying $D(A^{n+1}) = \sum_{i=0}^{n} A^i D(A)(A^*)^{n-i}$ for all $A \in A(\mathcal{H})$, is a Jordan $*$-derivation on $A(\mathcal{H})$. The answer is in affirmative sense. So, first we discuss this problem. Later we examine the behaviour of some algebraic identities in semiprime rings. To prove our main results, we need one of the prominent result due to Šemrl [6], which states the following:

Lemma 1.1 ([6, Theorem]). *Let $\mathcal{H}$ be real or complex Hilbert space with dim$(\mathcal{H}) > 1$ and $A(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose that $D : A(\mathcal{H}) \to B(\mathcal{H})$ is a linear Jordan $*$-derivation. Then there exists a unique linear operator $B \in B(\mathcal{H})$ such that $D(A) = AB - BA^*$ for all $A \in A(\mathcal{H})$.*

2. Main Results

Theorem 2.1. *Let $\mathcal{H}$ be a real or complex Hilbert space with dim$(\mathcal{H}) > 1$ and $A(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose that $D : A(\mathcal{H}) \to B(\mathcal{H})$ is a linear mapping satisfying $D(A^{n+1}) = \sum_{i=0}^{n} A^i D(A)(A^*)^{n-i}$ for all $A \in A(\mathcal{H})$, then D is a Jordan $*$-derivation on $A(\mathcal{H})$.*

Proof. Given that

$$D(A^{n+1}) \quad = \quad \sum_{i=0}^{n} A^i D(A)(A^*)^{n-i} \text{ for all } A \in A(\mathcal{H}). \tag{1}$$

First we consider that D in $F(\mathcal{H})$. Let A any operator in $F(\mathcal{H})$, and $P \in F(\mathcal{H})$ be an idempotent operator with the condition $AP = PA = A$. Replacing A by P in equation (1), we get

$$D(P) \quad = \quad D(P)P^* + (n-1)PD(P)P^* + PD(P). \tag{2}$$

Multiplying from right side by P^* to the above equation, we obtain $PD(P)P^* = 0$. Replacing A by $A + P$ in (1), we find that

$$D((A+P)^{n+1}) \quad = \quad \sum_{i=0}^{n} (A+P)^i D(A+P)((A+P)^*)^{n-i} \text{ for all } A \in A(\mathcal{H}). \tag{3}$$

On expanding, we have that

$$\begin{aligned}
&D\Big(A^{n+1} + \binom{n+1}{1} A^n P + \binom{n+1}{2} A^{n-1}P^2 + \cdots + P^{n+1}\Big) \\
&= \sum_{i=0}^{n} \Big\{A^i + \cdots + \binom{i}{i-2} A^2 P^{i-2} + \binom{i}{i-1} AP^{i-1} + P^i\Big\} D(x) \Big\{(A^*)^{n-i} \\
&+ \cdots + \binom{n-i}{n-i-2} (A^*)^2 P^{n-i-2} + \binom{n-i}{n-i-1} A^* P^{n-i-1} + P^{n-i}\Big\}.
\end{aligned}$$

Put (1) in an application to acquire

$$
\begin{aligned}
&D\Big(A^{n+1} + \Big(\begin{array}{c} n+1 \\ 1 \end{array}\Big)A^nP + \Big(\begin{array}{c} n+1 \\ 2 \end{array}\Big)A^{n-1}P^2 + \cdots + P^{n+1}\Big) \\
&= \sum_{i=0}^{n} A^iD(A)\Big\{\Big(\begin{array}{c} n-i \\ 1 \end{array}\Big)(A^*)^{n-i-1}P + \ldots + \Big(\begin{array}{c} n-i \\ n-i-2 \end{array}\Big)(A^*)^2P^{n-i-2} \\
&+\Big(\begin{array}{c} n-i \\ n-i-1 \end{array}\Big)A^*P^{n-i-1} + P^{n-i}\Big\} + \sum_{i=0}^{n}\Big\{\Big(\begin{array}{c} i \\ 1 \end{array}\Big)A^{i-1}k + \Big(\begin{array}{c} i \\ 2 \end{array}\Big)A^{i-2}P^2 + \cdots \\
&+\Big(\begin{array}{c} i \\ i-2 \end{array}\Big)A^2P^{i-2} + \Big(\begin{array}{c} i \\ i-1 \end{array}\Big)AP^{i-1} + P^i\Big\} D(x)\Big\{(A^*)^{n-i} + \Big(\begin{array}{c} n-i \\ 1 \end{array}\Big)(A^*)^{n-i-1}P + \ldots \\
&+\Big(\begin{array}{c} n-i \\ n-i-2 \end{array}\Big)(A^*)^2P^{n-i-2} + \Big(\begin{array}{c} n-i \\ n-i-1 \end{array}\Big)A^*P^{n-i-1} + P^{n-i}\Big\}.
\end{aligned}
$$

Which can be expressed as

$$Pf_1(A,P) + P^2f_2(A,P) + \cdots + P^nf_n(A,P) = 0 \text{ for all } A \in F(\mathcal{H}),$$

where $f_i(A,P)$ are the coefficients of $P^i{'}s$ for all $i = 1, 2, \cdots, n$. Now, replacing P by $1, 2, \cdots, n$ in turn and considering the resulting system of n homogeneous equations, we get that the resulting matrix of the system is a Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^n \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ n & n^2 & \ldots & n^n \end{pmatrix}.$$

Since the determinant of the matrix is equal to the product of positive integers, each of which is less then n, it follows immediately that $f_i(A,P) = 0$ for all $A \in F(\mathcal{H})$ and $i = 1, 2, \cdots, n$. Now, $f_n(A,P) = 0$ implies that

$$(n+1)D(A) = D(A) + nD(P)A^* + nD(A),$$

which yields that $D(P)A^* = 0$. Again, $f_{n-1}(A,P) = 0$ gives that

$$
\begin{aligned}
n(n+1)D(A^2) &= 2nD(A)A^* + n(n-1)D(P)(A^*)^2 \\
&\quad + n(n+1)AD(A) + n(n-1)D(A)A^*.
\end{aligned}
$$

That is,

$$(n+1)D(A^2) = 2D(A)A^* + (n-1)D(P)(A^*)^2 + (n+1)AD(A) + (n-1)D(A)A^* \text{ for all } A \in F(\mathcal{H}).$$

Since we have that $D(P)A^* = 0$. Using this in the above relation, we find that

$$
\begin{aligned}
(n+1)D(A^2) &= 2D(A)A^* + (n-1)D(P)(A^*)^2 \\
&\quad + (n+1)AD(A) + (n-1)D(A)A^* \\
&= (n+1)D(A)(A^*) + (n+1)AD(A).
\end{aligned}
$$

Therefore, we get

$$D(A^2) = D(A)A^* + AD(A). \tag{4}$$

Hence, D is Jordan $*$-derivation on $F(\mathcal{H})$. From Lemma 1.1, we conclude that $D(A) = AB - BA^*$. Now, the next target is show that (4) holds for all $A \in A(\mathcal{H})$. For this we define a mapping $D_0 : A(\mathcal{H}) \to B(\mathcal{H})$ by $D(A) = AB - BA^*$ and consider $D_1 = D - D_0$. In fact the mapping D_1 will be linear and satisfying equations (1). Other than linearity D_1 vanishes on $A(\mathcal{H})$. Let $A \in A(\mathcal{H})$ and P be an idempotent operator of rank 1. Let us define $S \in A(\mathcal{H})$ by $S = A + PAP - (AP + PA)$. Since $D_1(S) = D_1(A)$ and $SP = PS = 0$, then use equation (1) to have

$$
\begin{aligned}
D_1(A^{n+1}) &= \sum_{i=0}^{n} A^i D_1(A)(A^*)^{n-i} \\
\sum_{i=0}^{n} S^i D_1(S)(S^*)^{n-i} &= D_1(S^{n+1}) \\
&= D_1(S^{n+1} + P) \\
&= D_1(S + P)^{n+1} \\
&= \sum_{i=0}^{n} (S + P)^i D_1(S + P)((S + P)^*)^{n-i} \\
&= \sum_{i=0}^{n} (S + P)^i D_1(A)((S + P)^*)^{n-i} \\
&= \sum_{i=0}^{n} (S^i + P) D_1(A)((S^*)^{n-i} + P^*)) \\
&= \sum_{i=0}^{n} (S^i + P) D_1(A)((S^*)^{n-i} + P^*)) \\
&= \sum_{i=0}^{n} S^i D_1(A)(S^*)^{n-i}) + \sum_{i=0}^{n} S^i D_1(A) P^* \\
&\quad + \sum_{i=0}^{n} P D_1(A)(S^*)^{n-i}) + P D_1(A) P^*.
\end{aligned}
$$

This implies that

$$\sum_{i=0}^{n} S^i D_1(A) P^* + \sum_{i=0}^{n} P D_1(A)(S^*)^{n-i} + P D_1(A) P^* = 0. \tag{5}$$

Multiplying from left side by P and from right side by P^*, we obtain

$$P D_1(A) P^* = 0 \tag{6}$$

Using (6) in (5), we find

$$\sum_{i=0}^{n} S^i D_1(A) P^* + \sum_{i=0}^{n} P D_1(A)(S^*)^{n-i} = 0. \tag{7}$$

Multiplying (7) by P from left side yields that

$$\sum_{i=0}^{n} P D_1(A)(S^*)^{n-i} = 0. \tag{8}$$

Replace $2A$ in place of A to get

$$\sum_{i=0}^{n} 2^{n-i} P D_1(A)(S^*)^{n-i} = 0. \tag{9}$$

Multiplying (8) by 2^{n-1} and subtracting the relation so obtained from the above relation, we arrive at $\sum_{i=0}^{n} (2^{n-i-1} - 2^{n-1}) P D_1(A)(S^*)^{n-i} = 0$. Hence, $P D_1(A) = 0$. Since P is an arbitrary idempotent of rank 1, then $D_1(A) = 0$ for all $A \in A(\mathcal{H})$, which completes the proof of the theorem.

This would be interesting to analyze the problem in purely ring theoretic context as only algebraic concepts are used in formulation of the result given in the present paper. In view of this assessment, we conclude the following results:

Theorem 2.2. *Let $n \geq 1$ be a fixed integer and R be a $(n+1)!$ torsion free $*$-ring with identity. Suppose that $D : R \rightarrow R$ is an additive mapping satisfying $D(x^{n+1}) = \sum_{i=0}^{n} x^i D(x)(x^*)^{n-i}$ for all $x \in R$, then D is a Jordan $*$-derivation on R.*

Proof. The proof goes through using the same arguments as in the proof of Theorem 2.1. Theorem 2.2 is a corollary of Theorem 2.1 in [5].

Theorem 2.3. *Let $n \geq 1$ be a fixed integer and R be a $(n+1)!$ torsion free semiprime ring with identity. Suppose that $D : R \rightarrow R$ is an additive mapping satisfying $D(x^{n+1}) = \sum_{i=0}^{n} x^i D(x) x^{n-i}$ for all $x \in R$, then D is a derivation on R.*

Proof. We get the required result if we go through the identical arguments as in Theorem 2.1.

Acknowledgment: The researchers wish to extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post-Publishing Program 1.

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