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Left and right resolvents and new characterizations of left and right generalized Drazin invertible operators

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Abstract. Left and right resolvents of left and right generalized Drazin invertible operators are introduced in this paper. The construction of left and right resolvents allows us to find, in terms of the coefficients of Laurent series, new representation results for left and right generalized Drazin inverses and the associated spectral projections. Fundamental characterizations of left and right generalized Drazin invertible operators are also obtained, using essentially the range, the quasi-nilpotent part and the analytic core.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a nontrivial complex Hilbert space equipped with an inner product $\langle ., . \rangle$ and the corresponding norm $\|.\|$. By $M^{\perp}$ we denote the complement orthogonal of a subspace $M \subset \mathcal{H}$. The set of all bounded linear operators on $\mathcal{H}$ is $\mathcal{B}(\mathcal{H})$. Given $A \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ the kernel, the range and the spectrum of A, respectively. A^* is the adjoint operator of A. The identity operator will be denoted by I. The restriction operator of A to M is denoted by A_M and is such that $A_M x = Ax$ for all $x \in M$. The resolvent set $\rho(A) = \mathbb{C} \backslash \sigma(A)$ of $A \in \mathcal{B}(\mathcal{H})$ is the set of all complex number $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is invertible in $\mathcal{B}(\mathcal{H})$. The resolvent operator $R(\lambda, A) = (A - \lambda I)^{-1}$ is an analytic function on $\rho(A)$ since it satisfies the resolvent identity:

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A), \text{ for all } \lambda, \mu \in \rho(A). \tag{1}$$

We note in what follows $A_\lambda = A - \lambda I$ if $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. If A_λ is not bijective but does possess a left (resp. right) inverse, we can consider the left (resp. right) resolvent. Let $A \in \mathcal{B}(\mathcal{H})$ and Ω be an open set in the complex plane $\mathbb{C}$, the function $R_l : \Omega \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (resp. $R_r : \Omega \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$), is said to be a left (resp. right) resolvent of the operator A on Ω if the following conditions are satisfied:

(1) $R_l(\lambda, A) A_\lambda = I$ (resp. $A_\lambda R_r(\lambda, A) = I$) for all $\lambda \in \Omega$;

(2) $R_l(., A)$ (resp. $R_r(., A)$) satisfies the resolvent identity 1, for all $\lambda, \mu \in \Omega$.

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As an immediate consequence of the definition $R_l(.,A)$ and $R_r(.,A)$ are analytic on the hole Ω as they satisfy the resolvent identity. Left and right resolvents are special cases of the generalized resolvent that has been widely used in many fields such as spectrum theory and Fredholm operators theory, see e.g [4].

An operator $A \in \mathcal{B}(\mathcal{H})$ is bounded below if there exists some $c > 0$ such that $c\|x\| \leq \|Ax\|$, for every $x \in \mathcal{H}$. It is known that A is bounded below if and only if it is injective with closed range. Let us recall that in the Hilbert spaces setting, A is left invertible if and only if A is bounded below and A is right invertible if and only if A is surjective. A is said to be quasi-nilpotent if the spectrum $\sigma(A)$ consists of the set $\{0\}$. The approximate point spectrum $\sigma_{ap}(A)$ and the surjective spectrum $\sigma_{su}(A)$ of A are defined respectively by:

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : A_\lambda \text{ is not bounded below}\}$$

and

$$\sigma_{su}(A) = \{\lambda \in \mathbb{C} : A_\lambda \text{ is not surjective}\}.$$

Lemma 1.1. *([1]) Let $A \in \mathcal{B}(\mathcal{H})$, then A is bounded below (resp. surjective) if and only if A^* is surjective (resp. bounded below).*

Since $(A-\lambda I)^* = A^* - \overline{\lambda} I$, then, using the notation $\overline{\sigma(A)}$ for the set $\left\{\lambda : \overline{\lambda} \in \sigma(A)\right\}$, it follows from Lemma 1.1, that σ_{ap} and σ_{su} are dual to each other in the sens that $\sigma_{ap}(A^*) = \overline{\sigma_{su}(A)}$ and $\sigma_{su}(A^*) = \overline{\sigma_{ap}(A)}$. $\sigma_{ap}(A)$ and $\sigma_{su}(A)$, are non-empty compact subsets of $\mathbb{C}$ and the boundary $\partial\sigma(A) \subseteq \sigma_{su}(A) \cap \sigma_{ap}(A) \subseteq \sigma(A) = \sigma_{su}(A) \cup \sigma_{ap}(A)$.

We say that A is completely reduced by the pair (M,N), denoted as $(M,N) \in Red(A)$ or $A = A_M \oplus A_N$, if M and N are two closed A-invariant subspaces of $\mathcal{H}$ such that $\mathcal{H} = M \oplus N$ (that is $\mathcal{H} = M + N$ and $M \cap N = \{0\}$). In such case, it is easily seen that $\mathcal{N}(A) = \mathcal{N}(A_M) \oplus \mathcal{N}(A_N)$, $\mathcal{R}(A) = \mathcal{R}(A_M) \oplus \mathcal{R}(A_N)$ and $A^n = A_M^n \oplus A_N^n$, for all $n \in \mathbb{N}$. Moreover, A is bounded below (resp. surjective) if and only if A_M and A_N are bounded below (resp. surjective). We have respectively, $\sigma_{ap}(A) = \sigma_{ap}(A_M) \cup \sigma_{ap}(A_N)$ and $\sigma_{su}(A) = \sigma_{su}(A_M) \cup \sigma_{su}(A_N)$.

The quasi-nilpotent part $\mathcal{H}_0(A)$ of an operator $A \in \mathcal{B}(\mathcal{H})$ is defined by $\mathcal{H}_0(A) = \left\{x \in \mathcal{H} : \lim_{n\to\infty} \|A^n x\|^{1/n} = 0\right\}$. The analytic core of A, denoted by $\mathcal{K}(A)$, is the set of all $x \in \mathcal{H}$ for which there exist $\delta_x > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ satisfying $x_0 = x$, $Ax_{n+1} = x_n$ and $\|x_n\| \leq \delta_x^n \|x\|$ for all $n \in \mathbb{N}$. The basic properties of $\mathcal{H}_0(A)$ and $\mathcal{K}(A)$ are given in [1]. In particular, $\mathcal{H}_0(A)$ and $\mathcal{K}(A)$ are (not necessarily closed) subspaces of $\mathcal{H}$, $\mathcal{N}(A) \subset \mathcal{H}_0(A)$; for every $x \in \mathcal{H}$, $x \in \mathcal{H}_0(A)$ if and only if $Ax \in \mathcal{H}_0(A)$; $A(\mathcal{K}(A)) = \mathcal{K}(A)$; if M is a closed subspace of $\mathcal{H}$ and $A(M) = M$ then $M \subset \mathcal{K}(A)$; $\mathcal{H}_0(A) = \mathcal{H}$ if and only if A is quasi-nilpotent. If A is bounded below, then $\mathcal{H}_0(A) = \{0\}$, if A is quasi-nilpotent, then $\mathcal{K}(A) = \{0\}$ and if A is surjective then $\mathcal{K}(A) = \mathcal{H}$.

The concept of generalized Drazin invertibility on a Banach space was introduced by Koliha in [5]. It is defined as follows :

Definition 1.2. *An operator $A \in \mathcal{B}(\mathcal{H})$ is generalized Drazin invertible if there is an element $A^{gD} \in \mathcal{B}(\mathcal{H})$ such that:*

$$AA^{gD} = A^{gD}A,\ A^{gD}AA^{gD} = A^{gD} \text{ and } A\left(I - AA^{gD}\right) \text{ is quasi-nilpotent on } \mathcal{H}$$

A^{gD} is called the generalized Drazin inverse of A.

If A is generalized Drazin invertible then A^{gD} exists and is unique, $P = I - AA^{gD}$ is the spectral projection of A corresponding to $\{0\}$, $A + P$ is invertible and $AP = PA$ is quasi-nilpotent. It is well known that the generalized Drazin inverse of A exists if and only if 0 is not an accumulation point of $\sigma(A)$, or equivalently, if and only if $\mathcal{H}_0(A)$ and $\mathcal{K}(A)$ are both closed and $\mathcal{H} = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$, in such case $A_{\mathcal{K}(A)}$ is invertible and $A_{\mathcal{H}_0(A)}$ is a quasi-nilpotent operator if $\mathcal{H}_0(A) \neq \{0\}$, see [6]. If $A = A_{\mathcal{K}(A)} \oplus A_{\mathcal{H}_0(A)}$ is the described decomposition, then $A^{gD} = \left(A_{\mathcal{K}(A)}\right)^{-1} \oplus 0_{\mathcal{H}_0(A)}$ and there exists a punctured neighborhood Ω of 0 in $\mathbb{C}$ such that:

$$R(\lambda, A) = -\sum_{n=0}^{\infty} \lambda^{-n-1} A^n P + \sum_{n=0}^{\infty} \lambda^n \left(A^{gD}\right)^{n+1},\ \lambda \in \Omega. \tag{2}$$

Remark 1.3. *Since the definition above requires the existence of a quasi-nilpotent operator (i.e an operator having* $\{0\}$ *as spectrum) then we can not have* $\dim \mathcal{H} = 0$*, this justifies the exclusion of the trivial space* $\mathcal{H} = \{0\}$ *at the beginning.*

In addition, if $A \in \mathcal{B}(\mathcal{H})$ *is generalized Drazin invertible, it is not accurate to say that* $A_{\mathcal{H}_0(A)}$ *is quasi-nilpotent since* $\mathcal{H}_0(A)$ *is allowed to be* $\{0\}$*. But we have* $\sigma\left(A_{\mathcal{H}_0(A)}\right) \subset \{0\}$ *instead of* $\sigma\left(A_{\mathcal{H}_0(A)}\right) = \{0\}$*, since* $\sigma\left(A_{\mathcal{H}_0(A)}\right) = \{0\}$ *means that* $0 \in \sigma(A)$*, and in this case, invertible operators can not be generalized Drazin invertible which is completely false according to the definition 1.2.*

We think it is important to mention this imprecision here, as it appears in several papers dealing with left and right generalized Drazin invertible operators.

Now, consider two classes of operators, called left generalized Drazin invertible operators and right generalized Drazin invertible operators, introduced by Miloud, Benharrat and Messirdi in [7] which extends the class of generalized Drazin invertible operators. This two classes are defined as follows :

Definition 1.4. *We say that* $A \in \mathcal{B}(\mathcal{H})$ *is left generalized Drazin invertible if* $\mathcal{H}_0(A)$ *is closed and complemented with an A-invariant closed subspace* $M \subset \mathcal{H}$ *such that* $A(M)$ *is closed.*

The pair $(M, \mathcal{H}_0(A))$ *is called a left generalized Drazin decomposition of A and denoted by* $(M, \mathcal{H}_0(A)) \in lRed(A)$*.*

Definition 1.5. *We say that* $A \in \mathcal{B}(\mathcal{H})$ *is right generalized Drazin invertible if* $\mathcal{K}(A)$ *is closed and complemented with an A-invariant closed subspace* $N \subset \mathcal{H}$ *such that* $N \subseteq \mathcal{H}_0(A)$*.*

The pair $(\mathcal{K}(A), N)$ *is called a right generalized Drazin decomposition of A and denoted by* $(\mathcal{K}(A), N) \in rRed(A)$*.*

The following theorem gives some equivalent characterizations.

Theorem 1.6. *([2], [8]) Let* $A \in \mathcal{B}(\mathcal{H})$*, then the following assertions are equivalent:*

1) A is left (resp. right) generalized Drazin invertible;

2) there is a pair $(M, N) \in Red(A)$ *such that* A_M *is bounded below (resp. surjective) and* $\sigma(A_N) \subset \{0\}$ *;*

3) there exists a projection $P \in \mathcal{B}(\mathcal{H})$ *such that* $AP = PA$ *is quasi-nilpotent and* $A + P$ *is bounded below (resp. surjective).*

4) there exist two operators L (resp. R), $Q \in \mathcal{B}(\mathcal{H})$ *such that Q is quasi-nilpotent,*

$$\begin{aligned} ALA &= LA^2 = A - Q \text{ and } LAL = L^2A = L. \\ (\text{resp. } ARA &= A^2R = A - Q \text{ and } RAR = AR^2 = R). \end{aligned}$$

We say that L (resp. R) is a left (resp. right) generalized Drazin inverse of A (not necessarily unique).

$A^{\{lgD\}}$ (resp. $A^{\{rgD\}}$) denotes the set of all left (resp. right) generalized Drazin inverses of A.

Remark 1.7. *1)* $A \in \mathcal{B}(\mathcal{H})$ *is generalized Drazin invertible if and only if A is both left and right generalized Drazin invertible. In such case we have* $A^{\{lgD\}} = A^{\{rgD\}} = \left\{A^{gD}\right\}$*.*

2) If $(M, \mathcal{H}_0(A)) \in lRed(A)$ *and* $\mathcal{R}(A_M)$ *is dense in M, then A is generalized Drazin invertible.*

3) Assertion (3) in Theorem 1.6 involves $(\mathcal{N}(P), \mathcal{R}(P)) \in lRed(A)$ *(resp.* $\in rRed(A)$*) as A is left (resp. right) generalized Drazin invertible. In particular* $\mathcal{R}(P) = \mathcal{H}_0(A)$ *and* $\mathcal{K}(A) \subset \mathcal{N}(P)$ *(resp.* $\mathcal{R}(P) \subset \mathcal{H}_0(A)$ *and* $\mathcal{K}(A) = \mathcal{N}(P)$*).*

4) The link between assertions (3) and (4) in Theorem 1.6, keeping same notations, is given by :

$$\begin{aligned} P &= I - LA \text{ (resp. } P = I - AR), \\ Q &= AP. \end{aligned}$$

for some $L \in A^{\{lgD\}}$ *(resp.* $R \in A^{\{rgD\}}$*). At the opposite of the generalized Drazin decomposition, a left (resp. right) generalized Drazin decomposition of A if it exists is not necessarily unique.*

5) If $A = A_M \oplus A_N$ is a reduction of a left (resp. right) generalized Drazin invertible operator $A \in \mathcal{B}(\mathcal{H})$ as described in the assertion (2) in Theorem 1.6 and if A_M has a left (resp. right) inverse A^l (resp. A^r) on M, we obtain:

$$\begin{array}{rcl} A^l \oplus 0_N & \in & A^{\{lgD\}} \\ (\textit{resp. } A^r \oplus 0_N & \in & A^{\{rgD\}}). \end{array}$$

6) If A is a left (resp. right) generalized Drazin invertible operator then straightforward arguments give:

$$\begin{array}{rcl} \mathcal{R}(A) + \mathcal{H}_0(A) \textit{ is closed (resp. } \mathcal{R}(A) + \mathcal{H}_0(A) & = & \mathcal{H}), \\ \mathcal{K}(A) + \mathcal{H}_0(A) \textit{ is closed (resp. } \mathcal{K}(A) + \mathcal{H}_0(A) & = & \mathcal{H}). \end{array}$$

Various expressions and applications of the resolvent $R(\lambda, A)$ are known in the literature where A is generalized Drazin invertible operator see e.g. [9] and [3]. In this work we extend results of Djordjevic and Stanimirovic [3], where we investigate the existence of the left (resp. right) resolvents of left (resp. right) generalized Drazin invertible operators in Hilbert spaces by means of the operators $A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} A^*_{\overline{\lambda}}$ (resp. $A^*_{\overline{\lambda}} \left(A_\lambda A^*_{\overline{\lambda}}\right)^{-1} A_\lambda$). Then a representation for the left (resp. right) generalized Drazin inverses and spectral projections, in terms of the coefficients of Laurent series of the left (resp. right) resolvent, is established in this paper.

After giving in section 2 some preliminary results which our investigation will need, we present, in section 3, criteria for the existence of the left and the right generalized Drazin inverses, but also we give explicit expressions of the left and right resolvents and corresponding left and right generalized Drazin inverses. In addition, new characterizations of the left and right generalized Drazin invertibility are stated by establishing that the existence criteria presented are not only sufficient conditions, but also necessary ones. This result will be also investigated in order to formulate a second characterization of the left and right generalized Drazin invertibility. This last characterization is given essentially via the range, the quasi-nilpotent part and the analytic core.

2. Some useful results

The following lemma states the duality between the left and right generalized Drazin invertibility.

Lemma 2.1. *([2]) Let $A \in \mathcal{B}(\mathcal{H})$ then A is left (resp. right) generalized Drazin invertible, if and only if A^* is right (resp. left) generalized Drazin invertible, and if $(M, N) \in lRed(A)$ (resp. $(M, N) \in rRed(A)$) then $(N^\perp, M^\perp) \in rRed(A^*)$ (resp. $(N^\perp, M^\perp) \in lRed(A^*)$), in particular, $\mathcal{K}(A^*) = \mathcal{H}_0(A)^\perp$ (resp. $\mathcal{H}_0(A^*) = \mathcal{K}(A)^\perp$).*

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be left (resp. right) generalized Drazin invertible, then:*

$$\begin{array}{rcl} (\mathcal{R}(A) + \mathcal{H}_0(A))^\perp & = & \mathcal{K}(A^*) \cap \mathcal{N}(A^*) \\ (\textit{resp. } (\mathcal{K}(A) \cap \mathcal{N}(A))^\perp & = & \mathcal{R}(A^*) + \mathcal{H}_0(A^*)). \end{array}$$

Proof. In the left generalized Drazin invertibility case, $\mathcal{H}_0(A)$ and $\mathcal{R}(A) + \mathcal{H}_0(A)$ are closed in $\mathcal{H}$. Since A^* is right generalized Drazin invertible, $\mathcal{K}(A^*)$ is also closed in $\mathcal{H}$. Thus,

$$\begin{array}{rcl} (\mathcal{K}(A^*) \cap \mathcal{N}(A^*))^\perp & = & \overline{\mathcal{K}(A^*)^\perp + \mathcal{N}(A^*)^\perp} = \overline{\mathcal{K}(A^*)^\perp + \overline{\mathcal{R}(A)}} \\ & = & \overline{\mathcal{H}_0(A) + \mathcal{R}(A)} = \mathcal{R}(A) + \mathcal{H}_0(A). \end{array}$$

The second equation is simply the dual of the first one. □

The following properties of left and right generalized Drazin invertible operators, established by Miloud et al. in [7], will be required in the next section. We include a proof for the sake of illustrations.

Proposition 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$. If A is left (resp. right) generalized Drazin invertible, then 0 is not an accumulation point of $\sigma_{ap}(A)$ (resp. $\sigma_{su}(A)$)*

Proof. Suppose that A is left generalized Drazin invertible. If $(M,N) \in lRed(A)$, then, since $\sigma_{ap}(A) = \sigma_{ap}(A_M) \cup \sigma_{ap}(A_N)$, we have $0 \notin \sigma_{ap}(A_M)$ and $\sigma_{ap}(A_N) \subset \{0\}$, then 0 is at most an isolated point of $\sigma_{ap}(A)$ which also means that 0 is not an accumulation point of $\sigma_{ap}(A)$.

We may use duality to deduce the result for right generalized Drazin invertible operators. □

3. Left and right resolvents of left and right generalized Drazin invertible operators

Here, our first main objective is to investigate the left and the right generalized Drazin inverses via the left and the right resolvents in the algebra $\mathcal{B}(\mathcal{H})$. The idea, to describe the left and the right resolvents, is to use Proposition 2.3 and some classical results on operators of type A^*A and AA^* where $A \in \mathcal{B}(\mathcal{H})$. The operator A^*A is an important technical tool in operator theory, we will show its role in the characterization of left and right generalized invertibility. The operator A^*A is self-adjoint, $\mathcal{N}(A^*A) = \mathcal{N}(A)$ and if $\mathcal{R}(A)$ is closed in $\mathcal{H}$, we have $\mathcal{R}(A^*A) = \mathcal{R}(A^*)$.

Lemma 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be bounded below (resp. surjective). Then,*
1) A^*A (resp. AA^*) is invertible in $\mathcal{B}(\mathcal{H})$.
2) $A_l = (A^*A)^{-1} A^*$ (resp. $A_r = A^* (AA^*)^{-1}$) is a left (resp. right) inverse of A.
3) $AA_l = P_{\mathcal{R}(A)}$ (resp. $A_rA = P_{\mathcal{R}(A^*)}$) is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(A)$ (resp. $\mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$).

Proof. According to Lemma 1.1, we show the results for A bounded below, the dual case is obtained by replacing A by its adjoint.

1) If A is bounded below, since $\mathcal{H} = \mathcal{N}(A^*) \oplus \mathcal{R}(A)$, then $A^*_{\mathcal{R}(A)}$ is injective and so A^*A is injective too, on the other hand:

$$\mathcal{R}(A^*) = A^*(\mathcal{R}(A)) = \mathcal{R}(A^*A) = \mathcal{H}.$$

So, A^*A is surjective, and then bijective and by the closed graph theorem $(A^*A)^{-1}$ is bounded.

Assertions (2) and (3) are easy to verify since $(AA_l)^2 = AA_l$, $(AA_l)^* = AA_l$, $\mathcal{R}(AA_l) = \mathcal{R}(A)$ and if A is replaced by A^* we obtain that A_rA is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(A^*)$. □

To simplify our notations, we denote for $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \rho_{ap}(A) = \mathbb{C}\backslash\sigma_{ap}(A)$ (resp. $\lambda \in \rho_{su}(A) = \mathbb{C}\backslash\sigma_{su}(A)$), $A_{\lambda,l} = \left(A^*_{\overline{\lambda}}A_\lambda\right)^{-1} A^*_{\overline{\lambda}}$ and $P_{A,\lambda,l} = A_\lambda A_{\lambda,l}$ (resp. $A_{\lambda,r} = A^*_{\overline{\lambda}}\left(A_\lambda A^*_{\overline{\lambda}}\right)^{-1}$ and $P_{A,\lambda,r} = A_{\lambda,r}A_\lambda$). $P_{M,N}$ denotes a bounded projection according to the decomposition $\mathcal{H} = M \oplus N$ such that $\mathcal{R}(P_{M,N}) = M$ and $\mathcal{N}(P_{M,N}) = N$, when $N = M^\perp$, $P_{M,N}$ is simply denoted P_M.

$A_{\lambda,l}$ (resp. $A_{\lambda,r}$) is a left (resp. right) inverse of A_λ, and $P_{A,\lambda,l}$ (resp. $P_{A,\lambda,r}$) is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(A_\lambda)$ (resp. $\mathcal{R}(A^*_{\overline{\lambda}})$).

Lemma 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$, $\Omega \subset \rho_{ap}(A)$ be open and $L : \Omega \to \mathcal{B}(\mathcal{H})$ be a function such that $L(\lambda) A_\lambda = I$, then L satisfies the resolvent identity on Ω if and only if $\mathcal{N}(L(\lambda))$ is contant on Ω.*

Proof. The necessity of the condition is obvious. For the sufficiency, suppose that there is a closed subspace $N \in \mathcal{H}$ such that $\mathcal{N}(L(\lambda)) = N$ for every $\lambda \in \Omega$. Then, since $L(\lambda) A_\lambda = I$ we have $N \cap \mathcal{R}(A_\lambda) = \{0\}$, on the other hand:

$$x = (x - ALx) + (ALx)$$

which means that $N + \mathcal{R}(A_\lambda) = \mathcal{H}$, thus $N \oplus \mathcal{R}(A_\lambda) = \mathcal{H}$. Now, for every $x \in \mathcal{H}$ and $\lambda, \mu \in \Omega$ we have $L(\lambda) L(\mu) A_\mu A_\lambda = I$, then $L(\lambda) L(\mu)$ is a left inverse of $A_\mu A_\lambda$. Furthermore, $\mathcal{N}(L(\lambda) L(\mu)) = L(\mu)^{-1}(N) =$

$N \oplus A_\mu(N)$, then:

$$\begin{aligned} L(\mu) L(\lambda) \left(N \oplus A_\mu(N)\right) &= L(\mu) L(\lambda) A_\mu(N) \\ &= L(\mu) L(\lambda) (A_\lambda - (\mu - \lambda) I)(N) \\ &= L(\mu) (I - (\mu - \lambda) L(\lambda))(N) \\ &= L(\mu)(N) = \{0\}. \end{aligned}$$

Thus, $L(\lambda) L(\mu)$ and $L(\mu) L(\lambda)$ have same kernels, and since $A_\mu A_\lambda = A_\lambda A_\mu$ we have $L(\lambda) L(\mu) = L(\mu) L(\lambda)$, which means that $L(\mu)$ an $L(\lambda)$ commutes. Now we have:

$$\begin{aligned} (\lambda - \mu) L(\lambda) L(\mu) &= L(\lambda) L(\mu) \left(A_\mu - A_\lambda\right) \\ &= L(\lambda)(I - L(\mu) A_\lambda) \\ &= L(\lambda) - L(\mu) L(\lambda) A_\lambda \\ &= L(\lambda) - L(\mu). \end{aligned}$$

Which gives the result. □

Corollary 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$, $\Omega \subset \rho_{su}(A)$ be open and $R : \Omega \to \mathcal{B}(\mathcal{H})$ be a function such that $A_\lambda R(\lambda) = I$, then R satisfies the resolvent identity on Ω if and only if $\mathcal{R}(R(\lambda))$ is constant on Ω.*

Proof. Indeed, $\mathcal{R}(R(\lambda))$ constant $\Leftrightarrow \mathcal{N}(R(\lambda)^*)$ constant. □

Proposition 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible operator. Then,*

$$\lim_{\lambda \to 0} P_{A,\lambda,l} = P_{\mathcal{R}(A)+\mathcal{H}_0(A)}.$$

Proof. From Lemma 2.2, $(\mathcal{R}(A) + \mathcal{H}_0(A))^\perp = \mathcal{K}(A^*) \cap \mathcal{N}(A^*)$ and it follows that $\mathcal{N}\left(P_{\mathcal{R}(A)+\mathcal{H}_0(A)}\right) = \mathcal{K}(A^*) \cap \mathcal{N}(A^*)$. Since A is left generalized Drazin invertible, we deduce from Proposition 2.3 and Lemma 3.1, that there exists $r > 0$ such that $P_{A,\lambda,l}$ is an orthogonal projection with $\mathcal{R}(P_{A,\lambda,l}) = \mathcal{R}(A_\lambda)$, $\mathcal{N}(P_{A,\lambda,l}) = \mathcal{R}(A_\lambda)^\perp$ and $\mathcal{H} = \mathcal{R}(A_\lambda) \oplus \mathcal{R}(A_\lambda)^\perp = \mathcal{R}(A_\lambda) \oplus \mathcal{N}(A^*_{\overline{\lambda}})$, $0 < |\lambda| < r$. Since $\lambda \neq 0$ and $\sigma\left(A_{\mathcal{H}_0(A)}\right) \subset \{0\}$, then $A_\lambda(\mathcal{H}_0(A)) = \mathcal{H}_0(A)$ that is $\mathcal{H}_0(A) \subset \mathcal{R}(A_\lambda)$. On the other hand, we have for all $x \in \mathcal{H}$ and $0 < |\lambda| < r$:

$$P_{A,\lambda,l} A x = P_{A,\lambda,l} A_\lambda x + \lambda P_{A,\lambda,l} x = A_\lambda x + \lambda P_{A,\lambda,l} x.$$

Then, since $\left\|P_{A,\lambda,l}\right\| = 1$ we have $\lim\limits_{\lambda \to 0} \lambda P_{A,\lambda,l} x = 0$ and hence $\lim\limits_{\lambda \to 0} P_{A,\lambda,l} A x = Ax$, in $\mathcal{H}$.

Let $x \in \mathcal{K}(A^*) \cap \mathcal{N}(A^*)$, then $A^* x = 0$ and there exist $\delta_x > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$ satisfying $x_0 = x$, $A^* x_{n+1} = x_n$ and $\|x_n\| \leq \delta_x^n \|x\|$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} P_{A,\lambda,l} x &= A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} A^* x - \overline{\lambda} A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} x = -\overline{\lambda} A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} x \\ &= -\overline{\lambda} A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} A^* x_1 = -\left(\overline{\lambda} P_{A,\lambda,l} x_1 + \overline{\lambda}^2 A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} x_1\right) \\ &= -\left(\overline{\lambda} P_{A,\lambda,l} x_1 + \overline{\lambda}^2 P_{A,\lambda,l} x_2 + \overline{\lambda}^3 A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} x_2\right) \\ &= \ldots = -\left(P_{A,\lambda,l}\left[\overline{\lambda} x_1 + \overline{\lambda}^2 x_2 + \ldots + \overline{\lambda}^n x_n\right] + \overline{\lambda}^{n+1} A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1} x_n\right). \end{aligned}$$

Then we have:

$$\left\|P_{A,\lambda,l} x\right\| \leq \left(\delta_x^n |\lambda|^{n+1} \left\|A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda\right)^{-1}\right\| + \sum_{1 \leq j \leq n} |\delta_x \lambda|^j\right) \|x\|.$$

So if $r < \frac{1}{\delta_x}$, then $\lim_{n\to\infty} \delta_x^n |\lambda|^{n+1} \left\| A_\lambda \left(A^*_{\overline{\lambda}} A_\lambda \right)^{-1} \right\| = 0$, and taking the limits in last inequality we have:

$$\begin{aligned} \left\| P_{A,\lambda,l} x \right\| &\leq \|x\| \sum_{j\geq 1} |\delta_x \lambda|^j \\ &\leq \frac{\delta_x |\lambda| \|x\|}{1-\delta_x |\lambda|}. \end{aligned}$$

Then, $\lim_{\lambda\to 0} \left\| P_{A,\lambda,l} x \right\| = 0$. Finally, $\lim_{\lambda\to 0} P_{A,\lambda,l} x = P_{\mathcal{R}(A)+\mathcal{H}_0(A)} x$, for all $x \in \mathcal{H}$. At this level we have proved that $P_0 = P_{\mathcal{R}(A)+\mathcal{H}_0(A)}$ is the point-wise limit of $P_{A,\lambda,l}$ as $\lambda \to 0$. Now, we will show that it is in fact its uniform limit. Indeed, since $\mathcal{H} = (\mathcal{K}(A^*) \cap \mathcal{N}(A^*)) \oplus A(M) \oplus \mathcal{H}_0(A)$ where $(M, \mathcal{H}_0(A)) \in lRed(A)$, we can define the three bounded projections P_1, P_2 and P_3 onto $\mathcal{K}(A^*) \cap \mathcal{N}(A^*)$, $A(M)$ and $\mathcal{H}_0(A)$ respectively, such that $\mathcal{N}(P_i) = \mathcal{R}(P_j) \oplus \mathcal{R}(P_k)$, $i \neq j, i \neq k, j \neq k$ and $i, j, k \in \{1,2,3\}$. As A_M is left invertible on M, let L_M be a left inverse of A_M on M, then:

$$\begin{aligned} \left\| P_{A,\lambda,l} x - P_0 x \right\| &= \left\| (P_{A,\lambda,l} - P_0)(P_1 x + P_2 x + P_3 x) \right\| \\ &\leq \frac{\delta_{P_1 x} |\lambda|}{1-\delta_{P_1 x} |\lambda|} \|P_1 x\| + \left\| \lambda P_{A,\lambda,l} L_M P_2 x \right\| + \|P_3 x - P_3 x\| \\ &\leq \left(\frac{\delta_{P_1 x} |\lambda|}{1-\delta_{P_1 x} |\lambda|} \|P_1\| + |\lambda| \|L_M\| \|P_2\| \right) \|x\|. \end{aligned}$$

Moreover, since $\mathcal{K}(A^*)$ is closed and $A^*(\mathcal{K}(A^*)) = \mathcal{K}(A^*)$ then by the open mapping theorem $A^*_{\mathcal{K}(A^*)}$ is open, that is there exists $c > 0$ such that $B^1 \subset A^*\left(cB^1\right)$, where B^1 is the unit closed ball, then for every $x \in \mathcal{K}(A^*)$ there is some $x_1 \in \mathcal{K}(A^*)$ such that $A^* x_1 = x$ and $\|x_1\| \leq c\|x\|$. So, by induction, there exists for all $n \geq 1$, some $x_{n+1} \in \mathcal{K}(A^*)$ such that $A^* x_{n+1} = x_n$ and $\|x_{n+1}\| \leq c\|x_n\|$, which involves that $\|x_{n+1}\| \leq c^{n+1}\|x\|$. Then, if $x_0 = x$, and $C = \max\{1, c\}$, we have $\|x_n\| \leq C^n \|x\|$. The argument used above shows that if $r < \frac{1}{C}$ we have:

$$\sup_{x\in\mathcal{H}\setminus\{0\}} \frac{\left\| P_{A,\lambda,l} x - P_0 x \right\|}{\|x\|} \leq \frac{C|\lambda|}{1-C|\lambda|} \|P_1\| + |\lambda| \|L_M\| \|P_2\|.$$

This means that $\lim_{\lambda\to 0} P_{A,\lambda,l} = P_0 = P_{\mathcal{R}(A)+\mathcal{H}_0(A)}$ uniformly. □

Proposition 3.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a left generalized Drazin invertible operator. Then there exists a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that:*

$$P_{A,\lambda,l} (I - P_0 (I - P_{A,\lambda,l}))^{-1} = P_{\mathcal{R}(A_\lambda),\mathcal{K}(A^*)\cap\mathcal{N}(A^*)}, \text{ for all } \lambda \in \mathcal{V}.$$

Proof. As $\lim_{\lambda\to 0} P_0 (I - P_{A,\lambda,l}) = 0$ in $\mathcal{B}(\mathcal{H})$, then there exists a small enough punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that $\left\| P_0 (I - P_{A,\lambda,l}) \right\| < 1$, for all $\lambda \in \mathcal{V}$. Thus, $I - P_0 (I - P_{A,\lambda,l})$ is boundedly invertible and the inverse is given by the Neumann series:

$$(I - P_0 (I - P_{A,\lambda,l}))^{-1} = \sum_{n=0}^{\infty} (P_0 (I - P_{A,\lambda,l}))^n, \ \lambda \in \mathcal{V}.$$

Since $(I - P_{A,\lambda,l})^n P_{A,\lambda,l} = 0$, for all $n \geq 1$ and $\lambda \in \mathcal{V}$, we get:

$$(I - P_0 (I - P_{A,\lambda,l}))^{-1} P_{A,\lambda,l} = \sum_{n=0}^{\infty} (P_0 (I - P_{A,\lambda,l}))^n P_{A,\lambda,l} = P_{A,\lambda,l}$$

and

$$\left[P_{A,\lambda,l} (I - P_0 (I - P_{A,\lambda,l}))^{-1} \right]^2 = P_{A,\lambda,l} (I - P_0 (I - P_{A,\lambda,l}))^{-1}.$$

Furthermore,

$$P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}P_{A,\lambda,l}=P_{A,\lambda,l},$$

And since $I-P_0\left(I-P_{A,\lambda,l}\right)$ is invertible then:

$$\begin{aligned}\mathcal{R}\left(P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\right) &= \mathcal{R}\left(P_{A,\lambda,l}\right)\\ &= \mathcal{R}(A_\lambda),\text{ for all }\lambda\in\mathcal{V}.\end{aligned}\tag{3}$$

On the other hand, since $(I-P_0)\,P_0\left(I-P_{A,\lambda,l}\right)=0$ for all $n\geq 1$, we have:

$$\begin{aligned}& P_0P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\\ =\ & \left[\left(P_0P_{A,\lambda,l}+I-P_0\right)-I+P_0\right]\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\\ =\ & I-(I-P_0)\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\\ =\ & I-\sum_{n=0}^{\infty}(I-P_0)\left(P_0\left(I-P_{A,\lambda,l}\right)\right)^n\\ =\ & I-(I-P_0)=P_0,\ \lambda\in\mathcal{V}.\end{aligned}$$

So, $\mathcal{N}\left(P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\right)\subset\mathcal{N}\left(P_0\right).$

In addition, let $x\in\mathcal{N}\left(P_0\right)\cap\mathcal{R}(P_{A,\lambda,l})$,

$$\|x\|=\left\|(I-P_0)\,P_{A,\lambda,l}x\right\|\leq\left\|(I-P_0)\,P_{A,\lambda,l}\right\|\|x\|,\ \lambda\in\mathcal{V}.$$

If $x\neq 0$ we will have $\left\|((I-P_0))\,P_{A,\lambda,l}\right\|\geq 1$, which is absurd since $\lim\limits_{\lambda\to 0}\left\|((I-P_0))\,P_{A,\lambda,l}\right\|=0$. Therefore, $\mathcal{N}\left(P_0\right)\cap\mathcal{R}(P_{A,\lambda,l})=\{0\}$ and $\mathcal{N}\left(P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}\right)=\mathcal{N}\left(P_0\right)$, for all $\lambda\in\mathcal{V}$. Consequently, $\mathcal{N}\left(P_0\right)\oplus\mathcal{R}(P_{A,\lambda,l})=\mathcal{H}$, where $\mathcal{R}(P_{A,\lambda,l})=\mathcal{R}(A_\lambda)$ and $\mathcal{N}\left(P_0\right)=\mathcal{K}(A^*)\cap\mathcal{N}(A^*)$, for all $\lambda\in\mathcal{V}$. □

Let us now consider the operator $\mathcal{P}_{A,\lambda,l}=P_{A,\lambda,l}\left(I-P_0\left(I-P_{A,\lambda,l}\right)\right)^{-1}$. We obtain, by virtue of Proposition 3.5, the following corollary.

Corollary 3.6. *Let $A\in\mathcal{B}(\mathcal{H})$ be a left generalized Drazin invertible operator. Then there exists a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that for all $\lambda,\mu\in\mathcal{V}$:*

$$\mathcal{P}_{A,\lambda,l}\mathcal{P}_{A,\mu,l}=\mathcal{P}_{A,\lambda,l}.$$

Proof. We know, from Proposition 3.5, that $\mathcal{N}\left(\mathcal{P}_{A,\lambda,l}\right)=\mathcal{N}\left(P_0\right)$, so for all $\lambda,\mu\in\mathcal{V}$:

$$\mathcal{P}_{A,\lambda,l}\left(I-\mathcal{P}_{A,\mu,l}\right)=0$$

This completes the proof. □

Theorem 3.7. *Let $A\in\mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible. Then there exists a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that the operator $R_{lD}(\lambda,A)=A_{\lambda,l}\mathcal{P}_{A,\lambda,l}$ satisfies the following conditions for all $\lambda,\mu\in\mathcal{V}$:*

1) $R_{lD}(\lambda,A)\,A_\lambda=I$;

2) $R_{lD}(\lambda,A)-R_{lD}(\mu,A)=(\lambda-\mu)\,R_{lD}(\mu,A)\,R_{lD}(\lambda,A)$;

That is $R_{lD}(\lambda,A)$ is a left resolvent for A on $\mathcal{V}$.

Proof. By virtue of formula 3, there exists a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that $\mathcal{P}_{A,\lambda,l}A_\lambda=A_\lambda$ for all $\lambda\in\mathcal{V}$. Thus,

1)

$$\begin{aligned}R_{lD}(\lambda,A)\,A_\lambda &= A_{\lambda,l}\mathcal{P}_{A,\lambda,l}A_\lambda\\ &= A_{\lambda,l}A_\lambda=I,\ \lambda\in\mathcal{V}.\end{aligned}$$

We also obtain:

$$\begin{aligned} A_\lambda R_{lD}(\lambda, A) &= A_\lambda A_{\lambda,l}\mathcal{P}_{A,\lambda,l} \\ &= P_{A,\lambda,l}\mathcal{P}_{A,\lambda,l} = \mathcal{P}_{A,\lambda,l},\ \lambda \in \mathcal{V}. \end{aligned} \tag{4}$$

2) We have since $\mathcal{P}_{A,\lambda,l}$ is a projection $R_{lD}(\lambda, A)\mathcal{P}_{A,\lambda,l} = R_{lD}(\lambda, A)$, which means that $\mathcal{N}(P_0) \subset \mathcal{N}(R_{lD}(\lambda, A))$. On the other hand, from formula 4 we deduce that $\mathcal{N}(R_{lD}(\lambda, A)) \subset \mathcal{N}(P_0)$. So, $\mathcal{N}(R_{lD}(\lambda, A)) = \mathcal{N}(P_0)$ and now we can apply Lemma 3.2 to deduce that $R_{lD}(., A)$ satisfies the resolvent identity on a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$. □

From the resolvent identity, we deduce immediately the continuity of the operator-valued function $R_{lD}(\lambda, A)$ on $\mathcal{V}$, so $\lim\limits_{\mu \underset{\mu\in\mathcal{V}\setminus\{\lambda\}}{\longrightarrow} \lambda} R_{lD}(\mu, A) = R_{lD}(\lambda, A)$. Thus, for all $\lambda \in \mathcal{V}$,

$$\begin{aligned} \frac{d}{d\lambda} R_{lD}(\lambda, A) &= \lim_{\mu \underset{\mu\in\mathcal{V}\setminus\{\lambda\}}{\longrightarrow} \lambda} \frac{R_{lD}(\mu, A) - R_{lD}(\lambda, A)}{\mu - \lambda} = \lim_{\mu \underset{\mu\in\mathcal{V}\setminus\{\lambda\}}{\longrightarrow} \lambda} R_{lD}(\mu, A) R_{lD}(\lambda, A) \\ &= R_{lD}(\lambda, A) \lim_{\mu \underset{\mu\in\mathcal{V}\setminus\{\lambda\}}{\longrightarrow} \lambda} R_{lD}(\mu, A) = (R_{lD}(\lambda, A))^2, \end{aligned}$$

Theorem 3.8. *Let $A \in \mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible, then the residue $Res\left(\lambda^{-1}R_{lD}(\lambda, A), 0\right)$ of $\lambda \mapsto \lambda^{-1}R_{lD}(\lambda, A)$ at 0 is a left generalized Drazin inverse of A, i.e. $Res\left(\lambda^{-1}R_{lD}(\lambda, A), 0\right) \in A^{\{lgD\}}$.*

Proof. As $R_{lD}(\lambda, A)$ is analytic, then $R_{lD}(\lambda, A)$ admits in $\mathcal{V}$ a Laurent series expansion around 0, given by:

$$R_{lD}(\lambda, A) = \sum_{n=-\infty}^{\infty} \lambda^n A_n \tag{5}$$

where A_n are bounded linear operators on $\mathcal{H}$ and and the series $\sum\limits_{n=-\infty}^{\infty} \lambda^n A_n$ converges by the operator norm in $\mathcal{H}$. Using resolvent identity, we see that for $\lambda, \mu \in \mathcal{V}$ and positively-oriented small circles $\Gamma, \Gamma' \subset \mathcal{V}$ enclosing 0, we have, supposing Γ' is of small radius than Γ :

$$\begin{aligned} A_n A_p &= \left(\frac{1}{2\pi i}\int_\Gamma \frac{R_{lD}(\lambda, A)}{\lambda^{n+1}} d\lambda\right)\left(\frac{1}{2\pi i}\int_{\Gamma'} \frac{R_{lD}(\mu, A)}{\mu^{p+1}} d\mu\right) \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{R_{lD}(\lambda, A) R_{lD}(\mu, A)}{\lambda^{n+1}\mu^{p+1}} d\mu d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{R_{lD}(\lambda, A) - R_{lD}(\mu, A)}{\lambda^{n+1}\mu^{p+1}(\lambda - \mu)} d\mu d\lambda \end{aligned}$$

since $\lambda \neq \mu$. And then:

$$A_n A_p = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{R_{lD}(\lambda, A) - R_{lD}(\mu, A)}{\lambda^{n+2}\mu^{p+1}\left(1 - \frac{\mu}{\lambda}\right)} d\mu d\lambda. \tag{6}$$

Define $\tau_n = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$. Then we have:

$$\begin{aligned} \int_{\Gamma}\int_{\Gamma'} \frac{R_{lD}(\lambda, A)}{\lambda^{n+2}\mu^{p+1}\left(1-\frac{\mu}{\lambda}\right)} d\mu d\lambda &= \int_{\Gamma} \frac{R_{lD}(\lambda, A)}{\lambda^{n+2}} \int_{\Gamma'} \frac{\mu^{-p-1}}{1-\frac{\mu}{\lambda}} d\mu d\lambda \\ &= \int_{\Gamma} \frac{R_{lD}(\lambda, A)}{\lambda^{n+2}} \left(\frac{2\pi i \tau_p}{\lambda^p}\right) d\lambda \\ &= 2\pi i \tau_p \int_{\Gamma} \frac{R_{lD}(\lambda, A)}{\lambda^{n+p+2}} d\lambda \\ &= (2\pi i)^2 \tau_p A_{n+p+1}. \end{aligned} \tag{7}$$

On the other hand:

$$\begin{aligned} \int_{\Gamma}\int_{\Gamma'} \frac{R_{lD}(\mu, A)}{\lambda^{n+2}\mu^{p+1}\left(1-\frac{\mu}{\lambda}\right)} d\mu d\lambda &= \int_{\Gamma'} \frac{R(\mu, A)}{\mu^{p+1}} \int_{\Gamma} \frac{\lambda^{-n-2}}{\left(1-\frac{\mu}{\lambda}\right)} d\lambda d\mu \\ &= \int_{\Gamma'} \frac{R(\mu, A)}{\mu^{p+1}} \left(\frac{2\pi i \tau_{-n-1}}{\mu^{n+1}}\right) d\mu \\ &= 2\pi i \tau_{-n-1} \int_{\Gamma'} \frac{R(\mu, A)}{\mu^{n+p+2}} d\mu \\ &= (2\pi i)^2 \tau_{-n-1} A_{n+p+1}. \end{aligned} \tag{8}$$

Substituting 7 and 8 in 6 we get:

$$\begin{aligned} A_n A_p &= \left(\tau_p - \tau_{-n-1}\right) A_{n+p+1} \\ &= \left(\phi_n + \phi_p\right) A_{n+p+1} \end{aligned} \tag{9}$$

with $\phi_n = \begin{cases} \frac{1}{2} & \text{if } n \geq 0 \\ -\frac{1}{2} & \text{elsewhere} \end{cases}$. Hence, for $n = p = -1$ we obtain:

$$A_{-1}A_{-1} = -A_{-1},$$

thus $-A_{-1}$ is a projection and for all $n \in \mathbb{N}$, $n \geq 1$,

$$(A_{-1})^n = (-1)^{n-1} A_{-1}.$$

Furthermore, since $R_{lD}(\lambda, A) A_\lambda = I$ and $A_\lambda R_{lD}(\lambda, A) = \mathcal{P}_{A,\lambda,l}$ are analytic on $\mathcal{V}$, we get:

$$\mathcal{P}_{A,\lambda,l} = \sum_{n=-\infty}^{\infty} \lambda^n A_\lambda A_n = \sum_{n=-\infty}^{\infty} \lambda^n (AA_n - A_{n-1}), \tag{10}$$

$$I = \sum_{n=-\infty}^{\infty} \lambda^n A_n A_\lambda = \sum_{n=-\infty}^{\infty} \lambda^n (A_n A - A_{n-1}). \tag{11}$$

The singular parts of the two previous series are necessarily zero:

$$\sum_{n=1}^{\infty} \lambda^{-n} (AA_{-n} - A_{-n-1}) = 0, \ \sum_{n=1}^{\infty} \lambda^{-n} (A_{-n}A - A_{-n-1}) = 0.$$

So,

$$I = A_0 A - A_{-1} \text{ and } AA_{-n} = A_{-n}A = A_{-n-1} \text{ for all } n \geq 1.$$

We show by induction that, for all $n \in \mathbb{N}$,

$$A_{-n-1} = A^n A_{-1}. \tag{12}$$

Since $A_0A_{-1} = A_{-1}A_0 = 0$ and $I = A_0A - A_{-1}$, we can write:

$$A_0 = A_0(A_0A - A_{-1}) = A_0A_0A = (A_0A - A_{-1})A_0 = A_0AA_0.$$

Thus,

$$\begin{aligned} A_0 &= A_0A_0A = A_0AA_0, \\ AA_0A &= A + AA_{-1},\ A_0AA = A + A_{-1}A, \end{aligned}$$

and

$$A_0AA - A = A_{-1}A = AA_{-1} = AA_0A - A.$$

Consequently, it follows from Theorem 1.6 that A_0 is a left generalized Drazin inverse of A if AA_{-1} is quasi-nilpotent. Indeed,

$$(AA_{-1})^n = A^n (A_{-1})^n = (-1)^{n-1} A^n A_{-1} = (-1)^{n-1} A_{-n-1},\ n \geq 1$$

moreover the convergence of $\sum\limits_{n=1}^{\infty} \lambda^{-n} A_{-n}$ on $\mathcal{V}$ implies that $\|A_{-n}\| < |\lambda|^n$ as $n \to \infty$, so:

$$\lim_{n\to\infty} \left\|(AA_{-1})^n\right\|^{1/n} = \lim_{n\to\infty} \|A_{-n-1}\|^{1/n} \leq \lim_{n\to\infty} |\lambda|^{1+1/n},$$

as $\mathcal{V}$ is a small enough punctured neighborhood of 0, we can assume that it is included in the open unit disc of center 0, which gives finally $\lim\limits_{n\to\infty} |\lambda|^{1+1/n} = |\lambda|$ and then

$$\lim_{n\to\infty} \left\|(AA_{-1})^n\right\|^{\frac{1}{n}} \leq \min_{\lambda\in\mathcal{V}} |\lambda| = 0.$$

It is clear that A_{-1} is the residue of $R_{lD}(\lambda, A)$ and A_0 is that of $\lambda^{-1}R_{lD}(\lambda, A)$ at 0, i.e. $A_{-1} = Res\,(R_{lD}(\lambda, A), 0)$ and $A_0 = Res\left(\lambda^{-1}R_{lD}(\lambda, A), 0\right)$. □

Remark 3.9. *By formula 9 we get:*

$$A_{n+1} = A_nA_0 \textit{ for all } n \geq 0$$

Then we conclude by induction that:

$$A_n = A_0^{n+1} \textit{ for all } n \geq 0,$$

and substituting 12 into 5, we obtain:

$$R_{lD}(\lambda, A) = \sum_{n\geq 0} \lambda^n A_0^{n+1} + \sum_{n\geq 1} \lambda^{-n} A^{n-1} A_{-1}.$$

So, we derive an expression for a left resolvent analogous to formula 2.

Now let $(M, N) \in lRed(A)$, $P = P_{N,M}$ and L be a left generalized Drazin inverse of A such that $P = I - LA$. We have $L_M(A_M - \lambda I_M) = I_M - \lambda L_M$, then for sufficiently small $|\lambda|$, $L_M(A_M - \lambda I_M)$ is invertible and then we can construct an analytic function $S(\lambda, A)$ on a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that:

$$\begin{aligned} S(\lambda, A) &= (L_M(A_M - \lambda I_M))^{-1} L_M \oplus (A_N - \lambda I_N)^{-1} \\ &= \left((L_M(A_M - \lambda I_M))^{-1} \oplus (A_N - \lambda I_N)^{-1}\right)(L_M \oplus I_N) \\ &= (L(A - \lambda I)(I - P) + (A - \lambda I)P)^{-1}(L + P) \\ &= (L(I - P)A_\lambda + PA_\lambda)^{-1}(L + P) \\ &= (LA_\lambda + PA_\lambda)^{-1}(L + P) \\ &= ((L + P)A_\lambda)^{-1}(L + P). \end{aligned}$$

The relation $S(\lambda,A)A_\lambda = I$ is now obvious, and

$$\begin{aligned} S(\lambda,A) &= \left(\sum_{n\geq 0}\lambda^n L_M^{n+1}\right)\oplus\left(-\sum_{n\leq -1}\lambda^n A_N^{-n-1}\right) \\ &= \sum_{n\geq 0}\lambda^n L^{n+1}(I-P)+\sum_{n\leq -1}\left(-\lambda^n A^{-n-1}P\right) \\ &= \sum_{n\geq 0}\lambda^n L^{n+1}-\sum_{n\leq -1}\lambda^n A^{-n-1}P. \end{aligned}$$

Since $\mathcal{N}(S(\lambda,A)) = \mathcal{N}(L+P)$ for all $\lambda\in\mathcal{V}$, then by Lemma 3.2 $S(.,A)$ satisfies the resolvent identity on $\mathcal{V}$. On the other hand,

$$\begin{aligned} \lim_{\substack{\lambda\to 0\\ \lambda\in\mathcal{V}}} A_\lambda S(\lambda,A) &= \lim_{\substack{\lambda\to 0\\ \lambda\in\mathcal{V}}}\left[(A_M-\lambda I_M)(I_M-\lambda L_M)^{-1}L_M\oplus(A_N-\lambda I_N)(A_N-\lambda I_N)^{-1}\right] \\ &= A_M L_M\oplus I_N \end{aligned}$$

Now, since $(\mathcal{N}(A_{-1}),\mathcal{R}(A_{-1}))\in lRed(A)$ and if we take $L = A_0$ and $P = -A_{-1}$ we find:

$$R_{lD}(\lambda,A) = ((A_0-A_{-1})A_\lambda)^{-1}(A_0-A_{-1})$$

So, we have proved the following main result.

Theorem 3.10. *Let $A\in\mathcal{B}(\mathcal{H})$ be a left generalized Drazin invertible operator, then :*

$$R_{lD}(\lambda,A) = ((A_0-A_{-1})A_\lambda)^{-1}(A_0-A_{-1})$$

on a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$.

We now give a link between the residue of $R_{lD}(\lambda,A)$ and the spectral subspaces $\mathcal{H}_0(A)$ and $\mathcal{K}(A)$.

Theorem 3.11. *Let $A\in\mathcal{B}(\mathcal{H})$ be left generalized Drazin invertible, then:*

$$\mathcal{H}_0(A) = \mathcal{R}(Res(R_{lD}(\lambda,A),0)) \text{ and } \mathcal{K}(A)\subset\mathcal{N}(Res(R_{lD}(\lambda,A),0)).$$

Proof. The statement is an immediate consequence of $(\mathcal{N}(A_{-1}),\mathcal{R}(A_{-1}))\in lRed(A)$. □

We state similar results for right generalized Drazin invertible operators. Let $R_{rD}(\lambda,A) = \mathcal{P}_{A,\lambda,r}A_{\lambda,r}$, where:

$$\begin{aligned} &A_{\lambda,r} = A^*_{\overline{\lambda}}\left(A_\lambda A^*_{\overline{\lambda}}\right)^{-1},\ P_{A,\lambda,r} = A_{\lambda,r}A_\lambda,\ Q_0 = \lim_{\lambda\to 0}P_{A,\lambda,r}, \\ &\mathcal{P}_{A,\lambda,r} = (I-(I-P_{A,\lambda,r})Q_0)^{-1}P_{A,\lambda,r}. \end{aligned}$$

And let

$$\begin{aligned} &A_0 = Res\left(\lambda^{-1}R_{rD}(\lambda,A),0\right), \\ &A_{-1} = Res(R_{rD}(\lambda,A),0). \end{aligned}$$

Then we have the following result:

Theorem 3.12. *Let $A\in\mathcal{B}(\mathcal{H})$ be a right generalized Drazin invertible operator. Then there exists a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that the operator $R_{rD}(\lambda,A)$ satisfies the following six properties for all $\lambda,\mu\in\mathcal{V}$:*

1) $A_\lambda R_{rD}(\lambda,A) = I$;
2) $R_{rD}(\lambda,A)-R_{rD}(\mu,A) = (\lambda-\mu)R_{rD}(\mu,A)R_{rD}(\lambda,A)$;
That is $R_{lD}(\lambda,A)$ is a right resolvent for A on $\mathcal{V}$, and
3) $A_0\in A^{\{rgD\}}$;
4) $A_{-1} = AA_0-I$
5) $R_{lD}(\lambda,A) = (A_0-A_{-1})(A_\lambda(A_0-A_{-1}))^{-1}$
6) $\mathcal{R}(A_{-1})\subset\mathcal{H}_0(A)$ and $\mathcal{N}(A_{-1}) = \mathcal{K}(A)$.

Proof. All is an immediate consequence of the duality between left and right generalized Drazin invertibility. □

From the preceding theorems, we get the following corollary.

Corollary 3.13. *Let $A \in \mathcal{B}(\mathcal{H})$ be a generalized Drazin invertible operator, then:*

$$A^{gD} = Res\left((\lambda A_\lambda)^{-1}, 0\right),$$
$$\mathcal{R}\left(Res\left((A_\lambda)^{-1}, 0\right)\right) = \mathcal{H}_0(A) \text{ and } \mathcal{N}\left(Res\left((\lambda A_\lambda)^{-1}, 0\right)\right) = \mathcal{K}(A).$$

Remark 3.14. *For a left generalized Drazin invertible operator $A \in \mathcal{B}(\mathcal{H})$ we have first proved that the limit $\lim\limits_{\substack{\lambda \to 0 \\ \lambda \in \mathcal{V}}} P_{A,\lambda,l}$ exists. Moreover, to derive an explicit formula for a left generalized Drazin inverse of A, we have used only the fact that the limit $\lim\limits_{\substack{\lambda \to 0 \\ \lambda \in \mathcal{V}}} P_{A,\lambda,l}$ exists, and then we have constructed a left resolvent so that its residue is a left generalized Drazin inverse of A. So, if we apply also the duality, we have proved in fact the following new characterization:*

Theorem 3.15. *Let $A \in \mathcal{B}(\mathcal{H})$, then A is left (resp. right) generalized Drazin invertible if and only if there is a punctured neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}$ such that the limit $\lim\limits_{\substack{\lambda \to 0 \\ \lambda \in \mathcal{V}}} P_{A,\lambda,l}$ (resp. $\lim\limits_{\substack{\lambda \to 0 \\ \lambda \in \mathcal{V}}} P_{A,\lambda,r}$) exists.*

Lemma 3.16. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two bounded below (resp. surjective) operators having same ranges (resp. kernels) then:*

$$A\,(A^*A)^{-1}\,A^* = B\,(B^*B)^{-1}\,B^* \text{ (resp. } A^*\,(AA^*)^{-1}\,A = B^*\,(BB^*)^{-1}\,B).$$

Proof. $A\,(A^*A)^{-1}\,A^*$ (resp. $A^*\,(AA^*)^{-1}\,A$) is an orthogonal projection that depends only on $\mathcal{R}(A)$ (resp. $\mathcal{N}(A)$). □

Theorem 3.17. *Let $A \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*
1) A is left generalized Drazin invertible;
2) $\mathcal{H}_0(A)$ and $\mathcal{R}(A) + \mathcal{H}_0(A)$ are closed in $\mathcal{H}$.

Proof. Assertion (6) of Remark 1.7 asserts that (1) implies (2). It remains to show that (2)⇒(1). Suppose that $\mathcal{H}_0(A)$ and $\mathcal{R}(A) + \mathcal{H}_0(A) = W$ are closed in $\mathcal{H}$. $A_{\mathcal{H}_0(A)^\perp}$ is injective since $\mathcal{N}(A_{\mathcal{H}_0(A)^\perp}) = \mathcal{N}(A) \cap \mathcal{H}_0(A)^\perp \subset \mathcal{H}_0(A) \cap \mathcal{H}_0(A)^\perp = \{0\}$. Furthermore,

$$\begin{aligned} A(\mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A) &\subset \mathcal{R}(A) + \mathcal{H}_0(A) \\ &\subset A\,(\mathcal{H}_0(A) \oplus \mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A) \\ &\subset A(\mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A) \end{aligned}$$

hence, $\mathcal{R}(A) + \mathcal{H}_0(A) = A(\mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A)$. Moreover, since $x \in \mathcal{H}_0(A) \Leftrightarrow Ax \in \mathcal{H}_0(A)$ then $A(\mathcal{H}_0(A)^\perp) \cap \mathcal{H}_0(A) = \{0\}$, so let us define the operator S by:

$$\begin{aligned} S \;:\; & \mathcal{R}(A) + \mathcal{H}_0(A) \longrightarrow \mathcal{H}_0(A)^\perp \\ & Ax + y \longmapsto x;\; x \in \mathcal{H}_0(A)^\perp,\; y \in \mathcal{H}_0(A). \end{aligned}$$

If $(Ax_n + y_n, x_n)_{n\in\mathbb{N}}$ is a sequence in the graph of S which converges to (z, x) in $\mathcal{H}\times\mathcal{H}$, it's clear that $x \in \mathcal{H}_0(A)^\perp$, $(Ax_n)_{n\in\mathbb{N}}$ converges to Ax and $(y_n)_{n\in\mathbb{N}}$ converges to an element $y \in \mathcal{H}_0(A)$ with $z = Ax + y$. So, S is closed between the Hilbert spaces $\mathcal{R}(A) + \mathcal{H}_0(A)$ and $\mathcal{H}_0(A)^\perp$, it is then necessarily continuous.

Consequently, $S_{A(\mathcal{H}_0(A)^\perp)}$ is continuous and if $(Ax_n)_{n\in\mathbb{N}}$ is a sequence in $A(\mathcal{H}_0(A)^\perp)$ which converges to $y \in \mathcal{H}$, then $(SAx_n)_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ converges to an element $x \in \mathcal{H}_0(A)^\perp$. So, $(Ax_n)_{n\in\mathbb{N}}$ converges to $Ax = y \in A(\mathcal{H}_0(A)^\perp)$. This shows that $A(\mathcal{H}_0(A)^\perp)$ is closed in $\mathcal{H}$, $A_{\mathcal{H}_0(A)^\perp}$ is bounded below and the continuity of the projection $P_0 = P_{A(\mathcal{H}_0(A)^\perp),\mathcal{H}_0(A)} \in \mathcal{B}(W)$ since $\mathcal{R}(A) + \mathcal{H}_0(A) = A(\mathcal{H}_0(A)^\perp) \oplus \mathcal{H}_0(A)$. Furthermore, the below boundedness of $A_{\mathcal{H}_0(A)^\perp}$ implies that there is a constant $a > 0$ such that for all $x \in \mathcal{H}_0(A)^\perp$ and for any $\lambda \in \mathbb{C}$, $|\lambda| < \frac{a}{\|0_{W^\perp} \oplus P_0\|}$, $\|Ax\| \geq a\,\|x\|$, then:

$$\begin{aligned} \|(0_{W^\perp} \oplus P_0)\, A_\lambda x\| &= \|Ax - \lambda\,(0_{W^\perp} \oplus P_0)\, x\| \\ &\geq (a - |\lambda|\, \|0_{W^\perp} \oplus P_0\|)\, \|x\|\,. \end{aligned}$$

Thus, $P_0 A_\lambda x \neq 0$ for all $x \in \mathcal{H}_0(A)^\perp \setminus \{0\}$ and then we have $A_\lambda(\mathcal{H}_0(A)^\perp) \cap \mathcal{H}_0(A) = \{0\}$. It's clear that $A_{\mathcal{H}_0(A)^\perp} + I_{\mathcal{H}_0(A)}$ is injective with closed range, this operator is then bounded below, therefore there is a constant $b > 0$ such that:

$$\|S_\lambda x\| \geq (b - |\lambda|)\, \|x\|$$

for all $x \in \mathcal{H}$ and for any $\lambda \in \mathbb{C}$, $|\lambda| < b$, where $S_\lambda = \left(A_{\mathcal{H}_0(A)^\perp} \oplus I_{\mathcal{H}_0(A)}\right) - \lambda I$. On the other hand,

$$\mathcal{R}(S_\lambda) = \mathcal{R}(A_\lambda) = A_\lambda\,(\mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A)$$

which shows that $A_\lambda\,(\mathcal{H}_0(A)^\perp) + \mathcal{H}_0(A)$ is closed for $|\lambda| < b$. It now remains to show that $\lim\limits_{\substack{\lambda \longrightarrow 0 \\ 0<|\lambda|<b}} P_{A,\lambda,l}$ exists. Indeed, since S_λ is bounded below an has the same range as A_λ, we deduce, by continuity of S_λ and Lemma 3.16, that:

$$\begin{aligned} P_{A,\lambda,l} &= S_\lambda (S^*_{\overline{\lambda}} S_\lambda)^{-1} S^*_{\overline{\lambda}}, \\ \lim_{\substack{\lambda \longrightarrow 0 \\ 0<|\lambda|<b}} P_{A,\lambda,l} &= \lim_{\lambda \to 0} S_\lambda (S^*_{\overline{\lambda}} S_\lambda)^{-1} S^*_{\overline{\lambda}} = S_0 (S^*_0 S_0)^{-1} S^*_0. \end{aligned}$$

So, by Theorem 3.15 A is left generalized Drazin invertible. □

Theorem 3.18. *Let $A \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*
1) A is right generalized Drazin invertible;
2) $\mathcal{K}(A)$ is closed in $\mathcal{H}$ and $\mathcal{K}(A) + \mathcal{H}_0(A) = \mathcal{H}$.

Proof. Again assertion (6) of Remark 1.7 asserts that (1) implies (2). It remains to show that (2)⇒(1). If $\mathcal{K}(A)$ is closed in $\mathcal{H}$ and $\mathcal{K}(A) + \mathcal{H}_0(A) = \mathcal{H}$, then $\mathcal{K}(A) \cap \mathcal{N}(A)$ is closed and $A_{\mathcal{K}(A)} - \lambda I_{\mathcal{K}(A)}$ is surjective for $|\lambda| < c$, $c > 0$. Let $x \in \mathcal{H}_0(A)$ such that $A_\lambda x = y \in \mathcal{K}(A)$. If $0 < |\lambda| < c$, we have:

$$\begin{aligned} x &= \frac{Ax - y}{\lambda} = \frac{A\left(\frac{Ax-y}{\lambda}\right) - y}{\lambda} = \frac{A^2 x}{\lambda^2} - \frac{Ay}{\lambda^2} - \frac{y}{\lambda} \\ &= \ldots = \frac{A^n x}{\lambda^n} - \sum_{j=1}^{n} \frac{A^{j-1} y}{\lambda^j},\ n \in \mathbb{N}. \end{aligned}$$

As $x \in \mathcal{H}_0(A)$, then $\lim\limits_{n\to\infty} \frac{A^n x}{\lambda^n} = 0$ and $x = -\sum\limits_{j=1}^{\infty} \frac{A^{j-1} y}{\lambda^j}$. Since $y \in \mathcal{K}(A)$ and $\mathcal{K}(A)$ is closed we deduce that $x \in \mathcal{K}(A)$. Thus we observe that if $x \in \mathcal{H}_0(A)$ and $y \in \mathcal{K}(A)$ are such that $x + y \in \mathcal{N}(A_\lambda)$, then $A_\lambda x \in \mathcal{K}(A)$ and $x \in \mathcal{K}(A)$, so $x + y \in \mathcal{K}(A)$ and $\mathcal{N}(A_\lambda) \subset \mathcal{K}(A)$ for $|\lambda| < c$.

Let $T_\lambda \in \mathcal{B}(\mathcal{H})$ be the operator $\left(A_{\mathcal{K}(A)} \oplus I_{\mathcal{K}(A)^\perp}\right) - \lambda I$. Since T_0 is surjective, we can assume $c < 1$ and then T_λ is surjective for $|\lambda| < c$. We have $\mathcal{N}(T_\lambda) = \mathcal{N}(A_\lambda) \subset \mathcal{K}(A)$ for $|\lambda| < c$. It now remains to show that $\lim\limits_{\substack{\lambda \longrightarrow 0 \\ 0<|\lambda|<c}} P_{A,\lambda,r}$ exists. Indeed, since T_λ is surjective having the same kernel as A_λ, we deduce, by continuity of T_λ and Lemma 3.16, that:

$$\begin{aligned} P_{A,\lambda,r} &= T^*_{\overline{\lambda}} (T_\lambda T^*_{\overline{\lambda}})^{-1} T_\lambda, \\ \lim_{\substack{\lambda \longrightarrow 0 \\ 0<|\lambda|<c}} P_{A,\lambda,r} &= \lim_{\lambda \to 0} T^*_{\overline{\lambda}} (T_\lambda T^*_{\overline{\lambda}})^{-1} T_\lambda = T^*_0 (T_0 T^*_0)^{-1} T_0, \end{aligned}$$

therefore A is right generalized Drazin invertible. □

Remark 3.19. *1) Recalling duality principle between left and right generalized Drazin invertibility, one may conjecture after Theorem 3.17, that A is right generalized Drazin invertible if and only if $\mathcal{K}(A)$ and $\mathcal{N}(A) \cap \mathcal{K}(A)$ are closed in $\mathcal{H}$, which is then equivalent to the only condition that $\mathcal{K}(A)$ is closed. Unfortunately, this is not true as if A is the right shift then $\mathcal{K}(A) = \{0\}$ and 0 is an accumulation point of $\sigma_{su}(A)$.*

2) Recalling duality principle between left and right generalized Drazin invertibility, one may conjecture after Theorem 3.18, that A is left generalized Drazin invertible if and only if $\mathcal{H}_0(A)$ is closed and $\mathcal{K}(A) \cap \mathcal{H}_0(A) = \{0\}$, which is then equivalent to the only condition that $\mathcal{H}_0(A)$ is closed. Unfortunately, this is not true as if A is compact with an infinite spectrum, then $\mathcal{H}_0(A)$ is closed and 0 is an accumulation point of $\sigma_{ap}(A)$.

The approach used along this paper allows us to conjecture the following characterizations that seem dual to each other :

Conjecture 3.20. *Let $A \in \mathcal{B}(\mathcal{H})$, then:*

1. *A is left generalized Drazin invertible if and only if $\mathcal{R}(A) + \mathcal{H}_0(A)$ is closed and $\mathcal{K}(A) \cap \mathcal{H}_0(A) = \{0\}$.*
2. *A is right generalized Drazin invertible if and only if $\mathcal{K}(A) + \mathcal{H}_0(A) = \mathcal{H}$ and $\mathcal{K}(A) \cap \mathcal{N}(A)$ is closed.*

Example 3.21. *1) The left-shift on l^2 :*

$$L(x_1, x_2, x_3, ...) = (x_2, x_3, ...)$$

and the right-shift on l^2 :

$$R(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

are mutual adjoints. The spectra $\sigma(L)$ and $\sigma(R)$ are the closed unit disk. R is injective bounded linear operator on l^2 but not surjective. Thus, R is left invertible with left inverse the left shift operator L. So, R (resp. L) is a left (resp. right) generalized Drazin invertible operator on l^2 and $L \in R^{\{lgD\}}$ (resp. $R \in L^{\{rgD\}}$). Then:

$$LR = I \text{ and } R - \lambda I = R - \lambda LR = (I - \lambda L) R.$$

Further, the resolvent $R_{lD}(\lambda, R) = (I - \lambda L)^{-1} L$ is given by the Neumann series $\sum\limits_{n=0}^{\infty} \lambda^n L^{n+1}$ whenever $|\lambda| < \|L\|^{-1}$.

2) Let $A \in \mathcal{B}(l^2)$ be such that $Ax = \left(\frac{x_3}{1}, x_4, \frac{x_5}{2}, x_6, \frac{x_7}{3}, ...\right)$ for every $x = (x_n)_{n\in\mathbb{N}^*} \in l^2$. Let $\overset{0}{x} = \left(0, \overset{0}{x}_2, 0, \overset{0}{x}_4, 0; ...\right)$, and for every $k \in \mathbb{N}$ define $\overset{k}{x} = \left(\overset{k}{x}_n\right)_{n\in\mathbb{N}^*}$ such that:

$$\begin{aligned} \overset{k}{x}_{2i} &= 0,\ 1 \le i \le k \\ \overset{k}{x}_{2i+2k} &= \overset{0}{x}_{2i},\ i \ge 1 \\ \overset{k}{x}_{2i+1} &= 0,\ i \ge 0. \end{aligned}$$

Then we have : $A\overset{k+1}{x} = \overset{k}{x}$, for all $k \ge 0$ and $\left\|\overset{k}{x}\right\| = \left\|\overset{0}{x}\right\|$, thus $\overset{0}{x} \in \mathcal{K}(A)$, that is $M = \left\{x = (x_n)_{n\in\mathbb{N}^*} \in l^2 : x_{2n-1} = 0\right\} \subset \mathcal{K}(A)$. On the other hand, for $x = (x_1, 0, x_3, 0, x_5, ...)$ we have:

$$A^k x = \left(\frac{0! x_{2k+1}}{k!}, 0, \frac{1! x_{2k+3}}{(k+1)!}, 0, \frac{2! x_{2k+5}}{(k+2)!}, ...\right).$$

So,

$$\left\|A^k x\right\|^{\frac{1}{k}} \le (k!)^{-\frac{1}{k}} \|x\|^{\frac{1}{k}}.$$

Then, $x \in \mathcal{H}_0(A)$, that is $N = \left\{x = (x_n)_{n\in\mathbb{N}^*} \in l^2 : x_{2n} = 0\right\} \subset \mathcal{H}_0(A)$. But, $l^2 = M \oplus N$, then, $l^2 = \mathcal{K}(A) + H_0(A)$, and since $\mathcal{K}(A) \cap N = \{0\}$, $\mathcal{K}(A) = M$ is closed. So by Theorem 3.18, A is right generalized Drazin invertible.

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