Filomat 37:1 (2023), 317–334
https://doi.org/10.2298/FIL2301317A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Global existence and stability results for partial delay integro-differential equations with random impulses

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Abstract. We investigate the global existence, continuous dependence and exponential stability for mild solutions of a class of delay partial integro-differential equation with random impulsive moments. The results are obtained by using the Leray-Schauder alternative fixed point theory and Banach contraction principle.

1. Introduction

The study of global existence and qualitative properties, of a solution for partial differential equations and integro differential equations are very limited. Partial differential and partial integro-differential equations (PIDE) arise in many engineering and scientific disciplines. PIDE play very essential role in real world modelling problems with derivative and integral terms. Yong Chen [10] studied the second-order convergent IMEX scheme for solving the 2-dimensional PIDE with spatial delays arising in option pricing under the hard-to-borrow jump-diffusion models. There are problems in science and engineering that deal with PIDE's. Takács [29] studied qualitative properties of space-dependent SIR models with constant delay and their numerical solutions, by constructing a delay PIDE using biological sciences. Agarwal [2] studied stability of partial functional integro-differential equations and phase transition dynamics with the memory of visco-elasticity. Fractional PIDE with spatial-time delay is discussed in [8]. State delay for a first order hyperbolic PIDE is discussed in [43]. Second-order PIDE can be seen in [9]. Further, PIDEs are used in privacy-preserving, ensuring secure communication, ship course-keeping, population dynamics, mathematical physics, nuclear science, finance and heat transfer. For further reading and more details, refer [3, 4, 8, 12, 25, 27, 44] and the references therein.

Impulsive differential equations are well known in modelling problems from many areas of science and engineering. There has been much research activity concerning the theory of impulsive differential equations see [22, 26]. The impulses may exists at deterministic or random points. E. Hernández et al., [13–16] has studied the impulsive global partial differential equations and the references therein. Further, Sivasankaran et al. [28] studied the existence of global solutions for second-order impulsive abstract

2020 *Mathematics Subject Classification*. Primary 34A12, 60H99, 35R12, 37C75; Secondary 93E99.
Keywords. Delay partial integro-differential equations, random impulses, stability, Leray-Schauder alternative fixed point theorem, Contraction Principle.
Received: 27 November 2021; Revised: 22 November 2022; Accepted: 23 December 2022
Communicated by Miljana Jovanović
The First author acknowledge University Grant Commission for the financial support to the project F.No. 41-780/2012.
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partial differential equations using Leray–Schauder's alternative fixed point theorem. Vijayakumar et al. in [37, 38] proved the existence of global solutions for second-order impulsive differential equations with nonlocal conditions using Leray–Schauder's alternative fixed point theorem. Milan [23] studied the sufficient conditions for the existence of global solutions of nonlinear functional-differential evolution equations whose linear parts are infinitesimal generators of strongly continuous and analytic semigroups. Moreover, the fractional order PIDE with spatial-time and multiple delays are studied in [8, 12, 43]. There are lot of papers which investigate the properties of deterministic impulses see [5, 17] and the references therein.

Considerable attention have been given to the fixed impulses in the field of nonlinear differential systems, where the impulses do not always occur at fixed time points in the system state. There is a possibility that it occurs at random time points in the system state, since the real world system states are often subject to random changes. Moreover, the solutions follow a stochastic process when the impulse arrival time is taken at random. It is certainly different from the fixed impulse approach. However, a few research works have been done based on the stability of various differential systems, including random impulses. For example, Shujin Wu et. al.,[39–42] studied the qualitative properties of random impulsive differential system. In [6], the author studied the existence and exponential stability of a random impulsive semilinear functional differential equation through the fixed point technique under non-uniqueness. In [7, 30], the authors established the existence, uniqueness and stability results through Banach fixed point method for the system of random impulsive differential equations. In [31], the author studied the existence results for the random impulsive differential inclusions with delays. Further, Agarwal,R. P et. al.,[1], studied the exponential stability of differential equations with random impulses at random times. In [34], the problem of p^{th} moment global exponential stability for functional differential equations and scalar chaotic delayed equations under random impulsive effects have been studied. In [32, 33, 36], the study of unstable continuous time delay systems controlled by the random impulses has been investigated. Recently, in [19, 20] the authors studied the exponential stability of stochastic differential systems, when the impulse arrival time is taken at random. Till the date, there has been no result established based on exponential stability of random impulsive PIDEs. It is different from the stochastic differential equations. This is the motivation behind our study on random impulsive PIDEs in this paper. We have utilized the techniques developed in [11, 22, 24, 26] to establish our results.

The paper is organized as follows: In section 2, the useful notations, definitions and preliminary facts are briefly recalled. In section 3, we investigate the existence of mild solutions of partial delay integro-differential equations with random impulses by using Leray - Schauder alternative fixed point theory and Banach contraction principle. In section 4, we establish the global existence of solutions for random impulsive partial delay integro-differential equations by using Leray - Schauder alternative fixed point theory and Banach contraction principle. In section 5, we study the stability through continuous dependence on initial conditions. Finally in section 6, we study the exponential stability using Leray - Schauder alternative fixed point theory.

2. Preliminaries

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent exponentially distributed random variable with parameter λ, and each random variable τ_k is defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. For the sake of simplicity, we denote $\mathfrak{R}_\tau = [\tau, +\infty), \mathfrak{R}^+ = [0, +\infty)$.

We consider integrodifferential equation of the form

$$\begin{cases} x'(t) &= Ax(t) + \displaystyle\int_0^t F\Big(t, s, x(\sigma(s))\Big)ds, t \neq \xi_k, \quad t \geq \tau, \\ x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \;\; k = 1, 2, \cdots, \\ x_{t_0} &= \varphi \end{cases} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ with domain $D(A) \subset X$; the nonlinear operator $F : \Delta \times C \to X$, $C = C([-r, 0], X)$ is the set of continuous and bounded functions mapping $[-r, 0]$ into X with some given $r > 0$; $\sigma : \mathfrak{R}^+ \to \mathfrak{R}^+$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \cdots$, Here $t_0 \in R_\tau$ is an arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$; $b_k : D_k \to R$ for each $k = 1, 2, \cdots$; $x(\xi_k^-) = \lim_{t\uparrow\xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{t-r\le s\le t} |x(s)|$ for each t satisfying $\tau \le t \le T$ $\|\cdot\|$ is any given norm in X, here Δ denotes the set$\{(t, s) : 0 \le s \le t < \infty\}$.

$\{\mathcal{G}_t, t \ge 0\}$ denotes the simple counting process generated by $\{\xi_n\}$, that is, $\{\mathcal{G}_t \ge n\} = \{\xi_n \le t\}$, and $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\mathcal{G}_t, t \ge 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \mathcal{F}_t, X)$ denote the Hilbert space of all $\mathcal{F}_t$ - measurable square integrable random variables with values in X.

Assume that $T > t_0$ is any fixed time to be determined later and let $\mathcal{B}$ denote the Banach space $\mathcal{B}\Big([t_0 - r, T], L_2\Big)$, the family of all $\mathcal{F}_t$-measurable, C-valued random variables ψ with the norm

$$\|\psi\|_{\mathcal{B}} = \left(\sup_{t_0\le t\le T} E\|\psi\|_t^2\right)^{1/2}.$$

Let $L_2^0(\Omega, \mathcal{B})$ denote the family of all $\mathcal{F}_0$ - measurable, $\mathcal{B}$ - valued random variable φ.

Remark 2.1. *For any given time $t > 0$ and integer $n \ge 1$, ξ_n and $\mathcal{G}_t$ are related by $\{\mathcal{G}_t \ge n\} = \{\xi_n \le t\}$. It is understood that, $\{\xi_n \le t\}$ is the event that the n^{th} arrival occurs by time t, implies that $\mathcal{G}_t$, the number of arrivals by time t, must be at least n. Similarly, $\{\mathcal{G}_t \ge n\}$ implies $\{\xi_n \le t\}$, yielding the equality. The counting process $\{\mathcal{G}_t, t \ge 0\}$ is a stochastic process in time. The given filtration $\mathcal{F}_t$ represents the evolution of knowledge about the random system through time. The information at time t carried by filtration $\mathcal{F}_t$ determines the value of the random variable ξ_k.*

Lemma 2.2. *[1] The probability that there will be exactly k impulses until the time $t, t \ge t_0$,where impulse moments $\xi_k, k = 1, 2, \cdots$ follow exponential distribution with parameter λ, is given by the equality $P(I_{[\xi_k,\xi_{k+1})}(t)) = \frac{\lambda^k(t-t_0)^k}{k!} e^{-\lambda(t-t_0)}$, where, the events, $I_{[\xi_k,\xi_{k+1})}(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_{k+1}(\omega)\}, k = 1, 2, \cdots$.*

Remark 2.3. *In [1], the expected value of the solution $x(t)$ for the random impulsive differential equation is given as*

$$E[\|x(t)\|] = \sum_{k=0}^{\infty} E[\|x(t)\||I_{[\xi_k,\xi_{k+1})}(t)]P(I_{[\xi_k,\xi_{k+1})}(t)),$$

where the impulse moments $\xi_k, k = 1, 2, \cdots$ follow exponential distribution with parameter λ.

Definition 2.4. *[11, 24] A semigroup $\{S(t); t \ge t_0\}$ is said to be exponentially stable if there are positive constants $M \ge 1$ and $\gamma > 0$ such that $\|S(t)\| \le Me^{-\gamma(t-t_0)}$ for all $t \ge t_0$. A semigroup $\{S(t),\ t \ge t_0\}$ is said to be uniformly bounded if $\|S(t)\| \le M$ for all $t \ge t_0$, where $M \ge 1$ is some constant. If $M = 1$, then the semigroup is said to be contraction semigroup.*

Definition 2.5. *[21] A map $\mathcal{F}(t, s, x) : \Delta \times C \to X$, for all $t \in [\tau, T]$, $\mathcal{F}(t, \cdot, \cdot)$ satisfies $\mathcal{L}^2$-Caratheodory, if*
(i) $s \to \mathcal{F}(t, s, x)$ and is measurable for each $x \in C$;
(ii) $x \to \mathcal{F}(t, s, x)$ is continuous for almost all $t \in [\tau, T]$;
(iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1([\tau, T], \mathcal{R}^+)$ such that
$\sup_{E\|x\|^2\le m} E\|F(t, s, x)\|^2 \le \alpha_m(t)$, *for $t \in [\tau, T]$, a.e.*

Definition 2.6. *For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B},\ t_0 - r \le t \le T\}$ is called a mild solution to equation (1) in $(\Omega, P, \{\mathcal{F}_t\})$, if*
(i) *$x(t) \in X$ is $\mathcal{F}_t$−adapted for $t \ge t_0$;*

(ii) $x(t_0 + s) = \varphi(s) \in L_2^0(\Omega, \mathcal{B})$ *when* $s \in [-r, 0]$

$$\begin{aligned} x(t) \quad = \quad & \sum_{k=0}^{+\infty} \Big(\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu))) d\mu ds \\ & + \int_{\xi_k}^{t} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu))) d\mu ds \Big) I_{[\xi_k,\xi_{k+1})}(t), \ t \in [t_0, T], \end{aligned} \qquad (2)$$

where $\prod_{j=m}^{n}(\cdot) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i)$, *and* $I_A(\cdot)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & if \ \ t \in A, \\ 0, & if \ \ t \notin A. \end{cases}$$

Our existence theorem is based on the following theorem, which is a version of the topological transversality theorem.

Lemma 2.7. *Let B be a convex subset of a Banach space E and assume that* $0 \in B$. *Let* $F : B \to B$ *be a completely continuous operator and let* $U(F) = \{x \in B : x = \lambda\, Fx$ *for some* $0 < \lambda < 1\}$;
then either $U(F)$ is unbounded or F has a fixed point.

3. Existence of mild solution

In this section, we prove the existence theorem by using the following hypothesis.
(H_1): The function $F : [t_0, T] \times [t_0, T] \times C \to X$ is continuous, $F(t, s, 0) = 0$, and it satisfies the Lipschitz condition with respect to x, ie.,

$$E\|F(t,s,x_1) - F(t,s,x_2)\|^2 \le L(t,s,E\|x_1\|^2, E\|x_2\|^2) E\|x_1 - x_2\|_s^2, \ \ (t,s) \in \Delta, x_1, x_2 \in X,$$

where $L : [t_0, T] \times [t_0, T] \times \Re^+ \times \Re^+ \to \Re^+$ and is monotonically nondecreasing with respect to the second and third arguments.
(H_2): There exists a continuous function $p : [t_0, T] \times [t_0, T] \to (0, \infty)$ such that

$$E\|F(t,s,x)\|^2 \le p(t,s) H(E\|x\|_s^2), \ \ (t,s) \in \Delta, x \in X,$$

where $H : \Re^+ \to (0, \infty)$ is a continuous nondecreasing function.
(H_3): $\sigma : [t_0, T] \to [t_0, T]$, is a continuous functions such that $\sigma(t) \le t$.
(H_4): $E\left\{\|b_j(\tau_j)\|^2\right\} \le C$ for all $\tau_j \in D_j$, $\ j = 1, 2, \cdots .$, where $C > 0$.

Theorem 3.1. *If the hypothesis* $(H_2) - (H_4)$ *hold, then system* (1) *has a mild solution* $x(t)$, *defined on* $[t_0, T]$ *provided that the following inequality is satisfied*

$$M_1 \int_{t_0}^{T} e^{-\lambda(1-\max\{1,C\})(s-t_0)} p(s,s) ds \quad < \quad \int_{c_1}^{\infty} \frac{ds}{H(s)}, \qquad (3)$$

where $M_1 = 2M^2(T - t_0)^2$ *and* $\ c_1 = 2M^2 E\|\varphi\|^2$.

Proof. Let T be an arbitrary number $t_0 < T < +\infty$ satisfying (3) . We transform the problem (1) into a fixed point problem. We consider the operator $\Phi : \mathcal{B} \to \mathcal{B}$ defined by

$$\Phi x(t) = \begin{cases} \varphi(t - t_0), \qquad t \in [t_0 - r, t_0], \\ \displaystyle\sum_{k=0}^{+\infty} \left[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu))) d\mu ds \right. \\ \displaystyle \left. + \int_{\xi_k}^{t} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [t_0, T]. \end{cases}$$

In order to use the transversality theorem, first we establish the priori estimates for the solutions of the integral equation and $\lambda \in (0,1)$,

$$x(t) = \begin{cases} \lambda\, \varphi(t-t_0), \qquad t \in [t_0 - r, t_0], \\ \lambda \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \\ \quad + \int_{\xi_k}^{t} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \Big] I_{[\xi_k,\xi_{k+1})}(t), \quad t \in [t_0, T], \end{cases}$$

Thus by $(H_2)-(H_4)$, we have

$$\begin{aligned} E\|x(t)\|^2 \leq\ & 2E\Big[\sum_{k=0}^{+\infty} \big[\|\prod_{i=1}^{k} b_i(\tau_i)\|^2 \|S(t-t_0)\|^2 \|\varphi(0)\|^2 |I_{[\xi_k,\xi_{k+1})}(t)\big] P(I_{[\xi_k,\xi_{k+1})}(t)) \\ & + \Big[\sum_{k=0}^{+\infty} \big[\sum_{i=1}^{k} \|\prod_{j=i}^{k} b_j(\tau_j)\| \Big\{ \int_{\xi_{i-1}}^{\xi_i} \|S(t-s)\| \| \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu \|ds \Big\} \\ & + \int_{\xi_k}^{t} \|S(t-s)\| \| \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu \|ds \Big]^2 |I_{[\xi_k,\xi_{k+1})}(t) P(I_{[\xi_k,\xi_{k+1})}(t))\big]\Big], \end{aligned}$$

$$\begin{aligned} E\|x\|_t^2 \leq\ & 2M^2 E\|\varphi\|^2 \sum_{k=0}^{+\infty} \prod_{i=1}^{k} C \frac{\lambda^k (t-t_0)^k}{k!} e^{-\lambda(t-t_0)} \\ & + 2M^2(T-t_0) E\Big[\int_{t_0}^{t} \Big\| \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu \Big\|^2 ds\Big] \\ & \times \sum_{k=0}^{+\infty} \prod_{i=1}^{k} \max\{1,C\} \frac{\lambda^k (t-t_0)^k}{k!} e^{-\lambda(t-t_0)} \\ \leq\ & 2M^2 E\|\varphi\|^2 e^{-\lambda(1-C)(t-t_0)} \\ & + 2M^2(T-t_0) E\Big[\int_{t_0}^{t} \Big\| \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu \Big\|^2 ds\Big] \times e^{-\lambda(1-\max\{1,C\})(t-t_0)}, \end{aligned}$$

then

$$\begin{aligned} E\|x\|_t^2 \leq\ & 2M^2 e^{-\lambda(1-C)(t-t_0)} E\left[\|\varphi\|^2\right] \\ & + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)} (T-t_0) \int_{t_0}^{t} \left[\int_0^s E\left\|F(s,\mu,x(\sigma(\mu)))d\mu\right\|^2\right] ds \\ \leq\ & 2M^2 e^{-\lambda(1-C)(t-t_0)} E\left[\left\|\varphi\right\|^2\right] \\ & + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)} (T-t_0) \int_{t_0}^{t} \left[\int_0^s p(s,\mu) H(E\|x(\sigma(\mu))\|_s^2) d\mu\right] ds \\ \leq\ & 2M^2 e^{-\lambda(1-C)(t-t_0)} E\left[\left\|\varphi\right\|^2\right] \\ & + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)} (T-t_0) \int_{t_0}^{t} \left[\int_0^s p(s,\mu) H(E\|x(\mu)\|_s^2) d\mu\right] ds \\ \leq\ & 2M^2 e^{-\lambda(1-C)(t-t_0)} E\left[\left\|\varphi\right\|^2\right] \\ & + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)} (T-t_0)^2 \int_{t_0}^{t} p(s,s) H(E\|x\|_s^2) ds. \end{aligned}$$

Since the last term on the right hand side of the above inequality also increases in t, we have

$$\begin{aligned}\sup_{t_0\le v\le t} E\|x\|_v^2 &\le 2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\left\|\varphi\right\|^2\right]\\ &\quad +2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\int_{t_0}^{t}p(s,s)H(E\left[\|x\|_s^2\right])ds\\ &\le 2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\left\|\varphi\right\|^2\right]\\ &\quad +2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\int_{t_0}^{t}p(s,s)H(\sup_{t_0\le v\le s}E\left[\|x\|_v^2\right])ds.\end{aligned}$$

Consider the function $\ell(t)$ defined by

$$\ell(t)=\sup_{t_0\le v\le t}E\left[\|x\|_v^2\right],\qquad t\in[t_0,T].$$

Then, for any $t\in[t_0,T]$ it follows that

$$\ell(t)\le 2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\left\|\varphi\right\|^2\right]+2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\int_{t_0}^{t}p(s,s)H(\ell(s))ds. \tag{4}$$

Denoting the right hand side of the above inequality (4) by $u(t)$ we obtain that

$$\ell(t)\le u(t),\quad t\in[t_0,T],$$

$$u(t_0)=2M^2e^{-\lambda(1-C)(t_0-t_0)}E\left\|\varphi\right\|^2=c_1,$$

and

$$\begin{aligned}u'(t) &= 2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2p(t,t)H(\ell(t))\\ &\le 2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2p(t,t)H(u(t)),\qquad t\in[t_0,T].\end{aligned}$$

Then

$$\frac{u'(t)}{H(u(t))}\le 2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2p(t,t),\qquad t\in[t_0,T]. \tag{5}$$

Integrating (5) from t_0 to t and by applying the change of variable method, we obtain

$$\begin{aligned}\int_{u(t_0)}^{u(t)}\frac{ds}{H(s)} &\le 2M^2(T-t_0)^2\int_{t_0}^{t}e^{-\lambda(1-\max\{1,C\})(s-t_0)}p(s,s)ds\\ &\le 2M^2(T-t_0)^2\int_{t_0}^{T}e^{-\lambda(1-\max\{1,C\})(s-t_0)}p(s,s)ds\\ &< \int_{u(t_0)}^{\infty}\frac{ds}{H(s)},\quad t\in[t_0,T],\end{aligned} \tag{6}$$

where the last inequality is obtained by (3). From (6) and by mean value theorem, there is a constant η_1 such that $u(t)\le\eta_1$ and hence $\ell(t)\le\eta_1$. Since $\sup_{t_0\le v\le t}E\|x\|_v^2=\ell(t)$ holds for every $t\in[t_0,T]$, we have $\sup_{t_0\le v\le T}E\|x\|_v^2\le\eta_1$, where η_1 only depends on T, the functions p and H, and consequently

$$E\|x\|_{\mathcal{B}}^2=\sup_{t_0\le v\le T}E\|x\|_v^2\le\eta_1.$$

In the next steps, we will prove that Φ is continuous and completely continuous.

Step 1. We prove that Φ is continuous.

Let $\{x_n\}$ be a convergent sequence of elements x in $\mathcal{B}$. Then for each $t \in [t_0, T]$, we have

$$\begin{aligned}\Phi x_n(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu ds\Bigg]|I_{[\xi_k,\xi_{k+1})}(t).\end{aligned}$$

Thus,

$$\begin{aligned}&\Phi x_n(t) - \Phi x(t) \\ &= \sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)\{\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\}ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)\{\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\}ds\Bigg]|I_{[\xi_k,\xi_{k+1})}(t),\end{aligned}$$

and

$$\begin{aligned}&E\|\Phi x_n - \Phi x\|_t^2 \\ &\quad \le M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0) \\ &\quad \times \int_{t_0}^{t} E\|\{\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\}\|^2 ds \\ &\quad \longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus Φ is clearly continuous.

Step 2. We prove that Φ is completely continuous operator.

Denote

$$B_m = \left\{x \in \mathcal{B} \middle| \, \|x\|_{\mathcal{B}}^2 \le m\right\},$$

for some $m \ge 0$.

Step 2.1 We show that Φ maps B_m into an equicontinuous family.

Let $y \in B_m$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using hypotheses $(H_2) - (H_4)$ and condition (3), we have

$$\begin{aligned}&\Phi x(t_1) - \Phi x(t_2) \\ &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t_1-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \\ &+ \int_{\xi_k}^{t_1} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]|I_{[\xi_k,\xi_{k+1})}(t_1) \\ &- \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t_2-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t_2-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \\ &+ \int_{\xi_k}^{t_2} S(t_2-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]|I_{[\xi_k,\xi_{k+1})}(t_2).\end{aligned}$$

Thus,

$$\begin{aligned}
&\Phi x(t_1) - \Phi x(t_2)\\
=\ & \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t_1-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{\xi_k}^{t_1} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]\Big(I_{[\xi_k,\xi_{k+1})}(t_1) - I_{[\xi_k,\xi_{k+1})}(t_2)\Big)\\
&+\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)\Big(S(t_1-t_0) - S(t_2-t_0)\Big)\varphi(0)\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\Big(S(t_1-s)-S(t_2-s)\Big)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{\xi_k}^{t_1}\Big(S(t_1-s)-S(t_2-s)\Big)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{t_1}^{t_2} S(t_2-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]I_{[\xi_k,\xi_{k+1})}(t_2).
\end{aligned}$$

Then,

$$E\|\Phi x(t_1) - \Phi x(t_2)\|^2 \leq 2E\|I_1\|^2 + 2E\|I_2\|^2, \tag{7}$$

where

$$\begin{aligned}
I_1 =\ & \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t_1-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{\xi_k}^{t_1} S(t_1-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]\Big(I_{[\xi_k,\xi_{k+1})}(t_1) - I_{[\xi_k,\xi_{k+1})}(t_2)\Big),
\end{aligned}$$

and

$$\begin{aligned}
I_2 =\ & \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)\Big(S(t_1-t_0) - S(t_2-t_0)\Big)\varphi(0)\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\Big(S(t_1-s)-S(t_2-s)\Big)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{\xi_k}^{t_1}\Big(S(t_1-s)-S(t_2-s)\Big)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+\int_{t_1}^{t_2} S(t_2-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]I_{[\xi_k,\xi_{k+1})}(t_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E\|I_1\|^2 \leq\ & 2M^2E\|\varphi(0)\|^2\left[e^{-\lambda(1-C)(t_1-t_0)} - e^{-\lambda(1-C)(t_2-t_0)}\right]\\
&+2\Big[e^{-\lambda(1-\max\{1,C\})(t_1-t_0)} - e^{-\lambda(1-\max\{1,C\})(t_2-t_0)}\Big] \qquad (8)\\
&\times(t_1-t_0)^2E\int_{t_0}^{t_1}\|S(t_1-s)\|^2M^*H(m)ds\\
\to\ & 0 \quad \text{as} \quad t_2 \to t_1, \qquad (9)
\end{aligned}$$

where $M^* = \sup\{p(t,t) : t \in [t_0, T]\}$, and

$$\begin{aligned} E\|I_2\|^2 &\leq 3\Big[e^{-\lambda(1-C)(t_2-t_0)}\Big]\|S(t_1-t_0)-S(t_2-t_0)\|^2 E\|\varphi(0)\|^2 \\ &+3\Big[e^{-\lambda(1-\max\{1,C\})(t_1-t_0)}\Big] \\ &\times(t_1-t_0)E\int_{t_0}^{t_1}\|S(t_1-s)-S(t_2-s)\|^2\|\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2 ds \\ &+3\Big[e^{-\lambda(1-\max\{1,C\})(t_2-t_0)}\Big](t_2-t_1)E\int_{t_1}^{t_2}\|S(t_2-s)\|^2\|\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2 ds. \end{aligned}$$

Since there is $\delta > 0$ such that

$$\|S(t_1-t_0)-S(t_2-t_0)\| \leq \frac{\delta}{\sqrt{t_1-t_0}}\sqrt{t_1-t_2},$$

(see [18, Proposition 1]) and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology, we have

$$\|S(t_1-t_0)-S(t_2-t_0)\|^2 \to 0,\ \|S(t_1-s)-S(t_2-s)\|^2 \to 0 \text{ as } t_2 \to t_1.$$

Thus, Φ maps B_m into an equicontinuous family of functions.

Step 2.2 We show that ΦB_m is uniformly bounded.

From (3), $\|x\|^2_{\mathcal{B}} \leq m$ and by $(H_2)-(H_4)$, we get,

$$\begin{aligned} E\|(\Phi x)\|_t^2 &\leq 2M^2e^{-\lambda(1-C)(t-t_0)}E\|\varphi(0)\|^2 \\ &+2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)\int_{t_0}^t E\|\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2 ds \\ E\|(\Phi x)\|_t^2 &\leq 2M^2e^{-\lambda(1-C)(t-t_0)}E\|\varphi(0)\|^2 + 2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\|\alpha_m\|_{L^1}. \end{aligned}$$

This yields that the set $\{(\Phi x)(t),\ \|x\|^2_{\mathcal{B}} \leq m\}$ is uniformly bounded, so $\{\Phi B_m\}$ is uniformly bounded. We have already shown that ΦB_m is equicontinuous collection. Now it is sufficient, by the Arzela - Ascoli theorem, to show that Φ maps B_m into a precompact set in X.

Step 2.3 We show that ΦB_m is compact.

Let $t_0 < t \leq T$ be fixed and ϵ a real number satisfying $\epsilon \in (0, t-t_0)$, for $x \in B_m$. We define

$$\begin{aligned} (\Phi_\epsilon x)(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^k b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \\ &+\int_{\xi_k}^{t-\epsilon}S(t-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big]I_{[\xi_k,\xi_{k+1})}(t),\ t \in (t_0, t-\epsilon). \end{aligned} \tag{10}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon(t) = \{(\Phi_\epsilon x)(t) :\ x \in B_m\},$$

is precompact in X for every $\epsilon \in (0, t-t_0)$. By using $(H_2)-(H_4)$, (3) and $E\|x\|^2_\Gamma \leq m$, we obtain

$$E\|(\Phi x)-(\Phi_\epsilon x)\|_t^2 \leq M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\int_{t-\epsilon}^t M^*H(m)ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) :\ x \in B_m\}$. Hence the set $\{(\Phi x)(t) :\ x \in B_m\}$ is precompact in X. Therefore, Φ is a completely continuous operator.

Moreover, the set $U(\Phi) = \{x \in \mathcal{B} : x = \lambda\Phi x,\ \text{for some } 0 < \lambda < 1\}$ is bounded. Consequently, by Lemma 2.7, the operator Φ has a fixed point in $\mathcal{B}$. Therefore, the system (1) has a mild solution. Thus, the proof is completed. □

Now, we give another existence result for the system (1) by means of Banach contraction principle.

Theorem 3.2. *If the hypothesis* (H_1), (H_3) *and* (H_4) *holds then the initial value problem* (1) *has a unique mild solution on* $[t_0, T]$.

Proof. Consider the nonlinear operator $\Phi : \mathcal{B} \to \mathcal{B}$ defined as in Theorem 3.1,

$$\begin{aligned}
E\left\|\Phi x - \Phi y\right\|_t^2 &\leq 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0) \\
&\quad\times \int_0^t \Big\{ \int_0^s E\|F(s,\mu,x(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,y(\sigma(\mu)))d\mu\|^2 \Big\} ds \\
&\leq 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0) \\
&\quad\times \int_{t_0}^t \{ \int_0^s L\Big(s,\mu,E\|x(\sigma(\mu))\|^2, E\|y(\sigma(\mu))\|^2\Big) E\|x(\sigma(\mu)) - y(\sigma(\mu))\|_s^2 d\mu\} ds \\
&\leq 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0) \\
&\quad\times \int_{t_0}^t \{ \int_0^s L\Big(s,\mu,E\|x\|^2, E\|y\|^2\Big) E\|x(\mu) - y(\mu)\|_s^2 d\mu\} ds \\
&\leq 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \int_{t_0}^t L\Big(s,s,E\|x\|^2, E\|y\|^2\Big) E\|x-y\|_s^2 ds
\end{aligned}$$

.

Taking supremum over t, we get,

$$\left\|\Phi x - \Phi y\right\|_{\mathcal{B}}^2 \leq \Lambda(T)\|x-y\|_{\mathcal{B}}^2,$$

with $\Lambda(T) = 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \int_{t_0}^t L\Big(s,s,E\|x\|^2, E\|y\|^2\Big) ds.$

Then we can take a suitable T_1, $0 < T_1 < T$ sufficiently small such that $\Lambda(T_1) < 1$, then we get that Φ is a contraction on $\mathcal{B}_{T_1}$ ($\mathcal{B}_{T_1}$ denotes $\mathcal{B}$ with T substituted by T_1). Thus, by the well-known Banach fixed point theorem we obtain a unique fixed point $x \in \mathcal{B}_{T_1}$ for operator Φ, and hence $\Phi x = x$ is a mild solution of (1). This procedure can be repeated to extend the solution to the entire interval $[-r, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $[-r, T]$. □

4. Existence of global solutions

In this section we study the global existence of solutions for

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t F\Big(t,s,x(\sigma(s))\Big)ds, t \neq \xi_k, \quad t \in [t_0,\infty) \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \ k = 1,2,\cdots, \\ x_{t_0} = \varphi. \end{cases} \tag{11}$$

Definition 4.1. *A function* $x : [t_0,\infty) \to X$ *is called a mild solution of* (11) *if* $x/_{[t_0,T]} \in \mathcal{B}([t_0,T],X)$ *for every* $T \in (t_0,\infty)$,

$$\begin{aligned}
x(t) &= \sum_{k=0}^{+\infty} \Big(\prod_{i=1}^k b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \\
&\quad + \int_{\xi_k}^t S(t-s) \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds \Big) I_{[\xi_k,\xi_{k+1})}(t), t \in [t_0,\infty).
\end{aligned} \tag{12}$$

In order to obtain our results, we need to introduce some additional notations, definitions and technical remarks. It follows that $g : [t_0, \infty) \to R$ is a positive continuous and nondecreasing function such that $g(t_0) = 1$ and $\lim_{t\to\infty} g(t) = \infty$. In this section $\mathcal{B}([t_0, \infty), X), C_0(X), C_g^0(X)$ and $\mathcal{B}_g^0(X)$ are the spaces.

$\mathcal{B}([t_0, \infty), L_2) = \{x : [t_0, \infty) \to X : x/_{[t_0,T]} \in \mathcal{B}([t_0, T], X), \forall\, T \in (t_0, \infty), E\|x\|^2_{\mathcal{B}} = \sup_{t\geq t_0} E\|x(t)\|^2 < \infty\}$;

$C_0(X) = \left\{x \in C([t_0, \infty), X) : \lim_{t\to\infty} E\|x(t)\|^2 = 0\right\}$; $C_g^0(X) = \{x \in C([t_0, \infty), X) : \lim_{t\to\infty} \frac{E\|x(t)\|^2}{g(t)} = 0\}$;

$\mathcal{B}_g^0(X) = \{x \in \mathcal{B}([t_0, \infty), X) : \lim_{t\to\infty} \frac{E\|x(t)\|^2}{g(t)} = 0\}$, endowed with the norms $E\|x\|^2_\infty = \sup_{t\geq t_0} E\|x(t)\|^2$; $E\|x\|^2_0 = \sup_{t\geq t_0} E\|x(t)\|^2$; $E\|x\|^2_g = \sup_{t\geq t_0} \frac{E\|x(t)\|^2}{g(t)}$ and $E\|x\|^2_{\mathcal{B}_g} = \sup_{t\geq t_0} \frac{E\|x(t)\|^2}{g(t)}$ respectively.

We recall here the following results of compactness in these spaces [13]. We omit the proof.

Lemma 4.2. *A set $B \subset C_g^0(X)$ is relatively compact in C_g^0 if and only if,*
(a) B is equicontinuous;
(b) $\lim_{t\to\infty} \frac{E\|x(t)\|^2}{g(t)} = 0$, uniformly for $x \in B$;
(c) The set $B(t) = \{x(t) : x \in B\}$ is relatively compact in X, for every $t \geq t_0$.

Lemma 4.3. *A set $B \subset \mathcal{B}_g^0(X)$ is relatively compact in $\mathcal{B}_g^0(X)$ if and only if,*
(a) The set $B_T = \{x/_{[t_0,T]} : x \in B\}$ is relatively compact in $\Gamma([t_0, T]; X)$, for every $T \in (0, \infty)$,
(b) $\lim_{t\to\infty} \frac{E\|x(t)\|^2}{g(t)} = 0$, uniformly for $x \in B$.

Theorem 4.4. *Let the conditions (H_2),(H_3),(H_4) holds for every $T > 0$. Suppose, in addition that the following conditions are verified*
(a) For every $t > t_0$, the set $\{S(t) \int_0^s F(s, \mu, x(\sigma(\mu)))d\mu : s \in [t_0, t], x \in B_m(0, X)\}$,
[where $B_m(0, X)$ is a closed ball of radius $m > 0$ with center at the origin, in a Banach space X]
is relatively compact in X;
(b) For every $\hbar > 0$, $\lim_{t\to\infty} \frac{1}{g(t)} \int_{t_0}^t p(s, s)H[\hbar g(s)]ds = 0$,
(c) $\int_{t_0}^\infty M_1 e^{-\lambda(1-\max\{1,C\})(s-t_0)} p(s, s)ds < \int_{c_1}^\infty \frac{ds}{H(s)}$,
where $M_1 = 2M^2(T - t_0)^2$ and $c_1 = 2M^2E\|\varphi\|^2$. Then, there exists a mild solution for the system (11).

Proof. On the space $\mathcal{B}_g^0(X)$ we define the operator

$$\begin{aligned}
\Phi x(t) &= \sum_{k=0}^{+\infty}\Big(\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0)\\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\\
&+ \int_{\xi_k}^{t} S(t-s)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu ds\Big) I_{[\xi_k,\xi_{k+1})}(t),\ t\in[0,\infty).
\end{aligned}$$

We can observe that $E\|x(t)\|^2 \leq E\|x\|^2_{\mathcal{B}_g} g(t)$

$$\begin{aligned}
\frac{E\|\Phi x(t)\|^2}{g(t)} &\leq \frac{2M^2 e^{-\lambda(1-C)(t-t_0)}E\left[\|\varphi\|^2\right]}{g(t)}\\
&+ \frac{2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)}\int_{t_0}^{t} p(s,s)H(E\|x\|^2_{\mathcal{B}_g} g(s))ds.
\end{aligned}$$

Next we show that Φ satisfies all the conditions in Lemma 2.7.

Let $(x_n)_{n\in N}$ be a sequence in $\mathcal{B}_g^0(X)$ and $x \in \mathcal{B}_g^0(X)$ such that $x_n \to x$ in $\mathcal{B}_g^0(X)$. Let $\epsilon > 0$ be given and $\hbar = \sup_{n\in N} E\|x_n\|^2_{\mathcal{B}_g}$. From condition (b) there exists $L_1 > 0$ such that,

$$\begin{aligned}\frac{2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\|\varphi\|^2\right]}{g(t)} &+ \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)}\int_{t_0}^t p(s,s)H(\hbar g(s))ds \\ &< \frac{\epsilon}{2}, \quad t \geq L_1.\end{aligned}$$

From the Lebesgue-dominated convergence theorem, we infer that, $N_\epsilon \in N$ such that

$$\begin{aligned}E\Big\{\int_0^{L_1}\|\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2\Big\}ds \\ < \frac{\epsilon}{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}, \quad n \geq N_\epsilon,\end{aligned}$$

$$\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} \leq \int_0^{L_1}\Big\{E\|\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2\Big\}ds < \epsilon.$$

Therefore,

$$\sup\left\{\frac{E\|\Phi x_n(t) - \Phi x(t)\|^2}{g(t)} : t \in [0, L_1], n \geq N_\epsilon\right\} \leq \epsilon. \tag{13}$$

On the other hand $t \geq L_1$ and $n \geq N_\epsilon$, we find that

$$\begin{aligned}\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} &\leq \frac{\epsilon}{2} + \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)}\int_{L_1}^t p(s,s)H(E\|x_n - x\|^2_{\mathcal{B}_g}g(s))ds \\ &\leq \frac{\epsilon}{2} + \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)}\int_{L_1}^t p(s,s)H(2\hbar g(s))ds.\end{aligned}$$

So that,

$$\sup\Big\{\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} : t \geq L_1, n \geq N_\epsilon\Big\} \leq \epsilon. \tag{14}$$

From (13) and (14), we see that Φ is continuous. Next, we prove that Φ is completely continuous. Let $B_m = \{x \in \mathcal{B}/\|x\|^2_{\mathcal{B}} \leq m\}$. From the proof of Theorem 3.1, we establish that the set $\Phi(B_m)|_{[t_0,T]} = \{x|_{[t_0,T]} \in B_m : x \in B_m\}$ is relatively compact in $\mathcal{B}([t_0,T];X)$ for every $T \in (t_0,\infty)$. Moreover, for $x \in B_m$, we have that

$$\begin{aligned}\frac{E\|\Phi x(t)\|^2}{g(t)} \leq & \frac{2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\|\varphi\|^2\right]}{g(t)} \\ &+ \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)}\int_{t_0}^t p(s,s)H(E\|x\|^2_{\mathcal{B}_g}g(s))ds,\end{aligned}$$

where, from (b) we get that $\frac{E\|\Phi x\|_t^2}{g(t)} \to 0$ as $t \to \infty$ uniformly for $x \in B_m$. Now, Lemma 4.2 allows us to conclude that $\Phi(B_m)$ is relatively compact in $\mathcal{B}_g^0(X)$. Thus Φ is completely continuous.
We establish the priori estimates for the equation (12). For $t \geq t_0$, we get,

$$\begin{aligned}E\|\Phi x\|_t^2 \leq & 2M^2e^{-\lambda(1-C)(t-t_0)}E\left[\|\varphi\|^2\right] \\ &+2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2\int_{t_0}^t p(s,s)H(E\|x\|_s^2)ds.\end{aligned}$$

Denoting the right hand side of the above equation by $\hat{u}(t)$, we obtain

$$\hat{u}'(t) \leq 2M^2 p(t,t) e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 H(\hat{u}(t))$$

and hence,

$$\int_{c_1}^{\hat{u}(t)} \frac{ds}{H(s)} \leq \int_{t_0}^{\infty} M_1\, e^{-\lambda(1-\max\{1,C\})(s-t_0)} p(s,s)ds < \int_{c_1}^{\infty} \frac{ds}{H(s)}.$$

This inequality jointly with condition (c) allows us to affirm that Φ is bounded in $\mathcal{B}_g^0(X)$. By using Lemma 2.7, there exist a fixed point for Φ, and as a consequence a mild solution exists for (11). The proof is complete. □

Theorem 4.5. *Let the condition* $(H_1), (H_3)$ *and* (H_4) *be satisfied for every* $T > t_0$. *Then there exists a unique mild solution, provided that*

$$\hat{\wp} = 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \sup_{t\geq t_0} \frac{1}{g(t)} \int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big) g(s)ds < 1. \tag{15}$$

Proof.

$$\frac{E\|\Phi x(t)-\Phi y(t)\|^2}{g(t)} \leq \frac{2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2}{g(t)} \times \int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big) E\|x-y\|^2_{\mathcal{B}_g} g(s)ds$$

$$\begin{aligned}\|\Phi x(t)-\Phi y(t)\|^2_{\mathcal{B}_g} &\leq 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \\ &\quad\times \sup_{t\geq t_0} \frac{1}{g(t)} \int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big) E\|x-y\|^2_{\mathcal{B}_g} g(s)ds \\ &\leq \hat{\wp}\|x-y\|^2_{\mathcal{B}_g}.\end{aligned}$$

From (15), Φ is a contraction on $\mathcal{B}_g^0$. Hence, there exist a unique global fixed point for Φ in space $\mathcal{B}_g^0$ and this fixed point is the mild solution of the initial problem (11). □

5. Continuous dependence

Theorem 5.1. *Let* $x(t)$ *and* $\bar{x}(t)$ *be mild solution of system* (11) *with initial values* $\varphi(0)$ *and* $\bar{\varphi}(0) \in \mathcal{B}_g$ *respectively. If the assumption* (H_1), (H_3) *and* (H_4) *are satisfied then the mild solution of the system* (11) *is stable in the mean square.*

Proof. By the assumptions x and $\bar{x}$ are the two mild solutions of the system (11) for $t \in [t_0, \infty)$ then

$$\begin{aligned}\sup_{t\geq t_0} \frac{E\|x-\bar{x}\|_t^2}{g(t)} &\leq \sup_{t\geq t_0} \frac{2M^2 e^{-\lambda(1-C)(t-t_0)} E\|\varphi(0)-\bar{\varphi}(0)\|^2}{g(t)} \\ &\quad + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \\ &\quad \times \sup_{t\geq t_0} \frac{1}{g(t)} \int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big) E\|x-\bar{x}\|^2_{\mathcal{B}_g} g(s)ds, \\ \|x-\bar{x}\|^2_{\mathcal{B}_g} &\leq 2M^2 e^{-\lambda(1-C)(t-t_0)} E\|\varphi(0)-\bar{\varphi}(0)\|^2_{\mathcal{B}_g} \\ &\quad + 2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \\ &\quad \times \sup_{t\geq t_0} \frac{1}{g(t)} \int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big) E\|x-\bar{x}\|^2_{\mathcal{B}_g} g(s)ds.\end{aligned}$$

By applying Grownwalls inequality, we have

$$\|x-\bar{x}\|^2_{\mathcal{B}_g} \le 2M^2 e^{-\lambda(1-C)(t-t_0)} E\|\varphi(0)-\bar{\varphi}(0)\|^2_{\mathcal{B}_g}$$
$$\times exp(2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2 \sup_{t\ge t_0}\frac{1}{g(t)}\int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big)g(s)ds),$$
$$\|x-\bar{x}\|^2_{\mathcal{B}_g} \le \Im E\|\varphi(0)-\bar{\varphi}(0)\|^2_{\mathcal{B}_g},$$

where,

$$\Im = 2M^2 e^{-\lambda(1-C)(t-t_0)} exp(2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)^2$$
$$\times \sup_{t\ge t_0}\frac{1}{g(t)}\int_{t_0}^{t} L\Big(s,s,E\|x\|^2,E\|y\|^2\Big)g(s)ds).$$

Now given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{\Im}$ such that $\|\varphi(0)-\bar{\varphi}(0)\|^2_{\mathcal{B}_g} < \delta$. Then $\|x-\bar{x}\|^2_{\mathcal{B}_g} \le \epsilon$. Thus the difference between the mild solutions $x(t)$ and $\bar{x}(t)$ in the interval $[t_0,\infty)$ is small, provided that the change in the initial point $(t_0,\varphi(0))$ as well as in the function F do not exceed the prescribed amounts. This completes the proof. □

6. Exponential Stability

In this section, we will study the exponential stability of mild solution of the system (11). For any $\mathcal{F}_t$ adapted process $\phi(t) : [-r,\infty) \to \Re$ is almost surely continuous in t. For the purpose of stability we may assume that $F(t,s,0) = 0$ for any $t \ge t_0$ so that the system (11) gives a trivial solution. Moreover $\varphi(t) = \varphi(t-t_0)$ for $t \in [t_0-r,t_0]$ and $E\|\varphi\|^2_t \to 0$ as $t \to \infty$.

Definition 6.1. *[11] Eq.* (11) *is said to be exponentially stable in the quadratic mean if there exist positive constant* C_1 *and* $\hat{\lambda} > 0$ *such that*
$E\|x(t)\|^2 \le C_1 E\|\varphi\|^2 e^{-\hat{\lambda}(t-t_0)}, t \ge t_0$.

We now consider the following assumptions

(H_5) : $\mu H(\chi) \le H(\mu\chi)$ for all $\chi \in \Re^+$ where $\mu > 1$.

(H_6) : $\|S(t)\| \le Me^{-\gamma(t-t_0)}, t \ge t_0$ where $M \ge 1$, $\gamma > 0$.

Theorem 6.2. *Let the hypotheses of Theorem* 4.3 *and* (H_5-H_6) *hold. Then system* (11) *is exponentially stable in the quadratic mean if it satisfies the following inequality,*
(a) For every $t > t_0$*, the set* $\{S(t)\int_0^s F(s,\mu,x(\sigma(\mu)))d\mu : s \in [t_0,t], x \in \mathcal{B}_m(0,X)\}$ *is relatively compact in* X*;*
(b) For every $\hat{\hbar} > 0$, $\lim_{t\to\infty}\frac{1}{g(t)}\int_{t_0}^{t} e^{\gamma(s-t_0)}p(s,s)H[\hat{\hbar}g(s)]ds = 0$,
(c) $\int_{t_0}^{\infty} M_2 e^{-\lambda(1-\max\{1,C\})(s-t_0)}p(s,s)ds < \int_{c_2}^{\infty}\frac{ds}{H(s)}$,
where, $M_2 = \frac{2M^2}{\gamma}$ *and* $c_2 = 2M^2E\|\varphi\|^2$.

Proof. The proof is similar to the proof of Theorem 4.4, we define the operator Φ on the space $\mathcal{B}^0_g(X)$ and using $(H_2)-(H_6)$ we get,

$$E\|x\|^2_t \le 2M^2 e^{-\lambda(1-C)(t-t_0)}e^{-\gamma(t-t_0)}E\|\varphi\|^2$$
$$+\frac{2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}e^{-\gamma(t-t_0)}}{\gamma}\int_{t_0}^{t} e^{\gamma(s-t_0)}p(s,s)H(E\|x\|^2_s)ds,$$

$$e^{\gamma(t-t_0)}E\|x\|_t^2 \leq 2M^2e^{-\lambda(1-C)(t-t_0)}E\|\varphi\|^2 + \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma}\int_{t_0}^{t} p(s,s)H(e^{\gamma(s-t_0)}E\|x\|_s^2)ds.$$

Consider $l_1(t) = e^{\gamma(t-t_0)}E\|x\|_t^2$. For any $t \in [t_0, \infty)$,

$$l_1(t) \leq 2M^2e^{-\lambda(1-C)(t-t_0)}E\|\varphi\|^2 + \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma}\int_{t_0}^{t} p(s,s)H(l_1(s))ds.$$

Denote the right hand side of above inequality by $u_1(t)$, then $l_1(t) \leq u_1(t)$; $u_1(t_0) = 2M^2E\|\varphi\|^2 = c_2$, $u_1'(t) = \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma}p(t,t)H(u_1(t))$.

Hence $\frac{u_1'(t)}{H(u_1(t))} \leq \frac{2M^2e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma}p(t,t)$, integrating and making use of a change of variable we obtain

$$\int_{u_1(t_0)}^{u_1(t)} \frac{ds}{H(s)} \leq \int_{t_0}^{\infty} \frac{2M^2}{\gamma}\, e^{-\lambda(1-\max\{1,C\})(s-t_0)}p(s,s)ds < \int_{c_2}^{\infty} \frac{ds}{H(s)}.$$

This inequality along with the condition (c) of Theorem 6.2 allows us to affirm that Φ is bounded in $\mathcal{B}_g^0(X)$.

We will show that Φ is a completely continuous operator. First we prove that Φ is continuous.

Let $(x_n)_{n\in N}$ be a sequence in $\mathcal{B}_g^0(X)$ and $x \in \mathcal{B}_g^0(X)$ such that $x_n \to x$ in $\mathcal{B}_g^0(X)$. Let $\epsilon > 0$ be given $\hat{\hbar} = \sup_{n\in N} E\|x_n\|_{\mathcal{B}_g}^2$. From condition Theorem 6.2(b) there exists $L_1 > 0$ such that,

$$\frac{2M^2e^{-\gamma(t-t_0)}e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)}\int_{t_0}^{t} e^{\gamma(s-t_0)}p(s,s)H(2\hat{\hbar}g(s))ds < \frac{\epsilon}{2}, \quad t \geq L_1.$$

From the Lebesgue-dominated convergence theorem, we infer that $N_\epsilon \in N$ such that,

$$E\Big\{\int_0^{L_1} \Big\|\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\Big\|^2 ds\Big\} < \frac{\epsilon\,\gamma}{2M^2e^{-\gamma(t-t_0)}e^{-\lambda(1-\max\{1,C\})(t-t_0)}}, n \geq N_\epsilon.$$

Consequently, for $t \in [0, L_1]$ and $n \geq N_\epsilon$, we obtain that

$$\begin{aligned}\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} &\leq \frac{2M^2e^{-\gamma(t-t_0)}e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)} \\ &\times \int_0^{L_1} e^{\gamma(s-t_0)}\Big\{E\Big\|\int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\Big\|^2\Big\}ds \\ &< \epsilon,\end{aligned}$$

hence we get,

$$\sup\left\{\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} : t \in [0, L_1], n \geq N_\epsilon\right\} \leq \epsilon. \tag{16}$$

On the other hand $t \geq L_1$ and $n \geq N_\epsilon$, we find that

$$\begin{aligned}
\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} &\leq \frac{2M^2 e^{-\gamma(t-t_0)} e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)} \\
&\quad \times \int_{t_0}^{L_1} e^{\gamma(s-t_0)} E\| \int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2 ds \\
&\quad + \frac{2M^2 e^{-\gamma(t-t_0)} e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)} \\
&\quad \times \int_{L_1}^{t} e^{\gamma(s-t_0)} E\| \int_0^s F(s,\mu,x_n(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,x(\sigma(\mu)))d\mu\|^2 ds \\
&\leq \frac{\epsilon}{2} + \frac{2M^2 e^{-\gamma(t-t_0)} e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)} \\
&\quad \times \int_{L_1}^{t} e^{\gamma(s-t_0)} p(s,s) H(E\|x_n - x\|_{\mathcal{B}_g}^2 g(s)) ds \\
&\leq \frac{\epsilon}{2} + \frac{2M^2 e^{-\gamma(t-t_0)} e^{-\lambda(1-\max\{1,C\})(t-t_0)}}{\gamma\, g(t)} \int_{L_1}^{t} e^{\gamma(s-t_0)} p(s,s) H(2\hat{h} g(s)) ds.
\end{aligned}$$

Thus

$$\sup\left\{\frac{E\|\Phi x_n - \Phi x\|_t^2}{g(t)} : t \geq L_1, n \geq N_\epsilon\right\} \leq \epsilon. \tag{17}$$

From (16) and (17), we see that Φ is continuous.

Next, we prove that Φ is completely continuous. Let $B_m = \{x \in \mathcal{B}/\|x\|_{\mathcal{B}}^2 \leq m\}$. From the proof of Theorem 3.1 we establish that the set $\Phi(B_m)|_{[t_0,T]} = \{x|_{[t_0,T]} \in B_m : x \in B_m\}$ is relatively compact in $\mathcal{B}([t_0,T];X)$ for every $T \in (t_0,\infty)$. Moreover, for $x \in B_m$, we have that,

$$\begin{aligned}
\frac{E\|\Phi x(t)\|^2}{g(t)} &\leq \frac{2M^2 e^{-\gamma(t-t_0)} e^{-\lambda(1-C)(t-t_0)} E\left[\|\varphi\|^2\right]}{g(t)} \\
&\quad + \frac{2M^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)} e^{-\gamma(t-t_0)}}{\gamma g(t)} \int_{t_0}^{t} e^{\gamma(s-t_0)} p(s,s) H(E\|x\|_{\mathcal{B}_g}^2 g(s)) ds,
\end{aligned}$$

where, from Theorem 6.2(b) we get that $\frac{E\|\Phi x\|_t^2}{g(t)} \to 0$ as $t \to \infty$ uniformly for $x \in B_m$. Now, Lemma 4.2 allows us to conclude that $\Phi(B_m)$ is relatively compact in $\mathcal{B}_g^0(X)$. Thus Φ is completely continuous.

By Lemma 2.7 the operator Φ has a fixed point in $\mathcal{B}_g$. Therefore the system (11) has a mild solution which is exponentially mean square stable with $\varphi(t) = \varphi(t-t_0)$ when $t \in [t_0 - r, t_0]$ and $E\|\varphi\|_t^2 \to 0$ as $t \to \infty$. This completes the proof. □

Acknowledgements: The authors sincerely thank Professor R. Agarwal,Department of Mathematics, Texas A& M University-Kingsville, Kingsville, TX 78363, USA. e-mail: Ravi.Agarwal@tamuk.edu, for his valuable suggestion.

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