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Generalized Drazin-meromorphic pseudospectrum for multivalued linear relation

Kamel Mahfoudhi[a], Bilel Saadaoui[b], Vladimir Rakočević[c]

[a]*University of Sousse, Higher Institute of Applied Sciences and Technology of Sousse, Sousse, Tunisia*
[b]*Department of Mathematics, University of Sfax,Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia*
[c]*University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia*

Abstract. In this paperwe investigate the spectrum and the Drazin spectrum and their pseudo spectral analogues, for linear relations between Banach spaces and corresponding spectra, the generalized Drazin-meromorphic pseudospectrum. More specifically, the generalized Drazin-meromorphic pseudospectrum for a linear relations on a Banach space is studied. We also make several observations about the level set of the generalized Drazin-meromorphic pseudospectrum of linear relations. Furthermore, it is shown that pseudospectrum has no isolated points, has a finite number of connected components and each component contains an element from the generalized Drazin-meromorphic spectrum .

1. Introduction

This section contains some basic notions and results from the theory of linear relations given in [5] - [11, 13].

First, let us fix some notations. The symbols X, Y, Z stand for infinite dimensional Banach spaces over the same field $\mathbb{K}$ ($\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$). A multivalued linear operator or linear relation is a mapping $T \subset X \times Y$ which goes from a subspace $\mathcal{D}(T) \subset X$ called the domain of T, into the collection of nonempty subsets of Y such that

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for arbitrary scalars α_1, α_2 and $x_1,\ x_2 \in \mathcal{D}(T)$.

For $x \in X \backslash \mathcal{D}(T)$, we define $Tx = \emptyset$, with this convention, we have

$$\mathcal{D}(T) = \{x \in X : Tx \neq \emptyset\}.$$

The collection of linear relations as defined above will be denoted by $L\mathcal{R}(X, Y)$. A linear relation $T \in L\mathcal{R}(X, Y)$ is uniquely determined by and identified with its graph, $G(T)$, which is defined by

$$G(T) := \{(x, y) \in X \times Y : x \in \mathcal{D}(T),\ y \in Tx\}.$$

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Communicated by Dijana Mosić
Email addresses: kamelmahfoudhi72@yahoo.com (Kamel Mahfoudhi), saadaoui.bilel@hotmail.fr (Bilel Saadaoui), vrakoc@sbb.rs (Vladimir Rakočević)

The inverse of $T \in L\mathcal{R}(X,Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}.$$

If $T,\ S \in L\mathcal{R}(X,Y)$, then their algebraic sum $T+S$ is also a linear relation defined by

$$G(T+S) := \{(x, u+v) : (x,u) \in G(T),\ (x,v) \in G(S)\}.$$

Similarly, if $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$, then their product ST is also a linear relation defined by

$$G(ST) := \{(x,z) \in X \times Z : (x,y) \in G(T) \text{ and } (y,z) \in G(S) \text{ for some } y \in Y\}.$$

If M is a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$, then $T_{|M\cap\mathcal{D}(T)} := T_{|M}$ is defined by

$$G(T_{|M}) := \{(x,y) \in G(T) : x \in M\}.$$

The quotient map from Y onto $Y/\overline{T(0)}$ is denoted by Q_T. It is easy to see that $Q_T T$ is single valued so that we can define

$$\|Tx\| := \|Q_T Tx\| \text{ for all } x \in \mathcal{D}(T) \text{ and } \|T\| := \|Q_T T\|.$$

Let $T \in L\mathcal{R}(X,Y)$. The range of T is the subspace

$$R(T) := \{y : (x,y) \in G(T)\}$$

and T is called surjective if $R(T) = Y$. The subspace $T^{-1}(0)$ is denoted by $N(T)$ and T is called injective if $N(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator.

Now if T is both injective and surjective, then we say that T is bijective. T is said to be bounded below if the set is injective and open. Observe that

$$Tx = y + T(0), \text{ for any } y \in Tx.$$

We say that $T \in L\mathcal{R}(X,Y)$ is continuous if $\|T\| < \infty$; bounded if it is continuous and $\mathcal{D}(T) = X$, open if T^{-1} is continuous, equivalent if its minimum modulus $\gamma(T)$ is a positive number, where

$$\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|,\ x \in \mathcal{D}(T)\},$$

where $d(x, N(T))$ denotes the distance between x and $N(T)$.

A linear relation T is said to be closed if its graph is closed. Similarly, T is called closable if $\overline{T}(0) = T(0)$. $\overline{T}$ is defined by $G(\overline{T}) := \overline{G(T)}$.

Let $L(X)$ be the Banach algebra of all bounded linear operators on an infinite dimensional complex Banach space X. Recall that an operator $T \in L(X)$, is Drazin invertible if there is $S \in L(X)$ such that

$$TS = ST,\ \ STS = S,\ \ TST - T \text{ is nilpotent}.$$

The concept of the generalized Drazin invertible operators was introduced by J.Koliha [7]. An operator $T \in L(X)$ is generalized Drazin invertible in case there is $S \in L(X)$ such that

$$TS = ST,\ \ STS = S,\ \ TST - T \text{ is quasinilpotent}.$$

Recall that T is generalized Drazin invertible if and only if $0 \notin \operatorname{acc}\sigma(T)$, and this is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent.

Recently, Živković-Zlatanović and Cvetković [14, 15] further generalized the concept by replacing the third condition in the previous definitions by the condition that $TST - T$ is Riesz, and so it is introduced the concept of generalized Drazin-Riesz invertible operators. Also, Živković-Zlatanović and Duggal [6, 15] introduced the notion of generalized Drazin-meromorphic invertible by replacing the third condition with $TST - T$ is meromorphic.

In this paper, we further generalize this concept by introducing generalized Drazin invertible of a multivalued linear operator T.

Definition 1.1. $T \in L\mathcal{R}(X)$. *An element* $S \in L\mathcal{R}(X)$ *satisfying*

$$TS = ST + T(0), \quad STS = S \text{ and } TST = T + U$$

with, $U \in L\mathcal{R}(X)$ *meromorphic, is called a generalized Drazin-meromorphic inverse of* T *and it is denoted by* $T^{\diamond}$.

Now, we investigate corresponding resolvent. For $T \in L\mathcal{R}(X)$, the generalized Drazin-meromorphic resolvent

$$\rho_{\mathrm{g.D}\mathcal{M}}(T) = \{\lambda \in \mathbb{C} : \ (\lambda - T)^{\diamond} \text{ exists}\}.$$

The complement of the set $\rho_{\mathrm{g.D}\mathcal{M}}(T)$ is called generalized Drazin-meromorphic spectrum and it is defined as:

$$\sigma_{\mathrm{g.D}\mathcal{M}}(T) = \{\lambda \in \mathbb{C} : \ \lambda - T \text{ is not g.D}\mathcal{M} \text{ invertible}\},$$

where g.D$\mathcal{M}$ denote: generalized Drazin-meromorphic.

The map $(\lambda - T)^{\diamond}$ defined from $\rho_{\mathrm{g.D}\mathcal{M}}(T)$ to $L\mathcal{R}(X)$ is called the generalized Drazin-meromorphic resolvent map. In fact, that

$$\rho_{\mathrm{g.D}\mathcal{M},\varepsilon}(T) \ni \lambda \mapsto (\lambda - T)^{\diamond} \in L\mathcal{R}(X)$$

is continuous and even analytic. The Definition of the generalized Drazin-meromorphic pseudospectra of a multivalued linear operator T, for every $\varepsilon > 0$ is given by:

$$\sigma_{\mathrm{g.D}\mathcal{M},\varepsilon}(T) := \sigma_{\mathrm{g.D}\mathcal{M}}(T) \bigcup \left\{\lambda \in \mathbb{C} : \ \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon \right\},$$

where $\sigma_{\mathrm{g.D}\mathcal{M}}(T)$ is the generalized Drazin-meromorphic spectrum of linear relation T. By convention we write

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} = \infty,$$

if $(\lambda - T)^{\diamond}$ is unbounded or nonexistent, i.e., if λ is in $\sigma_{\mathrm{g.D}\mathcal{M}}(T)$. The generalized Drazin-meromorphic pseudoresolvent set of $T \in L\mathcal{R}(X)$ is defined as,

$$\rho_{\mathrm{g.D}\mathcal{M},\varepsilon}(T) := \rho_{\mathrm{g.D}\mathcal{M}}(T) \bigcap \left\{\lambda \in \mathbb{C} : \ \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} \leq \varepsilon \right\},$$

For more information about pseudospectrum of a multivalued linear operator see for example [1–4, 9, 10, 12].

The main aim of this paper is to offer a new definition of the generalized Drazin-meromorphic pseudospectrum of a multivalued linear operator T in Banach space, and we try to explain some properties of which (Theorem 2.1). Also we introduce some topological property (Theorem 2.5 and 2.6) and investigate and classify the possible cases, when the norm of the generalized Drazin-meromorphic resolvent map is not constant in an open connected subset of the generalized Drazin-meromorphic resolvent set (Theorem 3.3). Further attention is then devoted to the analysis of the level sets

$$O_{\varepsilon}(T) := \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} = \varepsilon \right\}.$$

The above set is called level set of the generalized Drazin-meromorphic pseudospectrum. Finally, we state some results relating the characterization of the generalized Drazin-meromorphic pseudospectrum (Theorems 2.6, 2.7 and 2.10) of a multivalued linear operator.

In this note, we focus attention on the generalized Drazin-meromorphic pseudospectra of a multivalued linear operator and its properties. The remainder of this paper is organized as follows. In Section 2, we first suggest a characterize for the generalized Drazin-meromorphic pseudospectrum of a multivalued linear operator. Then, in Section 3, we focus on the level set of the generalized Drazin-meromorphic pseudo spectrum, we also look at some topological property $\sigma_{g.\mathrm{D}\mathcal{M},\varepsilon}(T)$.

2. Generalized Drazin-meromorphic pseudospectra

In the following, we define and characterize the generalized Drazin-meromorphic pseudospectrum of a multivalued linear operator. We begin with the following definition.

Definition 2.1. *Given $\varepsilon > 0$ and $T \in L\mathcal{R}(X)$. The pseudospectrum of T is denoted by $\sigma_\varepsilon(T)$ and is defined to be the set*

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \tfrac{1}{\varepsilon}\right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$, (spectrum of T).

Definition 2.2. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. The generalized Drazin-meromorphic pseudospectrum of T is defined as,*

$$\sigma_{\text{g.D}\mathcal{M},\varepsilon}(T) := \sigma_{\text{g.D}\mathcal{M}}(T) \bigcup \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^\diamond\| : \begin{array}{c}(\lambda - T)^\diamond \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} > \varepsilon\right\},$$

By convention, we write

$$\inf\left\{\|(\lambda - T)^\diamond\| : \begin{array}{c}(\lambda - T)^\diamond \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} = \infty,$$

if $(\lambda - T)^\diamond$ is unbounded or nonexistent, i.e., if λ is in $\sigma_{\text{g.D}\mathcal{M}}(T)$.

The generalized Drazin-meromorphic pseudoresolvent set of a multivalued linear operator T is defined as,

$$\rho_{\text{g.D}\mathcal{M},\varepsilon}(T) := \rho_{\text{g.D}\mathcal{M}}(T) \bigcap \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^\diamond\| : \begin{array}{c}(\lambda - T)^\diamond \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \leq \varepsilon\right\}. \diamond$$

Definition 2.3. [5, Definition I.5.1] *Let $T \in L\mathcal{R}(X)$. A linear operator S is called a selection if*

$$T = \underbrace{S}_{\text{single valued part}} + T - T \text{ and } \mathcal{D}(T) = \mathcal{D}(S).$$

If S is a selection of T then we have for all $x \in \mathcal{D}(T)$

$$Tx = Sx + T(0).$$

Example 2.4. *Let I is selection of T, $S = I + T(0)$ and we suppose $T^2(0) = T(0)$, then S is a generalized Drazin-meromorphic inverse. Indeed, we have*

$$\begin{aligned} TS &= (I - T(0))(I + T(0)) \\ &= I^2 - T^2(0) \\ &= I - T(0) \\ &= I - T(0) + T(0) \\ &= ST + T(0), \end{aligned}$$

$$\begin{aligned} STS &= (I + T(0))(I - T(0))(I + T(0)) \\ &= (I + T(0))(I^2 - T^2(0)) \\ &= (I + T(0))(I - T(0)) \\ &= I^2 - T^2(0) \\ &= I - T(0) = S \end{aligned}$$

and

$$\begin{aligned} TST &= (I - T(0))(I + T(0))(I - T(0)) \\ &= (I - T(0))(I^2 - T^2(0)) \\ &= (I - T(0))(I - T(0)) \\ &= I^2 - 2T(0) + T^2(0) \\ &= I - 2T(0) + T(0) \\ &= I - T(0) \\ &= I + T(0) - 2T(0), \end{aligned}$$

where $U = -2T(0) = -2(T - T) \in L\mathcal{R}(X)$ *meromorphic .*

The following lemma collects some useful known properties of the multivalued linear operator. For more information about these notions, one can see [5].

Lemma 2.5. *Let* X *and* Y *be two vector spaces and let* $T \in L\mathcal{R}(X, Y)$*. Then*

(1) $\mathcal{D}(T^{-1}) = R(T)$; $\mathcal{D}(T) = R(T^{-1})$.

(2) *T is injective if, and only if,* $T^{-1}T = I_{\mathcal{D}(T)}$.

(3) *T is single valued if, and only if,* $T(0) = \{0\}$.

(4) $TT^{-1}y = y + T(0)$ $(y \in R(T))$ *and* $T^{-1}Tx = x + T^{-1}(0)$.

The following properties of the generalized Drazin-meromorphic pseudospectrum are easy to check from the definition of the generalized Drazin-meromorphic pseudospectrum.

Theorem 2.6. *Let* $T \in L\mathcal{R}(X)$ *and* $\varepsilon > 0$*. Then,*
(1) $\sigma_{\mathrm{g.D}\mathcal{M}}(T) = \bigcap\limits_{\varepsilon>0} \sigma_{\mathrm{g.D}\mathcal{M},\varepsilon}(T)$.
(2) *If* $0 < \varepsilon_1 < \varepsilon_2$*, then* $\sigma_{\mathrm{g.D}\mathcal{M},\varepsilon_2}(T) \subset \sigma_{\mathrm{g.D}\mathcal{M},\varepsilon_1}(T)$.

Proof. (1) Note that

$$\begin{aligned} \bigcap_{\varepsilon>0} \sigma_{\mathrm{g.D}\mathcal{M},\varepsilon}(T) &= \bigcap_{\varepsilon>0} \left(\sigma_{\mathrm{g.D}\mathcal{M}}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \inf \left\{ \|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon \right\} \right) \\ &= \sigma_{\mathrm{g.D}\mathcal{M}}(T) \bigcup \left(\bigcap_{\varepsilon>0} \left\{ \lambda \in \mathbb{C} : \inf \left\{ \|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon \right\} \right). \end{aligned}$$

It is sufficient to prove that

$$\bigcap_{\varepsilon>0} \left\{ \lambda \in \mathbb{C} : \inf \left\{ \|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon \right\} \subset \sigma_{\mathrm{g.D}\mathcal{M}}(T).$$

Let $\lambda \in \bigcap\limits_{\varepsilon>0} \left\{ \lambda \in \mathbb{C} : \inf \left\{ \|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon \right\}$, then we have

$$\inf \left\{ \|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon,$$

for all $\varepsilon > 0$. If $\varepsilon \longrightarrow +\infty$, we obtain

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{l}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} = +\infty.$$

Hence, $\lambda \in \sigma_{g.D\mathcal{M}}(T)$.

The proof of (2) is elementary. □

Theorem 2.7. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, the set $\sigma_{g.D\mathcal{M},\varepsilon}(T)$ is a closed subset of $\mathbb{C}$.*

Proof. Let $\lambda \in \overline{\rho_{g.D\mathcal{M},\varepsilon}(T)}$. Then, for $r \in]0, \varepsilon[$ we have

$$B_f(\lambda, r) \cap \rho_{g.D\mathcal{M},\varepsilon}(T) \neq \emptyset,$$

where $B_f(\lambda, r) = \{\mu \in \mathbb{C} \text{ such that } |\lambda - \mu| < r\}$.

So, there exists $\mu \in \rho_{g.D\mathcal{M},\varepsilon}(T)$ such that $|\lambda - \mu| \leq r$. Then

$$\mu \in \rho_{g.D\mathcal{M}}(T)$$

and $\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{l}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \leq \varepsilon$. Then, $(\lambda - T)$ is generalized Drazin-meromorphic invertible and there exists a generalized Drazin-meromorphic invertible $(\lambda - T)^{\diamond}$ such that the inequality

$$\|(\lambda - T)^{\diamond}\| \leq \varepsilon \quad \text{holds.}$$

Applying [5, Theorem. II.2.5] we have $\gamma((\mu - T)^{\diamond}) \geq \varepsilon$. Hence $(\mu - T)^{\diamond}$ is exists. But $|\lambda - \mu| \leq r < \varepsilon$, then by [5, Corollary III.7.7], $(\lambda - T)^{\diamond}$ is exists. It follows that,

$$\lambda \in \rho_{g.D\mathcal{M},\varepsilon}(T).$$

For $x \in \mathcal{D}(T)$, we have

$$\begin{aligned} \|(\lambda - T)^{\diamond}x\| &= \|(T - \mu + \mu - \lambda)^{\diamond}x\| \\ &\geq \|(T - \mu)^{\diamond}x\| - |\mu - \lambda|\|x\| \\ &\geq (\gamma((T - \mu)^{\diamond}) - |\mu - \lambda|)\|x\|, \end{aligned}$$

therefore

$$\gamma((\lambda - T)^{\diamond}) \geq \gamma((\mu - T)^{\diamond}) - |\mu - \lambda|.$$

Hence

$$\gamma((\lambda - T)^{\diamond}) \geq \varepsilon - r, \qquad \forall\, 0 < r < \varepsilon.$$

Then

$$\gamma((\lambda - T)^{\diamond}) \geq \varepsilon.$$

Consequently, applying [5, Theorem. II.2.5] we have

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{l}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \leq \varepsilon.$$

We deduce that,

$$\lambda \in \rho_{g.D\mathcal{M},\varepsilon}(T).$$

We observe that $\rho_{g.D\mathcal{M},\varepsilon}(T)$ is a closed set. □

Corollary 2.8. *The set $\sigma_{g.D\mathcal{M},\varepsilon}(T)$ is a non-empty compact subset of $\mathbb{C}$.*

Proof. From Theorem 2.7, it follows that $\sigma_{g.D\mathcal{M},\varepsilon}(T)$ is closed. Since

$$\sigma_{g.D\mathcal{M}}(T) \subset \sigma_{g.D\mathcal{M},\varepsilon}(T) \text{ and } \sigma_{g.D\mathcal{M},\varepsilon}(T) \subset \sigma_{\varepsilon}(T) \text{ is bounded.}$$

Thus $\sigma_{g.D\mathcal{M},\varepsilon}(T)$ is compact. $\square$

In the rest of the paper we investigate the relation between the generalized Drazin-meromorphic pseudospectrum and the usual generalized Drazin-meromorphic spectrum in a complex Banach space. To do this, we suppose that X is a Banach space satisfying the following property:

$(\mathcal{H})$: For all generalized Drazin-meromorphic invertible multivalued linear operator $T \in L\mathcal{R}(X)$ there exist $D \in L\mathcal{R}(X)$ such that $T(0) \subset D(0)$, $\mathcal{D}(T) \subset \mathcal{D}(D)$ and D is not generalized Drazin-meromorphic invertible and $\|T - D\| = \dfrac{1}{\|T^{\diamond}\|}$ with $T^{\diamond}$ a generalized Drazin-meromorphic inverse of T.

Our next example shows that $\|T - D\| = \dfrac{1}{\|T^{\diamond}\|}$.

Example 2.9. *Let X be a Banach space and, let $T, D \in L\mathcal{R}(X)$ where $T_1(0)$ is subspaces verify that $T_1^2(0) = T_1(0)$. Then, for all $n \in \mathbb{N}$ we consider*

$$T = \begin{pmatrix} I - T_1(0) & 0 \\ 0 & \frac{1}{n}(I - T_1(0)) \end{pmatrix} \text{ and } D = \begin{pmatrix} \frac{1}{n}(I - T_1(0)) & 0 \\ 0 & 0 \end{pmatrix}$$

We obtain that,

$$T^{\diamond} = \begin{pmatrix} I - T_1(0) & 0 \\ 0 & n(I - T_1(0)) \end{pmatrix} \text{ and } T - D = \begin{pmatrix} \frac{1}{n}(I - T_1(0)) & 0 \\ 0 & \frac{1}{n}(I - T_1(0)) \end{pmatrix}$$

This implies that,

$$\|T^{\diamond}\| = \max\left\{\|(I - T_1(0))\|, \|n(I - T_1(0))\|\right\} = n$$

and

$$\|T - D\| = \max\left\{\|\frac{1}{n}(I - T_1(0))\|, \|\frac{1}{n}(I - T_1(0))\|\right\} = \frac{1}{n}.$$

We conclude that

$$\|T - D\| = \frac{1}{n} = \frac{1}{\|T^{\diamond}\|}.$$

Suppose X is a complex Banach space with the following property $(\mathcal{H})$.

Theorem 2.10. *If $\lambda \in \sigma_{g.D\mathcal{M},\varepsilon}(T)$, then there exists $D \in L\mathcal{R}(X)$ such that $T(0) \subset D(0)$, $\mathcal{D}(T) \subset \mathcal{D}(D)$, $\varepsilon\|D\| \leq 1$ and*

$$\lambda \in \sigma_{g.D\mathcal{M}}(T + D).$$

Proof. Suppose $\lambda \in \sigma_{g.D\mathcal{M},\varepsilon}(T)$. We will discuss these two cases:
<u>1^{st} *case*</u> : If $\lambda \in \sigma_{g.D\mathcal{M}}(T)$, then it is sufficient to take $D = 0$.
<u>2^{nd} *case*</u> : If $\lambda \in \sigma_{g.D\mathcal{M},\varepsilon}(T) \backslash \sigma_{g.D\mathcal{M}}(T)$. Then,

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} \geq \varepsilon.$$

Hence, $(\lambda - T)$ is not generalized Drazin-meromorphic invertible and there exists a generalized Drazin-meromorphic invertible $(\lambda - T)^{\diamond}$ such that

$$\|(\lambda - T)^{\diamond}\| \geq \varepsilon.$$

By assumption, there exists $B \in L\mathcal{R}(X)$ such that $T(0) \subset B(0)$, $\mathcal{D}(T) \subset \mathcal{D}(B)$ and

$$\|\lambda - T - B\| = \frac{1}{\|(\lambda - T)^{\diamond}\|} \leq \frac{1}{\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\}}.$$

Let $D = \lambda - T - B$. Then

$$\|D\| = \frac{1}{\|(\lambda - T)^{\diamond}\|} \leq \frac{1}{\varepsilon}.$$

Since

$$\mathcal{D}(D) = \mathcal{D}(\lambda - T - B) = \mathcal{D}(T) \cap \mathcal{D}(B) = \mathcal{D}(B)$$

and

$$D(0) = T(0) + B(0) = B(0),$$

then

$$B = B + D - D \text{ and } D = B - B + D.$$

Thus, $D + (B - D) = \lambda - T - B + (B - D)$. This is equivalent to saying that $B = \lambda - (T + D)$ is not generalized Drazin-meromorphic invertible. So,

$$\lambda \in \sigma_{\text{g.D}\mathcal{M}}(T + D).$$

□

Theorem 2.11. *Let $T \in L\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If there is $D \in L\mathcal{R}(X)$ such that $T(0) \subset D(0)$, $\mathcal{D}(T) \subset \mathcal{D}(D)$, $\varepsilon\|D\| \leq 1$ and $\lambda \in \sigma_{\text{g.D}\mathcal{M}}(T + D)$. Then,*

$$\lambda \in \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T).$$

Proof. We assume that there exists $D \in L\mathcal{R}(X)$ such that $T(0) \subset D(0)$, $\mathcal{D}(T) \subset \mathcal{D}(D)$, $\varepsilon\|D\| < 1$ and $\lambda \in \sigma_{\text{G.D}\mathcal{M}}(T + D)$. We will discuss these two cases:

1st *case* : If $\lambda \in \sigma_{\text{G.D}\mathcal{M}}(T)$, then the conclusion follows trivially as

$$\sigma_{\text{g.D}\mathcal{M}}(T) \subset \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T).$$

2nd *case* : If $\lambda \notin \sigma_{\text{g.D}\mathcal{M}}(T)$. Then, $\lambda - T$ is not generalized Drazin-meromorphic invertible and $\lambda - T - D$ is not generalized Drazin-meromorphic invertible. Since $T(0) \subset D(0)$, $\mathcal{D}(T) \subset \mathcal{D}(D)$, then

$$\mathcal{D}((\lambda - T - D) - (\lambda - T)) = \mathcal{D}(T) \cap \mathcal{D}(D) = \mathcal{D}(D)$$

and

$$\Big((\lambda - T - D) - (\lambda - T)\Big)(0) = T(0) + D(0) = D(0).$$

Hence

$$(\lambda - T - D) - (\lambda - T) = D.$$

Therefore,

$$\frac{1}{\varepsilon} > \|D\| = \|(\lambda - T - D) - (\lambda - T)\| \geq \frac{1}{\|(\lambda - T)^{\diamond}\|}.$$

Thus, $\|(\lambda - T)^{\diamond}\| \geq \varepsilon$. Hence

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \geq \varepsilon.$$

That is, $\lambda \in \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$. □

Theorem 2.12. *Let $T \in L\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. Let $\lambda \in \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$ and let $\lambda_n \notin \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$ be such that $\lambda_n \to \lambda$ Then,*

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \textit{ a } \text{g.D}\mathcal{M} \\ \textit{inverse of } \lambda - T\end{array}\right\} = \infty.$$

Proof. Let $\delta \in \mathbb{R}$ and we suppose

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \le \delta.$$

Hence, $(\lambda - T)$ is generalized Drazin-meromorphic invertible and there exists a generalized Drazin-meromorphic invertible $(\lambda - T)^{\diamond}$ such that

$$\|(\lambda - T)^{\diamond}\| \le \delta.$$

Since $|\lambda_n - \lambda| \to 0$ for all $\lambda \in \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$, then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{array}{rcl} |\lambda_n - \lambda| & < & \dfrac{1}{\delta + 1} \\ & < & \dfrac{1}{\delta} \\ & \le & \dfrac{1}{\|(\lambda - T)^{\diamond}\|} \\ & \le & \dfrac{1}{\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\}} \text{ for all } n \ge n_0. \end{array}$$

Hence, $\lambda \notin \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$. This is a contradiction. □

Theorem 2.13. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, $\sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$ has no isolated points.*

Proof. Suppose $\sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$ has an isolated point μ. Then there exists an $\delta > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$ and there exists a generalized Drazin-meromorphic invertible $(\lambda - T)^{\diamond}$ such that

$$\|(\lambda - T)^{\diamond}\| < \varepsilon.$$

Let $\mu \in \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T) \backslash \sigma_{\text{G.D}\mathcal{M}}(T)$. Then, using the Hahn-Banach Theorem, there exist $x' \in X'$ such that

$$x'((\mu - T)^{\diamond}) = \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} \text{ with } \|x'\| = 1.$$

Now, we define

$$\begin{cases} \phi : \rho_{\text{g.D}\mathcal{M}}(T) \longrightarrow \mathbb{R}, \\ \qquad \lambda \longrightarrow \phi(\lambda) = x'((\lambda - T)^{\diamond}). \end{cases}$$

Since ϕ is is well-defined and continuous; in $B(\mu, \delta)$ and for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, we have

$$|\phi(\lambda)| = \left|x'((\lambda - T)^{\diamond})\right| \le \|(\lambda - T)^{\diamond}\| < \varepsilon.$$

But, $\phi(\mu) = \|(\mu - T)^{\diamond}\| \ge \varepsilon$. This contradicts the maximum modulus principle.
□

Theorem 2.14. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, for each $\lambda \in \sigma_{\text{g.D}\mathcal{M}}(T)$, there exists $r > 0$ such that $D(\lambda, r) \subseteq \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$.*

Proof. Let $\lambda \in \sigma_{g.D\mathcal{M}}(T)$ and suppose for every $r > 0$ such that $D(\lambda, r) \nsubseteq \sigma_{g.D\mathcal{M},\varepsilon}(T)$, then there exists a sequence $\lambda_n \to \lambda$ such that $\lambda_n \notin \sigma_{g.D\mathcal{M},\varepsilon}(T)$. it follows that $(\lambda_n - T)$ is generalized Drazin-meromorphic invertible and there exists a generalized Drazin-meromorphic invertible $(\lambda - T)^\diamond$ such that

$$\|(\lambda_n - T)^\diamond\| < \varepsilon.$$

From the continuity of the map

$$\begin{cases} \rho_{g.D\mathcal{M}}(T) \longrightarrow \mathbb{R}, \\ \qquad\quad \lambda \longrightarrow \|(\lambda - T)^\diamond\|. \end{cases}$$

We obtain that

$$\|(\lambda_n - T)^\diamond\| \to \|(\lambda - T)^\diamond\|.$$

implies that $\lambda - T$ is generalized Drazin-meromorphic invertible and therefore $\lambda \notin \sigma_{g.D\mathcal{M},\varepsilon}(T)$. We conclued that $\lambda \notin \sigma_{g.D\mathcal{M}}(T)$. A contradiction. □

3. Level sets of the generalized Drazin-meromorphic pseudospectrum for multivalued linear operator

We begin with the following definition.. For more information about level sets of pseudospectrum for linear operator see for example [8].

Definition 3.1. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Level set of the generalized Drazin-meromorphic pseudospectrum of T is defined as*

$$\mathcal{O}_\varepsilon(T) := \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^\diamond\| : \begin{array}{c} (\lambda - T)^\diamond \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} = \varepsilon\right\}.$$

Remark 3.2. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then,*

(1) $\mathcal{O}_\varepsilon(T) \subset \sigma_{g.D\mathcal{M},\varepsilon}(T)$.

(2) *If $T = \lambda I$ for some $\lambda \in \mathbb{C}$ then $\mathcal{O}_\varepsilon(T) = \emptyset$. In particular, $int\mathcal{O}_\varepsilon(T) = \emptyset$.*

Theorem 3.3. *Let $T \in L\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. Then, $\mathcal{O}_\varepsilon(T)$ is a compact subset of $\mathbb{C}$ with an uncountable number of elements.*

Proof. Since $\mathcal{O}_\varepsilon(T)$ is a closed subset of $\sigma_{g.D\mathcal{M},\varepsilon}(T)$, therefore $\mathcal{O}_\varepsilon(T)$ is compact. Suppose that $\mathcal{O}_\varepsilon(T)$ has a countable number of elements. Then, we select an isolated point $\mu \in \sigma_{G.D\mathcal{M},\varepsilon}(T)$. Hence, there exist an $r > 0$ such that

$$B(\mu, r) \cap \sigma_{g.D\mathcal{M}}(T) = \emptyset, \;\; B(\mu, r) \cap \sigma_{g.D\mathcal{M},\varepsilon}(T) \neq \emptyset \;\text{ and } B(\mu, r) \cap \rho_{g.D\mathcal{M},\varepsilon}(T) \neq \emptyset.$$

Now, we define the function ϕ by the following forms,

$$\begin{cases} \phi : \mathcal{V} \longrightarrow \mathbb{C}, \\ \quad \lambda \longrightarrow \inf\left\{\|(\lambda - T)^\diamond\| : \begin{array}{c} (\lambda - T)^\diamond \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\}, \end{cases}$$

where $\mathcal{V} = B(\mu, r) \setminus \mathcal{O}_\varepsilon(T)$. Ago ϕ is continuous and

$$\mathcal{V} = \left\{\lambda \notin \sigma_{g.D\mathcal{M}}(T) : \; \phi(\lambda) < \varepsilon\right\} \cup \left\{\lambda \notin \sigma_{g.D\mathcal{M}}(T) : \; \phi(\lambda) > \varepsilon\right\}.$$

This is a ambivalence to the fact that $\mathcal{V}$ is connected. □

Theorem 3.4. *Let $T \in L\mathcal{R}(X)$, $\varepsilon > 0$ and $\lambda \in \mathbb{C}$. Let Ω be an open subset of the unbounded component of $\rho_{g.D\mathcal{M},\varepsilon}(T)$. For some $M > 0$, we assume*

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \textit{ a g.D}\mathcal{M} \\ \textit{inverse of } \lambda - T \end{array}\right\} \leq M$$

for all $\lambda \in \Omega$, then

$$\inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \textit{ a g.D}\mathcal{M} \\ \textit{inverse of } \lambda - T \end{array}\right\} < M.$$

Proof. Let Ω_0 the unbounded component of $\rho_{G.D\mathcal{M},\varepsilon}(T)$. Then,

$$\left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} \geq M\right\} \text{ is bounded}$$

and we get

$$\left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} < M\right\} \cap \Omega_0 \neq \emptyset.$$

Now, let

$$\mu \in \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} < M\right\} \cap \Omega_0.$$

Then, the proof follows by applying [8, Theorem 2.1] to the analytic map, the open set Ω and to the point μ. □

Corollary 3.5. *Let $T \in L\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. The set $O_{\varepsilon}(T)$ has empty interior in the unbounded component of $\rho_{g.D\mathcal{M},\varepsilon}(T)$.*

Theorem 3.6. *Let $T \in L\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. Then, $\sigma_{g.D\mathcal{M},\varepsilon}(T)$ has a finite number of components and each of which contains an element of $\sigma_{g.D\mathcal{M}}(T)$.*

Proof. The final stage of knowledge transfer can be broken down into several steps:
Step 1 : Let Ω be a component of $\sigma_{g.D\mathcal{M},\varepsilon}(T)$. We first prove the following,

$$\text{If } \Omega \cap \left\{\lambda \in \mathbb{C} : inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} > \varepsilon\right\} \neq \emptyset,$$

then $\Omega \cap \sigma_{g.D\mathcal{M}}(T) \neq \emptyset$. Assume to the contrary that Ω is a component and

$$\Omega \cap \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} > \varepsilon\right\} \neq \emptyset$$

but $\Omega \cap \sigma_{g.D\mathcal{M}}(T) = \emptyset$. Consider the set

$$\Omega_1 = \Omega \backslash O_{\varepsilon}(T) = \Omega \cap O_{\varepsilon}(T)^c.$$

We see that

$$\Omega_1 \subseteq \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} > \varepsilon\right\} \subseteq O_{\varepsilon}(T)^c.$$

Since Ω is a component, $\mu \in \Omega$ and $B(\mu, r)$ is connected, we have

$$B(\mu, r) \subseteq \Omega.$$

From the definition of Ω_1, $B(\mu, r) \subseteq \Omega_1$, it follows that Ω_1 is open in $\mathbb{C}$. Let $\mu \in \Omega_1$. Using the Hahn-Banach Theorem, there exist $x' \in X'$ such that

$$x'(\mu - T) = \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array}\right\} \text{ with } \|x'\| = 1.$$

Now, we define

$$\begin{cases} \phi : \Omega_1 \longrightarrow \mathbb{C}, \\ \qquad \lambda \longrightarrow \phi(\lambda) = x'(\lambda - T). \end{cases}$$

Clearly ϕ is well defined, analytic and also continuous on $\overline{\Omega_1}$ (closure of Ω_1). For any boundary point λ of Ω_1 we have

$$\|(\lambda - T)^{\diamond}\| = \varepsilon,$$

hence $|\phi(\lambda)| \leq \varepsilon$ but at the point μ, we have

$$|\phi(\mu)| = |x'(\mu - T)| = \|(\mu - T)^{\diamond}\| > \varepsilon.$$

This is a contradiction to Maximum Modulus Theorem.

Step 2 : We claim that

$$\sigma_{\text{g.D}\mathcal{M}}(T) \subseteq \bigcup_{i=1}^{n} O_i.$$

Indeed, let $\lambda \in \sigma_{\text{g.D}\mathcal{M}}(T)$, then from Theorem 2.14, there exists $r > 0$ such that

$$D(\lambda, r) \subseteq \sigma_{\text{g.D}\mathcal{M},\varepsilon}(T) \text{ and } \left\{B(\lambda, r) : \lambda \in \sigma_{\text{g.D}\mathcal{M}}(T)\right\}$$

is an open cover for $\sigma_{\text{g.D}\mathcal{M}}(T)$. Since $\sigma_{\text{g.D}\mathcal{M}}(T)$ is compact, there exists $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that

$$\sigma_{\text{g.D}\mathcal{M}}(T) \subseteq \bigcup_{i=1}^{n} B(\lambda_i, r_i).$$

Consequently, there exists components $O_1, O_2, \cdots, O_n$ of $\sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$ and each O_i contains at least one $B(\lambda_i, r_i)$ such that

$$\sigma_{\text{g.D}\mathcal{M}}(T) \subseteq \bigcup_{i=1}^{n} B(\lambda_i, r_i) \subseteq \bigcup_{i=1}^{n} O_i.$$

Step 3 : We show that

$$\left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon\right\} \subseteq \bigcup_{i=1}^{n} O_i.$$

Let $\mu \in \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon\right\}$, then there exists $r > 0$ such that

$$B(\mu, r) \subseteq \left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : (\lambda - T)^{\diamond} \text{ is g.D}\mathcal{M} \text{ inverse of } \lambda - T\right\} > \varepsilon\right\}.$$

Hence $B(\mu, r) \subseteq \Omega_1$ for some connected component Ω_1 of $\sigma_{\text{g.D}\mathcal{M},\varepsilon}(T)$. We proved that $\Omega \cap \sigma_{\text{g.D}\mathcal{M}}(T) \neq \emptyset$. It follows that $\Omega \subseteq \bigcup_{i=1}^{n} O_i$. Thus

$$\left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c} (\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T \end{array} \right\} > \varepsilon\right\} \subseteq \bigcup_{i=1}^{n} O_i.$$

Since each O_i is closed in $\mathbb{C}$ and Theorem 2.13 and Step 2, we obtain

$$\begin{aligned}\overline{\left\{\lambda \in \mathbb{C} : \inf\left\{\|(\lambda - T)^{\diamond}\| : \begin{array}{c}(\lambda - T)^{\diamond} \text{ a g.D}\mathcal{M} \\ \text{inverse of } \lambda - T\end{array}\right\} > \varepsilon\right\}} &= \sigma_{\mathrm{G.D}\mathcal{M},\varepsilon}(T) \\ &\subseteq \bigcup_{i=1}^{n} O_i \ .\end{aligned}$$

The proof is therefore complete. □

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