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Error estimation by Schurer type $(\mathfrak{p},\mathfrak{q})$-Lorentz operator on a compact disk

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Abstract. In this manuscript, we construct a complex $(\mathfrak{p},\mathfrak{q})$ Lorentz-Schurer operator for $\mathfrak{q} > \mathfrak{p} > 1$ and discuss the approximation properties on a compact disk. Based on the Voronovskaja's type theorem and exact orders, we also obtain quantitative estimate by the $(\mathfrak{p},\mathfrak{q})$ Lorentz-Schurer operator attached to analytic functions in the compact disk.

1. Introduction

The concept of $\mathfrak{q}$-calculus emerged as a new interest in the area of approximation theory. The formalism depend its root in the field of approximation theory with the systematic study by several authors (For the sequential knowledge, the readers may refer to [1, 2, 5, 7, 9, 10, 18] and [20]). The emergence of $(\mathfrak{p},\mathfrak{q})$-analogue of different operators, for instance, Bernstein operators, Bernstein-Stancu operators and others (see [3, 4, 19, 22]) paved way for the more efficient concept of approximation by the $(\mathfrak{p},\mathfrak{q})$-analogue of a positive linear operator, which has been an active area of research lately. Now, we recall the following definitions essential for the present investigation: For any analytic function h from $\mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\}$ to $\mathbb{C}$, G.G. Lorentz introduced (see [8]) an important sequence of operators by

$$\mathcal{L}_\nu(h;w) = \sum_{k=0}^{\nu} \binom{\nu}{k} \left(\frac{w}{\nu}\right)^k h^{(k)}(0), \quad (\nu \in \mathbb{N},\ w \in \mathbb{C}). \tag{1}$$

The $\mathfrak{q}$-analogue of the Lorentz operators (for $\mathfrak{q} > 1$), studied by Gal (see [6]) is given by

$$\mathcal{L}_{\nu,\mathfrak{q}}(h;w) = \sum_{k=0}^{\nu} \mathfrak{q}^{\frac{k(k-1)}{2}} \binom{\nu}{k}_{\mathfrak{q}} \left(\frac{w}{[\nu]_{\mathfrak{q}}}\right)^k D_{\mathfrak{q}}^{(k)}(h)(0), \quad (\nu \in \mathbb{N},\ w \in \mathbb{C}). \tag{2}$$

The $(\mathfrak{p},\mathfrak{q})$ integer $[\nu]_{\mathfrak{p},\mathfrak{q}}$ is defined by

$$[\nu]_{\mathfrak{p},\mathfrak{q}} := \frac{\mathfrak{p}^\nu - \mathfrak{q}^\nu}{\mathfrak{p}-\mathfrak{q}}, \quad (\nu \in \mathbb{N}_0).$$

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The $(\mathfrak{p},\mathfrak{q})$-binomial expansion is given as

$$(ax+by)^{\nu}_{\mathfrak{p},\mathfrak{q}} := \sum_{k=0}^{\nu} \binom{\nu}{k}_{\mathfrak{p},\mathfrak{q}} a^{\nu-k} b^{k} x^{\nu-k} y^{k},$$

where, the coefficients $\binom{\nu}{k}_{\mathfrak{p},\mathfrak{q}}$ are given by

$$\binom{\nu}{k}_{\mathfrak{p},\mathfrak{q}} := \frac{[\nu]_{\mathfrak{p},\mathfrak{q}}!}{[k]_{\mathfrak{p},\mathfrak{q}}![\nu-k]_{\mathfrak{p},\mathfrak{q}}!}.$$

The $(\mathfrak{p},\mathfrak{q})$-derivative of a function h (differentiable at 0) is defined as:

$$D_{\mathfrak{p},\mathfrak{q}}\, h(x) = \begin{cases} \frac{h(\mathfrak{p}x)-h(\mathfrak{q}x)}{(\mathfrak{p}-\mathfrak{q})x}, & \text{if } x \neq 0 \\ h'(0), & \text{if } x = 0. \end{cases}$$

Details on $(\mathfrak{p},\mathfrak{q})$-calculus can be found in (see for details [11, 13, 15–17, 21]).

Throughout this paper we use the following standard notation: $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denotes the set of natural, non negative integer, integers, real, complex numbers respectively.

2. Construction of Operators

We define the $(\mathfrak{p},\mathfrak{q})$-analogue of Lorentz-Schurer operator on a compact disk for $\mathfrak{q} > \mathfrak{p} > 1$ as follows:

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h;w) = \sum_{k=0}^{\nu+\alpha} \mathfrak{q}^{\frac{k(k-1)}{2}} \binom{\nu}{k}_{\mathfrak{p},\mathfrak{q}} \left(\frac{w}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{k} D^{(k)}_{\mathfrak{p},\mathfrak{q}}(h)(0), \tag{3}$$

$$(\nu \in \mathbb{N},\ \alpha > 0 \text{ and fixed},\ w \in \mathbb{C}).$$

We remark that the operators in (3) reduce to the $\mathfrak{q}$-analogue of Lorentz operators (2) for $\mathfrak{p} = 1$ and $\alpha = 0$.

The upper approximation estimate of $(\mathfrak{p},\mathfrak{q})$-Lorentz-Schurer is given under the theorem below.

Theorem 2.1. Let $h : \mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\} \to \mathbb{C}$ be an analytic function in $\mathcal{D}_{\mathcal{R}}$, where $R > \mathfrak{q} > \mathfrak{p} > 1$ be such that $h(w) = \sum_{k=0}^{\infty} C_k\, w^k$, then

(i) Let $1 \leq r < \frac{\mathfrak{p}r_1}{\mathfrak{q}} < \frac{\mathfrak{p}R}{\mathfrak{q}}$ be arbitrary fixed and $|w| \leq r,\ \nu \in \mathbb{N}, \alpha > 0$, then we have the following upper estimate:

$$\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w)\right| \leq \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h),$$

where,

$$\mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h) = \frac{\mathfrak{p}(\mathfrak{q}-\mathfrak{p}+1)}{(\mathfrak{q}-\mathfrak{p})^2} \sum_{k=0}^{\infty} |C_k|\,(k+1) r_1^k < \infty.$$

(ii) Let $1 \leq r << r^* < \frac{\mathfrak{p}r_1}{\mathfrak{q}} < \frac{\mathfrak{p}R}{\mathfrak{q}}$ be arbitrary fixed with $|w| \leq r,\ \mu, \nu \in \mathbb{N}$, then the following approximation is given as

$$\left|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h^{(\mu)}(w)\right| \leq \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h) \frac{\mu!\, r^*}{(r^*-r)^{\mu+1}},$$

where,

$$\mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h) = \frac{\mathfrak{p}(\mathfrak{q}-\mathfrak{p}+1)}{(\mathfrak{q}-\mathfrak{p})^2} \sum_{k=0}^{\infty} |C_k|(k+1) r_1^k < \infty.$$

Proof. (i) For $e_i(w) = w^i \implies \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_0)(w) = 1$ and $\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_1)(w) = w$. Then

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) = \mathfrak{q}^{\frac{i(i-1)}{2}} \binom{\nu+\alpha}{i}_{\mathfrak{p},\mathfrak{q}} [i]_{\mathfrak{p},\mathfrak{q}}! \, \frac{w^i}{[\nu+\alpha]^i_{\mathfrak{p},\mathfrak{q}}}$$

$$(2 \leq i \leq \nu+\alpha, \forall\, i, \nu \in \mathbb{N}),$$

and by solving further, we find

$$\begin{aligned}\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) =& w^i \left(1 - \mathfrak{p}^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \\ &\times \left(1 - \mathfrak{p}^{\nu+\alpha-2} \frac{[2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right).\end{aligned}$$

It is obvious that for $i \geq \nu+\alpha+1$,

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) = 0.$$

Also, we can write

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{i=0}^{\infty} C_i \, \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w), \quad \forall\, |w| \leq r.$$

Thus, we have

$$\begin{aligned}&\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w)\right| \\ &\leq \sum_{i=0}^{\nu+\alpha} |C_i| \left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) - e_i(w)\right| + \sum_{i=\nu+\alpha+1}^{\infty} |C_i| \left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) - e_i(w)\right| \\ &\leq \sum_{i=2}^{\nu+\alpha} |C_i| r^i \left| \left(1 - p^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-2} \frac{[2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) - 1 \right| \\ &+ \sum_{i=\nu+\alpha+1}^{\infty} |C_i| r^i, \; \forall\, |w| \leq r.\end{aligned}$$

By using the result in [6], we have

$$\begin{aligned}1 - &\left(1 - \mathfrak{p}^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \left(1 - \mathfrak{p}^{\nu+\alpha-2} \frac{[2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \\ &\ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \leq \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{(i-1)[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}.\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=2}^{\nu+\alpha}|C_i|r^i\ \frac{1}{\mathfrak{p}^{i(\nu+\alpha)}}\\
&\quad\times\left|\left(1-\mathfrak{p}^{\nu+\alpha-1}\frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)\left(1-\mathfrak{p}^{\nu+\alpha-2}\frac{[2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)\ldots\left(1-\mathfrak{p}^{\nu+\alpha-(i-1)}\frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)-1\right|\\
&\quad\leq\frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\sum_{i=2}^{\nu+\alpha}|C_i|\ (i-1)\ [i-1]_{\mathfrak{p},\mathfrak{q}}\ \mathfrak{p}^{-(i-1)}\ r^i\\
&\quad\leq\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}(\mathfrak{q}-\mathfrak{p})}\sum_{i=2}^{\nu+\alpha}|C_i|\ i\left(\frac{\mathfrak{q}}{\mathfrak{p}}r\right)^i\\
&\quad\leq\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}(\mathfrak{q}-\mathfrak{p})}\sum_{i=2}^{\nu+\alpha}|C_i|\ (i+1)\ r_1^i\quad\left(\text{as }\ \frac{\mathfrak{q}r}{\mathfrak{p}}<r_1\right),
\end{aligned}$$

where, by the conditions already mentioned for h, we get $\sum_{i=2}^{\nu+\alpha}|C_i|\ (i+1)\ r_1^i<\infty$.

As h is analytic which implies that $C_i=\frac{h^{(i)}(0)}{i!}$, and by the Cauchy's estimate of C_i in the disk $|w|\leq r_1$, we have $|C_i|\leq\frac{K_{r_1}}{r-1}$, $\forall\ i\geq 0$, where

$$K_{r_1}=\max\{|h(x)|:|w|\leq r_1\}\leq\sum_{i=2}^{\nu+\alpha}|C_i|r^i\leq\sum_{i=2}^{\nu+\alpha}|C_i|\ (i+1)\ r_1^i:=R_{r_1}(h)<\infty.$$

Therefore,

$$\sum_{i=\nu+\alpha+1}^{\infty}|C_i|r^i\leq R_{r_1}(h)\left[\frac{r}{r_1}\right]^{\nu+\alpha+1}\sum_{i=0}^{\infty}\left(\frac{r}{r_1}\right)^i=R_{r_1}(h)\left[\frac{r}{r_1}\right]^{\nu+\alpha+1}\left(\frac{r_1}{r_1-r}\right)$$

$$=R_{r_1}(h)\left(\frac{r}{r_1-r}\right)\left[\frac{r}{r_1}\right]^{\nu+\alpha}\leq R_{r_1}(h)\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\frac{1}{(\mathfrak{q}-\mathfrak{p})^2},$$

and finally we have

$$\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)-h(w)\right|\leq\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\frac{(\mathfrak{q}-\mathfrak{p}+1)}{(\mathfrak{q}-\mathfrak{p})^2}R_{r_1}(h),\quad\forall\ \nu\in\mathbb{N}\text{ and }|w|\leq r.$$

(ii) Let us consider the circle Θ with center 0 and radius $r^*>r$. Also, we have that for any $|w|\leq r$ and $v\in\Theta$, $|v-w|\geq r^*-r$.

Now, for all $\nu\in\mathbb{N}$, it follows from Cauchy's formula that

$$\begin{aligned}
\left|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)-h^{(\mu)}(w)\right|&=\frac{\mu!}{2\pi}\left|\int_{\Theta}\frac{\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(v)-h(v)}{(v-w)^{\mu+1}}dv\right|\\
&\leq\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h)\ \frac{\mu!}{2\pi}\ \frac{2\pi r^*}{(r^*-r)^{\mu+1}}\\
&=\frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathcal{M}_{r_1,\mathfrak{p},\mathfrak{q}}(h)\ \frac{\mu!\ r^*}{(r^*-r)^{\mu+1}}.
\end{aligned}$$

Theorem 2.2. Let $h : \mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\} \to \mathbb{C}$ be an analytic function with $h(w) = \sum_{k=0}^{\infty} C_k w^k$, $\forall\, w \in \mathcal{D}_{\mathcal{R}}$ and $R > \mathfrak{q}^4 > \mathfrak{p}^4 > 1$, $1 \leq r < \frac{\mathfrak{p}^3 r_1}{\mathfrak{q}^3} < \frac{\mathfrak{p}^4 R}{\mathfrak{q}^4}$ be arbitrary fixed. Then for all $\nu \in \mathbb{N}$, $|w| \leq r$, we have

$$\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right| \leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}\, \mathcal{Q}_{r_1,\mathfrak{p},\mathfrak{q}}(h),$$

where $\mathcal{Q}_{r_1,\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}\mathfrak{q}-\mathfrak{q}+\mathfrak{p}}{(\mathfrak{p}-1)(\mathfrak{q}-\mathfrak{p})} \sum |C_k|(k+1)(k+2)^2 \left(\frac{\mathfrak{q}}{\mathfrak{p}} r_1\right)^k < \infty$.
□

Proof. For all $|w| \leq r, \nu \in \mathbb{N}$, we have

$$\begin{aligned}
&\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right| \\
&= \left|\sum_{k=0}^{\infty} C_k \left[\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) - e_k(w) + \mathfrak{p}^{\nu+\alpha-(k-1)}\, \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1}\, e_k(w)\right]\right| \\
&\leq \left|\sum_{k=0}^{\nu+\alpha} C_k \left[\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) - e_k(w) + \mathfrak{p}^{\nu+\alpha-(k-1)}\, \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1}\, e_k(w)\right]\right| \\
&+ \left|\sum_{k=\nu+\alpha+1}^{\infty} \mathfrak{p}^{\nu+\alpha-(k-1)}\, C_k\, w^k \left(\frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1} - 1\right)\right| \\
&\leq \left|\sum_{k=0}^{\nu+\alpha} C_k \left[\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) - e_k(w) + \mathfrak{p}^{\nu+\alpha-(k-1)}\, \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1}\, e_k(w)\right]\right| \\
&+ \sum_{k=\nu+\alpha+1}^{\infty} |C_k|\, r^k \left(\mathfrak{p}^{\nu+\alpha-(k-1)}\, \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1} - 1\right).
\end{aligned}$$

Here, we have by principle of mathematical induction with respect to k that

$$0 \leq \mathcal{E}_{\nu,\alpha,k,\mathfrak{p},\mathfrak{q}}(w) \leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}} \frac{(k+1)(k-2)^2}{(\mathfrak{q}-\mathfrak{p})} \left(\frac{\mathfrak{q} r_1}{\mathfrak{p}}\right)^k, \tag{4}$$

$\forall\, 2 \leq k \leq \nu+\alpha, \nu \in \mathbb{N}$ is arbitrary fixed and $|w| \leq r$, where

$$\begin{aligned}
\mathcal{E}_{\nu,\alpha,k,\mathfrak{p},\mathfrak{q}}(w) &= \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) - e_k(w) + \frac{\mathfrak{p}^{\nu+\alpha-(k-1)}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\, \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1}\, e_k(w) \\
&= \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) - e_k(w) + \frac{\mathfrak{p}^{\nu+\alpha-(k-1)}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \left([1]_{\mathfrak{p},\mathfrak{q}} + \ldots + [k-1]_{\mathfrak{p},\mathfrak{q}}\right)\, e_k(w).
\end{aligned}$$

Since, by principle of mathematical induction, we have

$$\frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{p}-1} = \left([1]_{\mathfrak{p},\mathfrak{q}} + \ldots + [k-1]_{\mathfrak{p},\mathfrak{q}}\right).$$

Then again, in view of the result in Theorem 2.1 (i), we can write $\mathcal{E}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(w) = 0$, $\forall\, \nu \in \mathbb{N}$ and we get the recurrence relation

$$\begin{aligned}
\mathcal{E}_{\nu,\alpha,k,\mathfrak{p},\mathfrak{q}}(w) \quad &= \quad -\frac{w^2}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \mathfrak{p}^{\nu+\alpha-(k-1)}\, D_{\mathfrak{p},\mathfrak{q}}[\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)] \\
&+ \quad \left(\frac{\mathfrak{p}-1}{\mathfrak{p}}\right) w[\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)] + \frac{w}{\mathfrak{p}}\, \mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w), \quad |w| \leq r.
\end{aligned}$$

Now, for $|w| \leq r$ and $3 \leq k \leq \nu + \alpha$ and with the use of mean value theorem, with $\|h\|_r = \max\{|h(w)| : |w| \leq r\}$, we write

$$\begin{aligned}
\left|\mathcal{E}_{\nu,\alpha,k,\mathfrak{p},\mathfrak{q}}(w)\right| &= \frac{r^2}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)}\ \|\left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)\right)'\|_{\frac{\mathfrak{q}r}{\mathfrak{p}}} \\
&+ \frac{(\mathfrak{p}-1)}{\mathfrak{p}} r \,\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)\|_{\frac{\mathfrak{q}r}{\mathfrak{p}}} + \frac{r}{\mathfrak{p}}\ \left|\mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w)\right| \\
&= \frac{r^2}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)}\ \frac{(k-1)}{\frac{\mathfrak{q}}{\mathfrak{p}}r}\ \|\left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)\right\|_{\frac{\mathfrak{q}r}{\mathfrak{p}}} \\
&+ \frac{(\mathfrak{p}-1)}{\mathfrak{p}} r \,\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_{k-1})(w) - e_{k-1}(w)\|_{\frac{\mathfrak{q}r}{\mathfrak{p}}} + \frac{r}{\mathfrak{p}}\ \left|\mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w)\right| \\
&= \frac{r^2}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)}\ \frac{(k-1)}{\frac{\mathfrak{q}r}{\mathfrak{p}}}\ \left(\frac{\mathfrak{q}r}{\mathfrak{p}}\right)^{k-1}\ \frac{(k-2)[k-2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)} \\
&+ \frac{(\mathfrak{p}-1)r}{\mathfrak{p}}\ \left(\frac{\mathfrak{q}r}{\mathfrak{p}}\right)^{k-1}\ \frac{(k-2)[k-2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)} + \frac{r}{\mathfrak{p}}\ \left|\mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w)\right| \\
&= \left\{\frac{r^2}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\mathfrak{p}^{\nu+\alpha-(k-1)}\frac{(k-1)}{\mathfrak{q}r/\mathfrak{p}} + \frac{(\mathfrak{p}-1)}{\mathfrak{p}}r\right\}\left(\frac{\mathfrak{q}r}{\mathfrak{p}}\right)^{k-1}\ \frac{(k-2)[k-2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \mathfrak{p}^{\nu+\alpha-(k-1)} \\
&+ \frac{r}{\mathfrak{p}}\ \left|\mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w)\right| \\
&\leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}(k+1)(k-2)[k-2]_{\mathfrak{p},\mathfrak{q}} r_1^k + r_1 \left|\mathcal{E}_{\nu,\alpha,k-1,\mathfrak{p},\mathfrak{q}}(w)\right|.
\end{aligned}$$

For $k = 1, 2, 3, \cdots$, we obtain the estimate

$$\begin{aligned}
\left|\mathcal{E}_{\nu,\alpha,k,\mathfrak{p},\mathfrak{q}}(w)\right| &\leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}} r_1^k \sum_{i=3}^{k}(i-1)(i-2)[i-2]_{\mathfrak{p},\mathfrak{q}} \\
&\leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}} \frac{(k+1)(k-2)^2}{(\mathfrak{q}-\mathfrak{p})}\left(\frac{\mathfrak{q}r_1}{\mathfrak{p}}\right)^k.
\end{aligned}$$

Further, we calculate

$$\left|\sum_{k=0}^{\nu+\alpha}|C_k|\left[\mathcal{L}_{\nu,\alpha,p,\mathfrak{q}}(e_k)(w) - e_k(w) + \mathfrak{q}^{\nu+\alpha-(k-1)}\ \frac{[k]_{\mathfrak{p},\mathfrak{q}}-[k]_{\mathfrak{q}}}{\mathfrak{p}-1}\ e_k(w)\right]\right|$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\nu+\alpha} |C_k|\ \left|\mathcal{E}_{\nu,\alpha,k,p,\mathfrak{q}}(w)\right| \\
&\leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}\frac{1}{(\mathfrak{q}-\mathfrak{p})}\sum_{k=0}^{\nu+\alpha}|C_k|(k+1)(k-2)^2\left(\frac{\mathfrak{q}r_1}{\mathfrak{p}}\right)^k \\
&\leq \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}\frac{1}{(\mathfrak{q}-\mathfrak{p})}\sum_{k=0}^{\nu+\alpha}|C_k|(k+1)(k+2)^2\left(\frac{\mathfrak{q}r_1}{\mathfrak{p}}\right)^k.
\end{aligned}$$

Finally, since $\left(\mathfrak{p}^{\nu+\alpha-(k-1)}\ \frac{[k]_{\mathfrak{p},\mathfrak{q}}-[k]_{\mathfrak{q}}}{\mathfrak{p}-1}-1\right)\geq 0,\ \forall\, k\geq \nu+\alpha+1$, in view of Theorem 2.1 (i), we get

$$
\begin{aligned}
&\sum_{k=\nu+\alpha+1}^{\infty}|C_k|r^k\left(\mathfrak{p}^{\nu+\alpha-(k-1)}\frac{[k]_{\mathfrak{p},\mathfrak{q}}-[k]_{\mathfrak{q}}}{\mathfrak{p}-1}-1\right)\\
&\quad\leq \sum_{k=\nu+\alpha+1}^{\infty}\mathfrak{p}^{\nu+\alpha-(k-1)}\ |C_k|\ r^k\frac{[k]_{\mathfrak{p},\mathfrak{q}}}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\\
&\quad\leq \sum_{k=\nu+\alpha+1}^{\infty}\mathfrak{p}^{\nu+\alpha-(k-1)}\ |C_k|\ \frac{1}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\frac{kq^k}{\mathfrak{p}^k(\mathfrak{q}-\mathfrak{p})}\\
&\quad\leq\frac{R_{r_1}(h)\mathfrak{p}^{\nu+\alpha+1}}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\sum_{k=\nu+\alpha+1}^{\infty}\frac{r^k}{r_1^k}\,\mathfrak{q}^k\mathfrak{p}^{-k}\\
&\quad\leq\frac{R_{r_1}(h)\mathfrak{p}^{\nu+\alpha+1}}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\sum_{k=\nu+\alpha+1}^{\infty}\left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k\ \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^{2k}\mathfrak{q}^k\mathfrak{p}^{-k}\\
&\quad\leq\frac{R_{r_1}(h)\mathfrak{p}^{\nu+\alpha+1}}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\left(\frac{r}{r_1}\right)^{\frac{(\nu+\alpha+1)}{3}}\sum_{k=0}^{\infty}\left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k\\
&\quad=\frac{R_{r_1}(h)\mathfrak{p}^{\nu+\alpha+1}}{(\mathfrak{p}-1)[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\left(\frac{r}{r_1}\right)^{\frac{\nu+\alpha}{3}}\ \frac{r^{\frac{1}{3}}}{(r_1^{\frac{1}{3}}-r^{\frac{1}{3}})}\\
&\quad\leq\frac{\mathfrak{p}^{2(\nu+\alpha)+2}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}\frac{R_{r_1}(h)}{(\mathfrak{p}-1)\ (\mathfrak{q}-\mathfrak{p})^2}\\
&\quad\leq\frac{\mathfrak{p}^{2(\nu+\alpha)+2}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}\ (\mathfrak{p}-1)\ (\mathfrak{q}-\mathfrak{p})^2}\sum_{k=0}^{\nu+\alpha}|C_k|\ (k+1)(k+2)^2\left(\frac{q}{\mathfrak{p}}r_1\right)^k,
\end{aligned}
$$

with the help of the inequalities

$$
[k]_{\mathfrak{p},\mathfrak{q}}\leq\frac{kq^k}{\mathfrak{p}^k},\ \frac{\mathfrak{p}^{\nu+\alpha}}{\mathfrak{q}^{\nu+\alpha}}\leq\frac{\mathfrak{p}^{\nu+\alpha}}{(\mathfrak{q}-\mathfrak{p})[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\ \text{ and }\ \frac{r^{1/3}}{(r_1^{1/3}-r^{1/3})}\leq\frac{\mathfrak{p}}{(\mathfrak{q}-\mathfrak{p})}.
$$

Thus, combining above estimates, we get

$$
\begin{aligned}
&\left|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)-h(w)+\frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right|\\
\leq&\frac{(\mathfrak{p}\mathfrak{q}-\mathfrak{q}+\mathfrak{p}-1)}{(\mathfrak{p}-1)\ (\mathfrak{q}-\mathfrak{p})^2}\ \frac{\mathfrak{p}^{2(\nu+\alpha)}}{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}\ \sum_{k=0}^{\nu+\alpha}|C_k|\ (k+1)(k+2)^2\ \left(\frac{q}{\mathfrak{p}}r_1\right)^k.
\end{aligned}
$$

We give the lower approximation estimate with the help of undermentioned Theorem:

Theorem 2.3. Let $h:\mathcal{D}_{\mathcal{R}}=\{w\in\mathbb{C}:|w|<R\}\to\mathbb{C}$ be an analytic function in $\mathcal{D}_{\mathcal{R}}$ where $h(w)=\sum_{k=0}^{\infty}C_kw^k,\ \forall\, w\in\mathcal{D}_{\mathcal{R}}$ and let $1\leq r<\frac{\mathfrak{q}^3r_1}{\mathfrak{p}^3}<\frac{\mathfrak{q}^4R}{\mathfrak{p}^4}$ be arbitrary fixed. If h is not a polynomial of degree ≤ 1, then for all $\nu\in\mathbb{N}$ and $|w|\leq r$, we have

$$
\left\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\right\|_r\geq\frac{p^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}C_{r,r_1,\mathfrak{p},\mathfrak{q}}(h),
$$

where, $C_{r,r_1,\mathfrak{p},\mathfrak{q}}(h)$ counts only on $h,\ r$ and r_1. Also, $\|h\|_r$ refers to $\max_{|w|\leq r}\{|h(w)|\}$.

□

Proof. For $|w| \leq r$ and $\nu \in \mathbb{N}$, we have

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) = \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}$$
$$\times\left\{-\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\left[\frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{2(\nu+\alpha)}}\left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathfrak{p}^{(\nu+\alpha)}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\right)\right]\right\}.$$

Using the inequality

$$\|F+G\|_r \geq \big|\|F\|_r - \|G\|_r\big| \geq \|F\|_r - \|G\|_r,$$

we have

$$\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h\|_r \geq \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}$$
$$\times\left\{\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\| - \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\left[\frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{2(\nu+\alpha)}}\left\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\right\|_r\right]\right\}.$$

Now, we get $\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)\|_r > 0$. Since by hypothesis, h is not a polynomial of degree ≤ 1 in $\mathcal{D}_{\mathcal{R}}$ it follows by supposing the contrary that

$$\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) = 0 \ \forall\ w\ \in \overline{\mathcal{D}_{\mathcal{R}}} = \{w \in \mathbb{C} : |w| \leq r\}.$$

Further calculation gives

$$\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) = w\,\frac{D_{\mathfrak{p},\mathfrak{q}}(h)(w) - h'(w)}{\mathfrak{p} - q},\ \ \mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) = 0,$$

implying

$$\mathcal{D}_{\mathfrak{p},\mathfrak{q}}(h)(w) = h'(w), \forall\ w \in \overline{\mathcal{D}_r}\backslash\{0\}.$$

Considering $h(w) = \sum\limits_{k=0}^{\infty} C_k w^k$, the above inequality gives $C_k = 0$, for all $k \geq 2$, showing h is linear in $\overline{\mathcal{D}_r}$, which contradicts with the hypothesis. Now, in view of Theorem 2.2 we have

$$\frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{p^{2(\nu+\alpha)}}\left\|\mathcal{L}_{\nu,\alpha,p,\mathfrak{q}}(h)(w) - h(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\right\|_r \leq \mathcal{Q}_{r_1,\mathfrak{p},\mathfrak{q}}(h),$$

where $\mathcal{Q}_{r_1,\mathfrak{p},\mathfrak{q}}(h)$ is a positive constant depending only on $h,\ r_1,\ \mathfrak{p}$ and $\mathfrak{q}$.
Since $\frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \to 0$ as $\nu \to \infty$, there exists an index $\nu_\circ$ depending only on $h,\ r,\ r_1,\ \mathfrak{p}$ and $\mathfrak{q}$ such that for all $\nu + \alpha > \nu_\circ$, we have

$$\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\| - \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\left[\frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{p^{2(\nu+\alpha)}}\left\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\right\|_r\right] \geq \frac{1}{2}\left\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)\right\|_r,$$

which surely implies that

$$\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h\|_r \geq \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\frac{1}{2}\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)\|_r, \quad \forall\ \nu + \alpha > \nu_0.$$

For $\nu+\alpha \in \{1, ..., \nu_0\}$, we have

$$\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r \;\geq\; \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\,\mathcal{M}_{r,r_1,\nu,\alpha,\mathfrak{p},\mathfrak{q}},$$

with

$$\mathcal{M}_{r,r_1,\nu,\alpha,\mathfrak{p},\mathfrak{q}} = \frac{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{\nu+\alpha}}\,\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r > 0.$$

Since $\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r = 0$ would imply that h is a linear function, a contradiction. Therefore

$$\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r \;\geq\; \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\,C_{r,r_1,\mathfrak{p},\mathfrak{q}}(h),$$

where

$$C_{r,r_1,\mathfrak{p},\mathfrak{q}}(h) = \min\left\{\mathcal{M}_{r,r_1,1,\mathfrak{p},\mathfrak{q}}(h), \ldots, \mathcal{M}_{r,r_1,\nu_\circ,\mathfrak{p},\mathfrak{q}}(h),\; \frac{1}{2}\|\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)\|\right\},$$

from which the proof easily follows. □

In view of Theorem 2.3, and (i) of Theorem 2.1, we deduce the following result:

Corollary 2.4. Let $h : \mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\} \to \mathbb{C}$ be analytic in $\mathcal{D}_{\mathcal{R}}$, be such that $h(w) = \sum_{k=0}^{\infty} C_k w^k$, $\forall\, w \in \mathcal{D}_{\mathcal{R}}$ and let $1 \leq r < \frac{\mathfrak{q}^3 r_1}{\mathfrak{p}^3} < \frac{\mathfrak{q}^4 R}{\mathfrak{p}^4}$ be arbitrary fixed. If h is not a polynomial of degree ≤ 1, then for all $\nu \in \mathbb{N}$ and $|w| \leq r$, we have

$$\|\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r \sim \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}},$$

where, the constants depends on $h,\ r,\ r_1,\ \mathfrak{p}$ and $\mathfrak{q}$, and are independent of ν.

3. Approximation results

We prove the undermentioned Theorems concerning the simultaneous approximation:

Theorem 3.1. Let $h : \mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\} \to \mathbb{C}$ be an analytic function in $\mathcal{D}_{\mathcal{R}}$ be such that $h(w) = \sum_{k=0}^{\infty} C_k w^k$, $\forall\, w \in \mathcal{D}_{\mathcal{R}}$ and let $1 \leq r < r^* < \frac{\mathfrak{p}^3 r_1}{\mathfrak{q}^3} < \frac{\mathfrak{p}^4 R}{\mathfrak{q}^4}$ be arbitrary fixed. If h is not a polynomial of degree $\leq \max\{1, \mu-1\}$, then for all $\nu \in \mathbb{N}$, we have

$$\|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h^{(\mu)}\|_r \sim \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}},$$

where the constants depend on $h,\ r,\ r_1,\ \mu,\ \mathfrak{p}$ and $\mathfrak{q}$, but are independent of ν.

Proof. We already have the upper estimate for $\|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h^{(\mu)}\|_r$ by Theorem 2.1 (ii), so it remains to find the lower estimate for $\|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h^{(\mu)}\|_r$.

Let Θ be the circle of radius r^* and center 0. The inequality $|v-w| \geq r^*-r$ holds for all $|w| \leq r$ and $v \in \Theta$. Applying the Cauchy's formula, we have

$$\left|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)-h^{(\mu)}(w)\right| = \frac{\mu!}{2\pi}\left|\int_{\Theta}\frac{\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(v)-h(v)}{(v-w)^{\mu+1}}dv\right|. \tag{5}$$

Now, in view of the proof of Theorem 2.1 (ii), for all $\upsilon \in \Theta$ and $\nu \in \mathbb{N}$, we have

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) = \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \times \left\{ -\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \left[\frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{2(\nu+\alpha)}} \left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) \right) \right] \right\}. \tag{6}$$

By (5) and (6), we get

$$\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h^{(\mu)}(h) = \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \left\{ \frac{\mu!}{2\pi i} \int_\Theta - \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{(\upsilon - w)^{\mu+1}} d\upsilon + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{\mu!}{2\pi i} \int_\Theta \frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}} \left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \mathfrak{p}^{\nu+\alpha} \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \right)}{\mathfrak{p}^{2(\nu+\alpha)} (\upsilon - w)^{\mu+1}} d\upsilon \right\}$$

$$= \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \times \left\{ [-\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)]^{(\mu)} + \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{\mu!}{2\pi i} \int_\Theta \frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}} \left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \mathfrak{p}^{\nu+\alpha} \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \right)}{\mathfrak{p}^{2(n+\alpha)} (\upsilon - w)^{\mu+1}} d\upsilon \right\}.$$

Thus

$$\|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}} - h^{(\mu)}\|_r \geq \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \left\{ \left\| - [\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)]^{(\mu)} \right\|_r - \left\{ \frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \left\| \frac{\mu!}{2\pi} \int_\Theta \frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}} \left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \mathfrak{p}^{\nu+\alpha} \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \right)}{\mathfrak{p}^{2(\nu+\alpha)} (\upsilon - w)^{\mu+1}} d\upsilon \right\|_r \right\}.$$

Now in view of Theorem 2.2, $\forall\ \nu + \alpha \in \mathbb{N}$, we get

$$\begin{aligned} &\left\| \frac{\mu!}{2\pi} \int_\Theta \frac{[\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}} \left(\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) - h(w) + \mathfrak{p}^{\nu+\alpha} \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \right)}{\mathfrak{p}^{2(\nu+\alpha)} (\upsilon - w)^{\mu+1}} d\upsilon \right\|_r \\ &\leq \frac{\mu!}{2\pi} \frac{2\pi r^* [\nu+\alpha]^2_{\mathfrak{p},\mathfrak{q}}}{(r^* - r)^{\mu+1} \mathfrak{p}^{2(\nu+\alpha)}} \left\| \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h + \mathfrak{p}^{\nu+\alpha} \frac{\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \right\|_{r^*} \\ &\leq Q_{r_1,\mathfrak{p},\mathfrak{q}}(h) \frac{\mu!\, r^*}{(r^* - r)^{\mu+1}}. \end{aligned}$$

But we have $\left\| -[\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)]^{(\mu)} \right\|_{r^*} > 0$, by hypothesis on h.
Therefore, it would follow by supposing the contrary that

$$[\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)]^{(\mu)}(w) = 0 \quad (\forall\, |w| \leq r^*),$$

where, in view of Theorem 2.2, we write

$$\mathcal{S}_{\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=2}^{\infty} \mathfrak{q}^{\nu+\alpha-(k-1)} C_k \frac{[k]_{\mathfrak{p},\mathfrak{q}} - [k]_{\mathfrak{q}}}{\mathfrak{q} - 1} w^k = \sum_{k=2}^{\infty} \mathfrak{q}^{\nu+\alpha-(k-1)} C_k \left([1]_{\mathfrak{p},\mathfrak{q}} + \dots + [k-1]_{\mathfrak{p},\mathfrak{q}} \right) w^k.$$

Supposing $\mu = 1$, from

$$\mathcal{S}'_{\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=2}^{\infty} \mathfrak{q}^{\nu+\alpha-(k-1)} C_k\, k\, \left([1]_{\mathfrak{p},\mathfrak{q}} + \dots + [k-1]_{\mathfrak{p},\mathfrak{q}} \right) w^{k-1} = 0, \quad (\forall\, |w| \leq r^*),$$

it would follow that $C_k = 0$, for all $k \geq 2$, eventually which shows h would be a polynomial of degree $1 = \max\{1, \mu - 1\}$, that is a contradiction with the hypothesis.

Now taking $\mu = 2$, we get

$$\mathcal{S}''_{\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=2}^{\infty} \mathfrak{q}^{\nu+\alpha-(k-1)}\, C_k\, k\,(k-1)\ \left([1]_{\mathfrak{p},\mathfrak{q}} + \ldots + [k-1]_{\mathfrak{p},\mathfrak{q}}\right) w^{k-2} = 0, \quad (\forall |w| \leq r^*),$$

which would leads to a contradiction.

Now, taking $\mu > 2$, for all $|w| \leq r^*$, we get

$$\mathcal{S}^{(\mu)}_{\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=\mu}^{\infty} \mathfrak{q}^{\nu+\alpha-(k-1)}\, C_k\, k\,(k-1)\ldots(k-\mu+1)$$
$$\times \left([1]_{\mathfrak{p},\mathfrak{q}} + \ldots + [k-1]_{\mathfrak{p},\mathfrak{q}}\right) w^{k-\mu} = 0,$$

implying that $C_k = 0$, for all $k \geq \mu$, again a contradiction with the hypothesis.

Lastly, for h analytic in $\mathcal{D}_{\mathcal{R}}$ that is

$$h(w) = \sum_{k=0}^{\infty} C_k w^k, \quad (\forall w \in \mathcal{D}_{\mathcal{R}}),$$

and for any $\mu \in \mathbb{N},\ \mu \geq 2$, we define the iterates of complex Lorentz-Schurer operators $\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)$, by

$$\mathcal{L}^{(1)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) = \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)$$

and

$$\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) = \mathcal{L}_{\nu,\alpha,\mathfrak{p},q}[\mathcal{L}^{(\mu-1)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)](w).$$

Since by recurrence for all $\mu \geq 1$,

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=0}^{\infty} C_k \mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w),$$

thus we get

$$\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w) = \sum_{k=0}^{\infty} C_k \mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w),$$

where, for $k = 0, 1$ and $k \geq \nu + \alpha + 1$, we respectively have

$$\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) = 1,\ \mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) = w,\ \mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_k)(w) = 0.$$

For $2 \leq k \leq \nu$,

$$\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(e_i)(w) = \left(1 - \mathfrak{p}^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{\mu}$$
$$\times \left(1 - \mathfrak{p}^{\nu+\alpha-2} \frac{[2]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},q}}\right)^{\mu} \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{\mu}\ w^k,$$

□

Theorem 3.2. Let $h : \mathcal{D}_{\mathcal{R}} = \{w \in \mathbb{C} : |w| < R\} \to \mathbb{C}$ is analytic in $\mathcal{D}_{\mathcal{R}}$ with $R > \mathfrak{p} > \mathfrak{q} > 1$ and $1 \leq r < \frac{\mathfrak{p}r_1}{\mathfrak{q}} < \frac{\mathfrak{p}R}{\mathfrak{q}}$ be arbitrary fixed such that $h(w) = \sum_{k=0}^{\infty} C_k w^k, \forall\, w \in \mathcal{D}_{\mathcal{R}}$, then we have the upper estimate

$$\left|\mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h\right|_r \leq \frac{\mu\, p^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{\mathfrak{q}-\mathfrak{p}+1}{(\mathfrak{q}-\mathfrak{p})^2} \sum_{k=0}^{\infty} |C_k|(k+1) r_1^k. \tag{7}$$

Moreover,

$$\text{if } \lim_{\nu\to\infty} \frac{\mu_{\nu+\alpha}\,\mathfrak{p}^{\nu+\alpha}}{[\nu]_{\mathfrak{p},\mathfrak{q}}} = 0, \text{ then } \lim_{\nu\to\infty} \left\|\mathcal{L}^{(\mu_\nu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h) - h\right\|_r = 0.$$

Proof. We have for all $|w| \leq r$,

$$\left|h(w) - \mathcal{L}^{(\mu)}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)(w)\right| \leq \sum_{k=2}^{\nu+\alpha} |C_k| r^k$$
$$\times \left[1 - \left(1 - \mathfrak{p}^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{\mu} \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{p,\mathfrak{q}}}\right)^{\mu}\right] + \sum_{k=\nu+\alpha+1}^{\infty} |C_k| r^k.$$

Denoting

$$\mathcal{A}_{k,\nu,\alpha} = \left(1 - \mathfrak{p}^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right) \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[k-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right),$$

we get

$$1 - \mathcal{A}^{\mu}_{k,\nu,\alpha} = (1 - \mathcal{A}_{k,\nu,\alpha})(1 + \mathcal{A}_{k,\nu,\alpha} + \mathcal{A}^2_{k,\nu,\alpha} + \ldots + \mathcal{A}^{\mu-1}_{k,\nu,\alpha}) \leq \mu(1 - \mathcal{A}_{k,\nu,\alpha}),$$

and since

$$1 - \mathcal{A}_{k,\nu,\alpha} \leq \mathfrak{p}^{\nu+\alpha-(k-1)} \frac{(k-1)[k-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \quad (\forall\, |w| \leq r),$$

we therefore obtain

$$\sum_{k=2}^{\nu+\alpha} |C_k| r^k \left[1 - \left(1 - p^{\nu+\alpha-1} \frac{[1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{\mu} \ldots \left(1 - \mathfrak{p}^{\nu+\alpha-(i-1)} \frac{[i-1]_{\mathfrak{p},\mathfrak{q}}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^{\mu}\right] + \sum_{k=\nu+\alpha+1}^{\infty} |C_k| r^k$$

$$\leq \mu \sum_{k=2}^{\infty} |C_k| r^k \mathfrak{p}^{1-(k-1)}\,[1 - \mathcal{A}_{k,\nu,\alpha}] \leq \frac{\mu\,\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \sum_{k=2}^{\infty} |C_k| r^k (k-1)[k-1]_{\mathfrak{p},\mathfrak{q}} r^k$$
$$\leq \frac{\mu\,\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \sum_{k=2}^{\infty} |C_k| r^k \frac{k\mathfrak{q}^k/\mathfrak{p}^k}{\mathfrak{q}-\mathfrak{p}} \leq \frac{\mu\,\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{1}{\mathfrak{q}-\mathfrak{p}} \sum_{k=2}^{\infty} |C_k|(k+1)\left(\frac{\mathfrak{q}}{p} r\right)^k$$
$$\leq \frac{\mu\,\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{1}{\mathfrak{q}-\mathfrak{p}} \sum_{k=2}^{\infty} |C_k|(k+1)(r_1)^k.$$

Also, in view of the proof of Theorem 2.1, we get the estimate

$$\sum_{k=n+\alpha+1}^{\infty} |C_k| r^k \leq \frac{\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} \frac{\sum_{k=0}^{\infty} |C_k|(k+1)(r_1)^k}{(\mathfrak{q}-\mathfrak{p})^2} \leq \frac{\mu\,\mathfrak{p}^{\nu+\alpha+1}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}} . \frac{\sum_{k=0}^{\infty} |C_k|(k+1)(r_1)^k}{(\mathfrak{q}-\mathfrak{p})^2}.$$

Collecting now all the estimates and considering that

$$\frac{1}{\mathfrak{q}-\mathfrak{p}}+\frac{1}{(\mathfrak{q}-\mathfrak{p})^2}=\frac{\mathfrak{q}-\mathfrak{p}+1}{(\mathfrak{q}-\mathfrak{p})^2},$$

we get the desired result.
As

$$\frac{\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\sim\frac{\mathfrak{p}^{\nu+\alpha}}{\mathfrak{q}^{\nu+\alpha}},$$

we conclude, if $\lim_{\nu\to\infty}\frac{\mu_{\nu+\alpha}\mathfrak{p}^{\nu+\alpha}}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}=0$, that

$$\lim_{\nu\to\infty}\|\mathcal{L}^{(\mu_{\nu+\alpha})}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h)-h\|_r=0.$$

□

4. Concluding Remark

In this paper we have intruduced $(\mathfrak{p},\mathfrak{q})$-analogue of Lorentz-Schurer operator, is given by the following equation:

$$\mathcal{L}_{\nu,\alpha,\mathfrak{p},\mathfrak{q}}(h;w)=\sum_{k=0}^{\nu+\alpha}\mathfrak{q}^{\frac{k(k-1)}{2}}\binom{\nu}{k}_{\mathfrak{p},\mathfrak{q}}\left(\frac{w}{[\nu+\alpha]_{\mathfrak{p},\mathfrak{q}}}\right)^k D^{(k)}_{\mathfrak{p},\mathfrak{q}}(h)(0), \tag{8}$$

$$(\nu\in\mathbb{N},\ \alpha>0 \text{ and fixed},\ w\in\mathbb{C}).$$

• If we put $\mathfrak{p}=1$ in equation (8), $(\mathfrak{p},\mathfrak{q})$-analogue of Lorentz-Schurer operator reduces to the $\mathfrak{q}$-analogue of the Lorentz-Schurer operators and it is given by the following equation:

$$\mathcal{L}_{\nu,\alpha,\mathfrak{q}}(h;w)=\sum_{k=0}^{\nu+\alpha}\mathfrak{q}^{\frac{k(k-1)}{2}}\binom{\nu}{k}_{\mathfrak{q}}\left(\frac{w}{[\nu+\alpha]_{\mathfrak{q}}}\right)^k D^{(k)}_{\mathfrak{q}}(h)(0),$$

$$(\nu\in\mathbb{N},\ \alpha>0 \text{ and fixed},\ w\in\mathbb{C}).$$

• For the case $\alpha=0$ in above equation, we get $\mathfrak{q}$-analogue of the Lorentz operators:

$$\mathcal{L}_{\nu,\mathfrak{q}}(h;w)=\sum_{k=0}^{\nu}\mathfrak{q}^{\frac{k(k-1)}{2}}\binom{\nu}{k}_{\mathfrak{q}}\left(\frac{w}{[\nu]_{\mathfrak{q}}}\right)^k D^{(k)}_{\mathfrak{q}}(h)(0),\quad(\nu\in\mathbb{N},\ w\in\mathbb{C}).$$

• If we take $\alpha=0$, $\mathfrak{p}=\mathfrak{q}=1$ in equation (8), then we get very famous simple Lorentz operator introduced by G.G. Lorentz, and is given by

$$\mathcal{L}_{\nu}(h;w)=\sum_{k=0}^{\nu}\binom{\nu}{k}\left(\frac{w}{\nu}\right)^k h^{(k)}(0),\quad(\nu\in\mathbb{N},\ w\in\mathbb{C}).$$

Henceforth, our operators given by equation (8), can be seen as a generalization as well as modification of the well known operators of $\mathfrak{q}$-analogue of the Lorentz-Schurer operators. As $\frac{1}{n}<\frac{1}{[n]_q}$, that is sequaence $\frac{1}{[n]_q}$ is sharply converge against the sequence $\frac{1}{n}$.

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