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A note on the Moore-Penrose inverse of block matrices

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Abstract. Motivated by the representation for the Moore-Penrose inverse of the block matrix over a $*$-regular ring presented in [R.E. Hartwig and P. Patrício, When does the Moore-Penrose inverse flip? Operators and Matrices, 6(1):181-192, 2012], we show that the formula of the Moore-Penrose inverse is the same as the expression given by [Nieves Castro-González, Jianlong Chen and Long Wang, Further results on generalized inverses in rings with involution, Elect. J. Linear Algebra, 30:118-134, 2015].

1. Introduction

Representations and characterizations of the Moore-Penrose inverse (abbr. MP-inverse) for matrices over various settings attract wide interest from many scholars. In 2012, Hartwig [5] obtained new expressions for the MP-inverse of the matrix $\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ over a $*$-regular ring, extending some well known results for complex matrices. However, in order to guarantee the existence and to be able to give a formula of block matrices over a ring, the extra conditions on the ring are assumed. In [2, 3], Deng investigated the existence of MP-inverse of block operator valued triangular matrices with specified properties on a Hilbert space. In [6], necessary and sufficient conditions for the existence of the MP-inverse of the companion matrix in the form $\begin{pmatrix} 0 & a \\ I_n & b \end{pmatrix}$ over an arbitrary ring are considered and the formulae for the MP-inverse of the companion matrix are established. In [1], Castro-González obtained some characterizations on the existence of MP-inverse of block matrices over a ring in terms of the invertibility of elements, and the expressions of such MP-inverses were given. In this article, we show that the formula of MP inverse which was given by [1, Theorem 4.7] is the same as the expression given by (10)-(19) in [5, Section 2.2] for the MP-inverse of a 2×2 lower triangular matrix over a $*$-regular ring.

We recall that $*$ is an involution in R, if it is a map $* : R \to R$ such that for all $a, b \in R$:

$$(a^*)^* = a, \ (a + b)^* = a^* + b^* \text{ and } (ab)^* = b^* a^*.$$

Throughout this paper, R is an associative ring with unity and involution $*$. Let $M_{m\times n}(R)$ denote the set of $m \times n$ matrices over R. For any matrix $A = (a_{ij}) \in M_{m\times n}(R)$, $A^* \in M_{n\times m}(R)$ stands for $(\bar{A})^T$ where $\bar{A} = (a^*_{ij})$. A matrix $A \in M_{m\times n}(R)$ is said to be Moore-Penrose invertible with respect to $*$ if the equations

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$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

have a unique common solution. Such a solution, when exists, is denoted by $A^\dagger$. From now on, $R^\dagger$ stands for the set of all MP-invertible elements in R. Following [4], a ring R is said to satisfy the k-term star-cancellation (SC_k) if

$$a_1^* a_1 + \cdots + a_k^* a_k = 0 \Rightarrow a_1 = \cdots = a_k = 0$$

for any $a_1, \cdots, a_k \in R$. Note that a ring satisfying SC_1 is known as a $*$-cancellable ring. A ring is said to be $*$-regular if it is regular and $*$-cancellable. It is well-known that R is a $*$–regular ring if and only if every element in R is MP-invertible, and that $M_{2\times 2}(R)$ is a $*$–regular ring if and only if R is a regular ring satisfying SC_2 (see [4]).

2. Main results

Hartwig [5] derived the representations for the MP-inverse of the matrix

$$M = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \tag{1}$$

over $\tilde{R}$. In order to guarantee the existence and to be able to give a formula of $M^\dagger$, the following extra conditions on the regular $\tilde{R}$ are assumed:

(1) $\tilde{R}$ satisfy the SC_2.

(2) For each $r \in \tilde{R}$, there exists $c \in \tilde{R}$ such that $1 + r^* r = c^* c = cc^*$.

Under these hypothesis, the following result was obtained in [5].

Lemma 2.1. *[5] Let M as in (1), where $a, d \in \tilde{R}^\dagger$. And $\tilde{R}$ satisfy the above two conditions. Then*

$$M^\dagger = \begin{pmatrix} p & q \\ s & r \end{pmatrix}.$$

where

$$\begin{aligned}
p &= \xi a^* - (1 + \eta^*\eta)^{-1}\eta^* d^\dagger b \xi a^*,\\
s &= -(1 + \eta\eta^*)^{-1} d^\dagger b \xi a^*,\\
q &= \xi b^*(1 - dd^\dagger) + (1 + \eta^*\eta)^{-1}\eta^* d^\dagger[1 - b\xi b^*(1 - dd^\dagger)],\\
r &= (1 + \eta\eta^*)^{-1} d^\dagger[1 - b\xi^* b^*(1 - dd^\dagger)],
\end{aligned}$$

in which

$$\begin{aligned}
\xi &= t(1 + x^*x)^{-1}t^* + (\zeta^*\zeta)^\dagger,\\
x &= (1 - \zeta\zeta^\dagger)(1 - dd^\dagger)ba^\dagger,\\
t &= [1 - \zeta^\dagger(1 - dd^\dagger)b]a^\dagger,\\
\eta &= d^\dagger b(1 - a^\dagger a - \zeta^\dagger\zeta),\\
\zeta &= (1 - dd^\dagger)b(1 - a^\dagger a).
\end{aligned}$$

In fact, we write $e = 1 - dd^\dagger$ and $f = 1 - a^\dagger a$. Then $\zeta = ebf$, and it is easy to check that $\zeta^\dagger e = \zeta^\dagger = f\zeta^\dagger$. Indeed, $\zeta^\dagger e = \zeta^\dagger\zeta\zeta^\dagger e = \zeta^\dagger(e\zeta\zeta^\dagger)^* = \zeta^\dagger(\zeta\zeta^\dagger)^* = \zeta^\dagger$. Similarly, we can obtain $f\zeta^\dagger = \zeta^\dagger$. This leads to

$$t = [1 - \zeta^\dagger(1 - dd^\dagger)b]a^\dagger = (1 - \zeta^\dagger b)a^\dagger.$$

Note that $af = 0$ implies that $a\zeta^* = a(ebf)^* = af(eb)^* = 0$, and consequently

$$a\zeta^\dagger = a(\zeta^\dagger\zeta)^*\zeta^\dagger = a\zeta^*(\zeta^\dagger)^*\zeta^\dagger = 0.$$

Similarly, $\zeta a^\dagger = \zeta a^*(a^\dagger)^* a^\dagger = (a\zeta^*)^*(a^\dagger)^* a^\dagger = 0$.

Write $g = 1 - \zeta\zeta^\dagger$ and $h = 1 - \zeta^\dagger\zeta$. Then $x = geba^\dagger$ and $ege = e(1 - \zeta\zeta^\dagger)e = eg$, where the second identity is due to the fact that $\zeta^\dagger e = \zeta^\dagger$.

Base on the above equations, we have the following claims:

Claim 1. $1 + x^*x = u$, where $u = 1 + (ba^\dagger)^* egba^\dagger$ (See [1, Theorem 2.4.10]). Indeed,

$$1 + x^*x = 1 + (geba^\dagger)^*(geba^\dagger) = 1 + (ba^\dagger)^* egeba^\dagger = 1 + (ba^\dagger)^* egba^\dagger = u. \tag{2}$$

Claim 2. $1 + \eta\eta^* = v$, where $v = 1 + d^\dagger bhf(d^\dagger b)^*$ (See [1, Theorem 2.4.10]). Indeed, on account of $a\zeta^\dagger = 0$ and $\zeta a^\dagger = 0$, we conclude that $(1 - a^\dagger a - \zeta^\dagger\zeta)^2 = 1 - a^\dagger a - \zeta^\dagger\zeta$. Thus,

$$\begin{aligned} \eta\eta^* &= d^\dagger b(1 - a^\dagger a - \zeta^\dagger\zeta)(1 - a^\dagger a - \zeta^\dagger\zeta)(d^\dagger b)^* \\ &= d^\dagger b(1 - a^\dagger a - \zeta^\dagger\zeta)(d^\dagger b)^* \\ &= d^\dagger b(1 - \zeta^\dagger\zeta)(1 - a^\dagger a)(d^\dagger b)^* = d^\dagger bhf(d^\dagger b)^*. \end{aligned} \tag{3}$$

Claim 3. $\xi a^* = tu^{-1}$ (See [5, Section 2.2 (20)]).

Claim 4. $\xi b^* e = (1 - \zeta^\dagger b)a^\dagger u^{-1}(ba^\dagger)^* eg + \zeta^\dagger$.

Note that $eg = e(1 - \zeta\zeta^\dagger) = e - e\zeta\zeta^\dagger = e - \zeta\zeta^\dagger$. Then we can obtain $ebt = egba^\dagger$. Indeed, since $t = (1 - \zeta^\dagger b)a^\dagger$, $f\zeta^\dagger = \zeta^\dagger$ and $\zeta = ebf$, we have

$$\begin{aligned} ebt &= eb(1 - \zeta^\dagger b)a^\dagger = (e - eb\zeta^\dagger)ba^\dagger = (e - ebf\zeta^\dagger)ba^\dagger \\ &= (e - \zeta\zeta^\dagger)ba^\dagger = egba^\dagger. \end{aligned}$$

By $eg = ege$, $(eg)^* = (ege)^* = eg$, and $(\zeta^*\zeta)^\dagger = \zeta^\dagger(\zeta^\dagger)^*$, then we get

$$\begin{aligned} \xi b^* e &= [tu^{-1}t^* + (\zeta^*\zeta)^\dagger]b^* e \\ &= tu^{-1}(ebt)^* + \zeta^\dagger(\zeta^\dagger)^* b^* e \\ &= tu^{-1}(egba^\dagger)^* + \zeta^\dagger(eb\zeta^\dagger)^* \\ &= tu^{-1}(ba^\dagger)^*(eg)^* + \zeta^\dagger(ebf\zeta^\dagger)^* \\ &= tu^{-1}(ba^\dagger)^* eg + \zeta^\dagger \\ &= (1 - \zeta^\dagger b)a^\dagger u^{-1}(ba^\dagger)^* eg + \zeta^\dagger. \end{aligned}$$

The next theorem, a main result of this paper, shows that the formula of MP inverse which was given by [1, Theorem 4.7] is the same as the expression given by (10)-(19) in [5, Section 2.2] for the Moore-Penrose inverse of a 2×2 lower triangular matrix over a $*$-regular ring.

Theorem 2.2. *Let R be a ring and M as in (1) and let $a, d \in R^\dagger$. If $\zeta^\dagger$ exists, then $M^\dagger$ exist if and only if $u = 1 + (ba^\dagger)^* egba^\dagger$ and $v = 1 + d^\dagger bhf(d^\dagger b)^*$ are invertible, where $e = 1 - dd^\dagger$, $f = 1 - a^\dagger a$, $g = 1 - \zeta\zeta^\dagger$ and $h = 1 - \zeta^\dagger\zeta$. In this case,*

$$M^\dagger = \begin{pmatrix} (1 - hf(d^\dagger b)^* v^{-1} d^\dagger b)\sigma & \gamma \\ -\rho ba^\dagger u^{-1} & \rho(1_l - ba^\dagger u^{-1}(ba^\dagger)^* eg) \end{pmatrix} = \begin{pmatrix} p & q \\ s & r \end{pmatrix},$$

where

$$\begin{aligned} &\rho = v^{-1} d^\dagger (1 - b\zeta^\dagger), \\ &\sigma = (1 - \zeta^\dagger b)a^\dagger u^{-1}, \\ &\gamma = \zeta^\dagger + hf(d^\dagger b)^* \rho(1 - ba^\dagger u^{-1}(ba^\dagger)^* eg) + \sigma(ba^\dagger)^* eg, \end{aligned}$$

and p, q, r, s as in Lemma A.

Proof. In view of Claim 1, Claim 2 and [1, Theorem 3.7], we have $1 + x^*x$ and $1 + \eta\eta^*$ are invertible if and only if $M^\dagger$ exists. This, we only have to verify two matrices have the equal corresponding elements.

Step one: We prove $p = (1 - hf(d^\dagger b)^* v^{-1} d^\dagger b)\sigma$.
Indeed, by Claim 2, $1 + \eta\eta^* = v$. Since $(1+\eta^*\eta)\eta^* = \eta^*(1+\eta\eta^*)$ and v is invertible, we can obtain $(1+\eta^*\eta)^{-1}\eta^* = \eta^*(1+\eta\eta^*)^{-1}$, it is due to the fact that $1+\eta^*\eta$ is invertible if and only if $1+\eta\eta^*$ is invertible. Therefore,

$$\begin{aligned} p &= \xi a^* - (1+\eta^*\eta)^{-1}\eta^* d^\dagger b\xi a^* \\ &= \xi a^* - \eta^*(1+\eta\eta^*)^{-1} d^\dagger b\xi a^* \\ &= \xi a^* - \eta^* v^{-1} d^\dagger b\xi a^* \end{aligned}$$

Note that $\eta^* = (1 - a^\dagger a - \zeta^\dagger\zeta)(d^\dagger b)^* = (1-\zeta^\dagger\zeta)(1-a^\dagger a)(d^\dagger b)^* = hf(d^\dagger b)^*$.
This gives that

$$p = \xi a^* - hf(d^\dagger b)^* v^{-1} d^\dagger b\xi a^*.$$

By Claim 3, $\xi a^* = tu^{-1} = (1-\zeta^\dagger b)a^\dagger u^{-1} = \sigma$. So we get

$$p = [1 - hf(d^\dagger b)^* v^{-1} d^\dagger b]\sigma.$$

Step two: We prove $s = -\rho b a^\dagger u^{-1}$. Indeed,
note that $\xi a^* = tu^{-1}$, we can obtain

$$\begin{aligned} s &= -(1+\eta\eta^*)^{-1} d^\dagger b\xi a^* \\ &= -v^{-1} d^\dagger b\xi a^* \\ &= -v^{-1} d^\dagger b t u^{-1} \\ &= -v^{-1} d^\dagger b(1-\zeta^\dagger b)a^\dagger u^{-1} \\ &= -v^{-1} d^\dagger (1 - b\zeta^\dagger) b a^\dagger u^{-1} \\ &= -\rho b a^\dagger u^{-1} \end{aligned}$$

Step three: We prove $r = \rho[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg]$. Indeed,

$$\begin{aligned} r &= (1+\eta\eta^*)^{-1} d^\dagger (1 - b\xi^* b^*(1-dd^\dagger)) \\ &= v^{-1} d^\dagger (1 - b\xi^* b^* e) \\ &= v^{-1} d^\dagger (1 - bc^\dagger - b(1-c^\dagger b)a^\dagger u^{-1}(ba^\dagger)^* eg) \\ &= v^{-1} d^\dagger (1 - bc^\dagger - (1-bc^\dagger)ba^\dagger u^{-1}(ba^\dagger)^* eg) \\ &= v^{-1} d^\dagger (1 - bc^\dagger)[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg] \\ &= \rho[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg] \end{aligned}$$

Step four: We show that $q = \zeta^\dagger + \sigma(ba^\dagger)^* eg + hf(d^\dagger b)^*\rho[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg]$.
By Claim 4, we have

$$\xi b^* e = \zeta^\dagger + (1-\zeta^\dagger b)a^\dagger u^{-1}(ba^\dagger)^* eg = \zeta^\dagger + \sigma(ba^\dagger)^* eg. \tag{4}$$

Since

$$\begin{aligned} \eta^* &= [d^\dagger b(1 - a^\dagger a - \zeta^\dagger\zeta)]^* = (1 - a^\dagger a - \zeta^\dagger\zeta)(d^\dagger b)^* \\ &= (1-\zeta^\dagger\zeta)(1-a^\dagger a)(d^\dagger b)^* = hf(d^\dagger b)^* \end{aligned}$$

and $\xi^* = \xi$, this implies that

$$\eta^* v^{-1} d^\dagger (1 - b\xi b^* e) = \eta^* r = hf(d^\dagger b)^*\rho[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg], \tag{5}$$

the last identity due to Step 3. In view of (2.4) and (2.5), by direct computation, we have

$$\begin{aligned} q &= \xi b^*(1-dd^\dagger) + (1+\eta^*\eta)^{-1}\eta^* d^\dagger[1 - b\xi b^*(1-dd^\dagger)] \\ &= \xi b^* e + \eta^*(1+\eta\eta^*)^{-1} d^\dagger (1 - b\xi b^* e) \\ &= \xi b^* e + \eta^* v^{-1} d^\dagger (1 - b\xi b^* e) \\ &= c^\dagger + \sigma(ba^\dagger)^* eg + hf(d^\dagger b)^*\rho[1 - ba^\dagger u^{-1}(ba^\dagger)^* eg] \end{aligned}$$

So, we can obtain that

$$\begin{pmatrix} (1 - hf(d^\dagger b)^* v^{-1} d^\dagger b)\sigma & \gamma \\ -\rho b a^\dagger u^{-1} & \rho(1_l - b a^\dagger u^{-1}(b a^\dagger)^* e g) \end{pmatrix} = \begin{pmatrix} p & q \\ s & r \end{pmatrix}.$$

The proof is complete. □

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