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Abundant semigroups with weakly simplistic RGQA transversals

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Abstract. As the *real* common generalisations of both orthodox transversals and adequate transversals in abundant semigroups, the concept of *refined generalised quasi-adequate transversals*, briefly, *RGQA transversals* was introduced by Kong and Wang. In this paper, for the RGQA transversal, the necessary and sufficient condition for the sets I and Λ to be bands is investigated. It is demonstrated that the sets I and Λ are both bands if and only if the RGQA transversal is weakly simplistic. Moreover, the RGQA transversal S^o being weakly simplistic is different from S^o being a quasi-ideal nor the abundant semigroup S satisfying the regularity condition. Finally, by means of a quasi-adequate semigroup and a band, the structure theorem for an abundant semigroup with a weakly simplistic RGQA transversal is established.

1. Introduction and preliminaries

Let S^o be a subsemigroup of the regular semigroup S. Then S^o is called an *inverse transversal* of S if S^o is an inverse subsemigroup of S and contains exactly one inverse of each element of S, that is, $|V_{S^o}(a)| = 1$, where $V_{S^o}(a)$ denotes the intersection of $V(a)$ and S^o. This concept was first introduced by Blyth and McFadden [1] in 1982. Afterwards, this class of regular semigroups attracted many semigroup researchers' attention and a deal of important results were obtained (see [1-4] and their references). Let $I = \{aa^o : a \in S, a^o \in V_{S^o}(a)\}$ and $\Lambda = \{a^oa : a \in S, a^o \in V_{S^o}(a)\}$. In 1997, Tang [4] showed that if S is a regular semigroup with an inverse transversal S^o, then both I and Λ are bands with I a left regular band and Λ a right regular band. These two bands play a key role in the study of regular semigroups with inverse transversals. Other important subsets of S are $R = \{x \in S : x^ox = x^ox^{oo}\}$ and $L = \{x \in S : xx^o = x^{oo}x^o\}$. Both R and L are subsemigroups with R left inverse and L right inverse. The concept of *orthodox transversals* was introduced by Chen [5] as a generalisation of inverse transversals, and an excellent structure theorem for regular semigroups with quasi-ideal orthodox transversals was established. Chen and Guo [6] considered the general case of orthodox transversals and investigated some properties associated with the sets I and Λ. In [7,8], Kong and Zhao introduced two interesting sets R and L and established the structure theorems for regular semigroups

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with quasi-ideal orthodox transversals. In 2014, Kong [9] introduced the concept of *generalised orthodox transversals* and Kong and Meng [10] acquired the characterization for a generalised orthodox transversal to be an orthodox transversal. In [11], Kong introduced the weakly simplistic orthodox transversal and obtained the result that I and Λ are both bands if and only if the orthodox transversal S^o is weakly simplistic.

The concept of *adequate transversals*, was introduced by El-Qallali [12] in the class of abundant semigroups. Chen, Guo and Shum [13,14] obtained some important results about quasi-ideal adequate transversals. Afterwards, Kong [15] explored some properties concerned with adequate transversals. Kong and Wang [16] considered the product of quasi-ideal adequate transversals and proposed the open problem of the isomorphism of adequate transversals. The concept of *quasi-adequate transversals* was introduced by Ni [17] and followed by Luo, Kong and Wang [18,19], their work mainly focused on the structure and the properties of multiplicative quasi-adequate transversals. Unfortunately, quasi-adequate transversals are neither the generalisation of orthodox transversals nor adequate transversals. Inspired by the characterization of orthodox transversals [10], the concept of *refined generalised quasi-adequate transversal*, briefly, *RGQA transversal* was introduced by Kong and Wang [20]. It was demonstrated that RGQA transversals are the *real* common generalisations of both orthodox transversals and adequate transversals in the abundant semigroups. The product of quasi-ideal RGQA transversals was explored [21] and generalised to quasi-Ehresmann transversals [22].

In this article, we continue along the line of [3, 11, 20] by studying the equivalent condition of the sets I and Λ to be bands as for abundant semigroups with RGQA transversals. It is shown that the sets I and Λ are both bands if and only if the RGQA transversal is weakly simplistic. A structure theorem for an abundant semigroup with a weakly simplistic RGQA transversal is also established. The related results concerning orthodox transversals and adequate transversals are generalised and enriched.

It is worth remarking that the RGQA transversal being weakly simplistic is different from a quasi-ideal nor the abundant semigroup S satisfying the regularity condition. As for an abundant semigroup S with adequate transversals S^o, even if the adequate transversal S^o is a quasi-ideal of S, I and Λ need not be subsemigroups, see Example 2.7 in [13]. If S is a regular semigroup with an orthodox transversal S^o, then S is certainly an abundant semigroup satisfying the regularity condition and S^o is an RGQA transversal of S, but in general, I and Λ are not necessary bands. Let S be an abundant semigroup with an RGQA transversal S^o. If S satisfies the regularity condition and S^o is a quasi-ideal of S, it is easy to check that both I and Λ are bands. But these conditions are a little stronger. As for orthodox transversals, it is shown that in [11], I and Λ are both bands if and only if the orthodox transversal S^o is weakly simplistic. The second author gave Example 2.4 in [11] illustrating that weakly simplistic orthodox transversal S^o is not necessarily a quasi-ideal of S. As for adequate transversals, it happens that both I and Λ are bands, but the abundant semigroup S does not satisfy the regular condition, see Example 1 in [15].

The so called Miller-Clifford theorem will be used frequently.

Lemma 1.1 [23] (1) *Let e and f be $\mathcal{D}$-equivalent idempotents of the semigroup S. Then each element a in $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$ with $aa' = e$ and $a'a = f$;*
(2) *Let a, b be elements of the semigroup S. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ has an idempotent.*

Definition 1.1 [5] Let S^o be an orthodox subsemigroup of be the regular semigroup S. Then S^o is called an *orthodox transversal* of S, if the following two conditions are satisfied:
(1) For all $a \in S$, $V_{S^o}(a) \neq \emptyset$;
(2) For any $a, b \in S$, if $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Lemma 1.2 [10] *Let S^o be an orthodox subsemigroup of the regular semigroup S. If $V_{S^o}(a) \neq \emptyset$ for any $a \in S$, then S^o is an orthodox transversal of S if and only if*

$$(\forall a, b \in S) \quad [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

A subsemigroup T of S is called a *quasi-ideal* of S, if $TST \subseteq T$.

In this article, for semigroups S and S^o, we denote the set of idempotents of S and S^o by E and E^o respectively if no confusion. If the product of any two regular elements in S is also regular, then S is said to *satisfy the regularity condition*.

On a semigroup S the relation $\mathcal{L}^*$ is defined by $a\ \mathcal{L}^*\ b$ if and only if $\{\forall x, y \in S^1, ax = ay \Leftrightarrow bx = by\}$ and the relation $\mathcal{R}^*$ is defined dually. Obviously, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ a left congruence with $\mathcal{L}\subseteq\mathcal{L}^*, \mathcal{R}\subseteq\mathcal{R}^*$. It is easy to see if a, b are regular elements of S, then $a\ \mathcal{L}^*\ b$ ($a\ \mathcal{R}^*\ b$) if and only if $a\ \mathcal{L}\ b$ ($a\ \mathcal{R}\ b$). A semigroup is called *abundant* [24] if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup S is called *quasi-adequate* [25] (*adequate*) if its idempotents form a band (semilattice). Let S be an abundant semigroup and U an abundant subsemigroup of S. Then U is a $*$-subsemigroup of S if and only if $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$.

Lemma 1.3 [24] *Let e be an idempotent of a semigroup S. Then for $a \in S$, the following conditions are equivalent:*
(1) $a\ \mathcal{L}^*\ e$ ($a\ \mathcal{R}^*\ e$);
(2) $a = ae$ ($ea = a$) *and for all* $x, y \in S^1, ax = ay$ ($xa = ya$) *implies* $ex = ey$ ($xe = ye$).

Lemma 1.4 [17] *Let S be an abundant semigroup and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e\ \mathcal{L}\ y^+, f\ \mathcal{R}\ y^*$ for some $y^+, y^* \in E$, then $e\ \mathcal{R}^*\ x$ and $f\ \mathcal{L}^*\ x$.*

Definition 1.2 [12] Let S^o be a $*$-adequate subsemigroup of an abundant semigroup S. Then S^o is called an *adequate transversal* of S, if for any $x \in S$ there exist idempotents $e, f \in S$ and a unique element $\overline{x} \in S^o$ such that $x = e\overline{x}f$, where $e\ \mathcal{L}\ \overline{x}^+$ and $f\ \mathcal{R}\ \overline{x}^*$. It can be shown that e and f are uniquely determined by x and S^o (see [12] for detail).

Let S be an abundant semigroup and S^o a quasi-adequate $*-$subsemigroup of S. Then S^o is called a *generalised quasi-adequate transversal* of S if $C_{S^o}(x) = \{\overline{x} \in S^o \mid x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\} \neq \emptyset$. Let

$$I_x = \{i_x \in E \mid (\exists \overline{x} \in C_{S^o}(x))\, x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\},$$

$$\Lambda_x = \{\lambda_x \in E \mid (\exists \overline{x} \in C_{S^o}(x))\, x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\},$$

$$I = \bigcup_{x\in S} I_x, \quad \Lambda = \bigcup_{x\in S} \Lambda_x.$$

In [17], Ni called the generalised quasi-adequate transversal S^o a *quasi-adequate transversal* of S if it satisfies $(\forall e \in E)\ (\forall g \in E^o)$, $C_{S^o}(e)C_{S^o}(g) \subseteq C_{S^o}(ge)$ and $C_{S^o}(g)C_{S^o}(e) \subseteq C_{S^o}(eg)$.

Lemma 1.5 [20] *If S is an abundant semigroup with a generalised quasi-adequate transversal S^o, then $I = \{e \in E :$ $(\exists e^* \in E^o)\ e\ \mathcal{L}\ e^*\}$* and $\Lambda = \{f \in E : (\exists f^+ \in E^o)\ f\ \mathcal{R}\ f^+\}$with $I \cap \Lambda = E^o$.
Let $R = \{x \in S : (\exists \lambda_x \in \Lambda_x)\ \lambda_x \in E^o\}$ and $L = \{a \in S : (\exists i_a \in I_a)\ i_a \in E^o\}$. Then $R = \{x \in S : (\exists l \in E^o)\ x\ \mathcal{L}^*\ l\}$ and $L = \{a \in S : (\exists h \in E^o)\ a\ \mathcal{R}^*\ h\}$ with $R \cap L = S^o, E(R) = I$ and $E(L) = \Lambda$.

Definition 1.3 [20] Let S^o be a generalised quasi-adequate transversal of the abundant semigroup S. If for all $a, b \in RegS$, $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$ implies that $V_{S^o}(a) = V_{S^o}(b)$, then S^o is called a *refined generalised quasi-adequate transversal*, briefly, *RGQA transversal* of S.

Lemma 1.6 [20] *Let S^o be a generalised quasi-adequate transversal of the abundant semigroup S. Then S^o is refined if and only if $IE^o, E^o\Lambda \subseteq E$ and for all $i \in I, \lambda \in \Lambda, e^o \in E^o$, if $e^oi, \lambda e^o$ are regular, then they are idempotent.*

Lemma 1.7 [20] *Let S be an abundant semigroup with an RGQA transversal S^o.*
(1) *If $C_{S^o}(a) \cap E^o \neq \emptyset$ or $V_{S^o}(a) \cap E^o \neq \emptyset$, then $C_{S^o}(a) = V_{S^o}(a) \subseteq E^o$.*
(2) *If $C_{S^o}(a) \cap C_{S^o}(b) \neq \emptyset$ and $a\ \mathcal{L}^*\ b, a\ \mathcal{R}^*\ b$, then $a = b$.*

2. Weakly simplistic RGQA transversals

In this section, the concept of left simplistic, simplistic, left weakly simplistic, weakly simplistic and left quasi-ideal RGQA transversals are introduced and some interesting equivalence conditions for an RGQA transversal to be left simplistic, simplistic, left weakly simplistic, weakly simplistic and left quasi-ideal are obtained.

Definition 2.1 Let S^o be an RGQA transversal of the abundant semigroup S. Then S^o is called *left simplistic* (resp. *right simplistic*) if $S^oIS^o \subseteq S^o$ (resp. $S^o\Lambda S^o \subseteq S^o$); and *simplistic* if it is both left simplistic and right simplistic.

Theorem 2.1 *Let S^o be an RGQA transversal of the abundant semigroup S. Then the following conditions are equivalent:*

(1) *S^o is left simplistic;*
(2) *$E^oI \subseteq S^o$;*
(3) *$S^oI \subseteq S^o$;*
(4) *S^o is a right ideal of R;*
(5) *R is a subsemigroup and $S^oS \subseteq L$;*
(6) *R is a subsemigroup and L is a right ideal of S;*
(7) *R is a subsemigroup and $\Lambda I \subseteq L$.*

Proof. (1) $\Longrightarrow$ (2). For any $i \in I$, there exists $i^* \in E^o$ such that $i^* \mathcal{L}\ i$. Thus for any $e^o \in E^o, e^o i = e^o i i^* \in S^oIS^o \subseteq S^o$.

(2) $\Longrightarrow$ (3). For any $s^o \in S^o$, there exists $s^{o*} \in E^o$ such that $s^{o*} \mathcal{L}^*\ s^o$. Then for any $i \in I$ we have $s^o i = s^o \cdot s^{o*} i \in s^o \cdot E^oI \subseteq s^oS^o \subseteq S^o$.

(3) $\Longrightarrow$ (4). For any $s^o \in S^o, x \in R$, it is easy to see $s^o x = s^o i_x \overline{x} \lambda_x \in S^oI \cdot S^oE^o \subseteq S^o \cdot S^o \subseteq S^o$.

(4) $\Longrightarrow$ (5). For any $x, y \in R$, there exists $e^o \in E^o$ such that $x \mathcal{L}^*\ e^o$ and so $xy \mathcal{L}^*\ e^o y \in S^oR \subseteq S^o$. Thus $xy \in R$ and R is a subsemigroup.

For any $x \in S, s^o \in S^o$, we have $s^o x \mathcal{R}^*\ s^o i_x \in S^oI \subseteq S^oR \subseteq S^o$ and so $s^o x \in L$. Consequently, $S^oS \subseteq L$.

(5) $\Longrightarrow$ (6). It only need to show that L is a right ideal of S. For any $x \in S$ and $a \in L$, $ax = i_a \overline{a} \lambda_a x \in E^oS^o\lambda_a x \subseteq S^oS \subseteq L$ and so $LS \subseteq L$.

(6) $\Longrightarrow$ (7). It follows from (6) that $\Lambda I \subseteq LI \subseteq L$.

(7) $\Longrightarrow$ (1). For any $a^o, b^o \in S^o$ and $i \in I$, there exists $a^{o*} \in E^o$ such that $a^{o*} \mathcal{L}^*\ a^o$, we have

$$a^{o*} i \in E^oI \subseteq \Lambda I \subseteq L, \quad a^{o*} i \in E^oI \subseteq RR \subseteq R.$$

Thus $a^{o*} i \in L \cap R = S^o$ and so $a^o i b^o = a^o(a^{o*} i) b^o \in a^o S^o b^o \subseteq S^o$, that is, S^o is left simplistic. □

Combining Theorem 2.1 with its dual, it is easy to see the following result.

Theorem 2.2 *Let S^o be an RGQA transversal of the abundant semigroup S. Then the following statements are equivalent:*

(1) *$S^oI \subseteq S^o$, $\Lambda S^o \subseteq S^o$;*
(2) *S^o is simplistic, that is, $S^oIS^o \subseteq S^o$ and $S^o\Lambda S^o \subseteq S^o$;*
(3) *S^o is a quasi-ideal of S, that is, $S^oSS^o \subseteq S^o$;*
(4) *$\Lambda I \subseteq S^o$;*
(5) *$S^oR \subseteq S^o$, $LS^o \subseteq S^o$;*
(6) *$E^oI \subseteq S^o$, $\Lambda E^o \subseteq S^o$;*
(7) *$SS^o \subseteq R$, $S^oS \subseteq L$;*
(8) *R is a left ideal and L is a right ideal of S;*
(9) *$LR \subseteq S^o$.*

Theorem 2.3 *Let S be an abundant semigroup with an RGQA transversal S^o. Then the following statements are equivalent:*

(1) *S^o is left simplistic and weakly multiplicative (i.e. ΛI is a regular subset and $V_{S^o}(\Lambda I) \subseteq E^o$);*
(2) *$\Lambda I \subseteq \Lambda$ and R is a subsemigroup of S.*

Proof. (1) $\Longrightarrow$ (2). If (1) holds, then $\Lambda I \subseteq L$ and R is a subsemigroup by Theorem 2.2. For any $l \in \Lambda, i \in I$, since $li \in L$ and regular, there exist $(li)^o \in V_{S^o}(li)$, $(li)^{oo} \in V_{S^o}((li)^o)$ such that $li = (li)^{oo}(li)^o li$. It follows from S^o is weakly multiplicative that $(li)^o \in E^o$ and so $(li)^{oo} \in E^o$. Thus, by Lemma 1.7 $li = (li)^{oo}(li)^o li \in E^o\Lambda \subseteq E$. It is easy to see in this case, $E^o\Lambda \subseteq \Lambda$ and so $\Lambda I \subseteq \Lambda$.

(2) $\Longrightarrow$ (1). Suppose that $\Lambda I \subseteq \Lambda$, then $V_{S^o}(\Lambda I) \subseteq V_{S^o}(\Lambda) \subseteq E^o$ and so S^o is weakly multiplicative. Since $\Lambda \subseteq L$, we have $\Lambda I \subseteq L$. It follows from R is a subsemigroup and Theorem 2.1 that S^o is left simplistic. □

Theorem 2.4 *Let S be an abundant monoid with an RGQA transversal S^o. Then the following statements are equivalent:*

(1) *S^o is left simplistic;*

(2) *$L = S$.*

Proof. (1) $\Longrightarrow$ (2). From S is a monoid we deduce that $1 \in S^o$. Thus by S^o is left simplistic, for each $i \in I$, $i = 1 \cdot i \cdot 1 \in S^o I S^o \subseteq S^o$, and so $I \subseteq I \cap S^o = E^o$. Consequently, for every $x \in S$, $x \mathcal{R}^* e_x \in I = E^o$ and so $x \in L$. Therefore $S = L$.

(2) $\Longrightarrow$ (1). For any $i \in I$, there exists $i^* \in E^o$ such that $i^* \mathcal{L} i$. It follows from $S = L$ that for $i \in I$, there exists $h \in E^o$ such that $h \mathcal{R}^* i$. Thus $h \mathcal{R}^* i \mathcal{L} i^*$ and so by Proposition 2.2 in [20] $i \in S^o$. Therefore $I \subseteq S^o$ and $S^o I S^o \subseteq S^o S^o S^o \subseteq S^o$, thus S^o is left simplistic. □

As a consequence of Theorem 2.4 and its dual, we have

Corollary 2.1 *Let S be an abundant monoid with an RGQA transversal S^o. Then S^o is a quasi-ideal of S if and only if $S = S^o$.*

Theorem 2.5 *Let S be an abundant monoid with an RGQA transversal S^o with $E^o I \subseteq RegS$. Then the following statements are true:*

(1) *$i_k^* i_{k-1}^* \cdots i_1^* \in V_{S^o}(i_1 i_2 \cdots i_k)$, where $i_l^* \in E^o$ and $i_l^* \mathcal{L}\, i_l$, $l = 1, 2, \cdots k$;*

(2) *the semiband $\langle I \rangle$ generated by I is a subband of S.*

Dually, if S^o is an RGQA transversal of an abundant semigroup S with $\Lambda E^o \subseteq RegS$, then the semiband $\langle \Lambda \rangle$ generated by Λ is a subband of S.

Proof. (1). Certainly, this is true for $k = 1$. Now, if it is true for $k = s - 1$ and we will show that it is also true for $k = s$. Let $i_1, i_2, \cdots, i_k \in I$. Then we have $i_s^* \cdots i_2^* \in V_{S^o}(i_2 \cdots i_s)$ by the hypothesis. It follows from $E^o I \subseteq RegS$ and Lemma 1.7 that

$$i_1^* \cdot (i_2 \cdots i_s)(i_s^* \cdots i_2^*) \in E^o I \subseteq E, \quad (i_2 \cdots i_s)(i_s^* \cdots i_2^*) \cdot i_1^* \in I E^o \subseteq E.$$

Thus

$$\begin{aligned}
& (i_s^* \cdots i_2^*) i_1^* \cdot i_1^* (i_2 \cdots i_s) \cdot (i_s^* \cdots i_2^*) i_1^* \\
= \;& (i_s^* \cdots i_2^*)((i_2 \cdots i_s)(i_s^* \cdots i_2^*) i_1^*)((i_2 \cdots i_s)(i_s^* \cdots i_2^*) i_1^*) i_1^* \\
= \;& (i_s^* \cdots i_2^*)((i_2 \cdots i_s)(i_s^* \cdots i_2^*) i_1^*) i_1^* \\
= \;& (i_s^* \cdots i_2^*)(i_2 \cdots i_s)(i_s^* \cdots i_2^*) i_1^* i_1^* \\
= \;& (i_s^* \cdots i_2^*) i_1^*
\end{aligned}$$

and

$$\begin{aligned}
& i_1^* (i_2 \cdots i_s) \cdot (i_s^* \cdots i_2^*) i_1^* \cdot i_1^* (i_2 \cdots i_s) \cdot \\
= \;& i_1^* (i_1^* (i_2 \cdots i_s)(i_s^* \cdots i_2^*))(i_1^* (i_2 \cdots i_s)(i_s^* \cdots i_2^*))(i_2 \cdots i_s) \\
= \;& i_1^* (i_1^* (i_2 \cdots i_s)(i_s^* \cdots i_2^*))(i_2 \cdots i_s) \\
= \;& i_1^* (i_2 \cdots i_s),
\end{aligned}$$

and so $i_s^* \cdots i_2^* i_1^* \in V_{S^o}(i_1^* i_2 \cdots i_s)$. Therefore

$$\begin{aligned}(i_s^* \cdots i_2^* i_1^*)(i_1 i_2 \cdots i_s)(i_s^* \cdots i_2^* i_1^*) &= (i_s^* \cdots i_2^*)(i_1^* i_1)((i_2 \cdots i_s)(i_s^* \cdots i_2^*) i_1^*) \\ &= (i_s^* \cdots i_2^* i_1^*)(i_1^* i_2 \cdots i_s)(i_s^* \cdots i_2^* i_1^*) = i_s^* \cdots i_2^* i_1^*\end{aligned}$$

and

$$\begin{aligned}(i_1 i_2 \cdots i_s)(i_s^* \cdots i_2^* i_1^*)(i_1 i_2 \cdots i_s) &= i_1(i_1^* i_2 \cdots i_s)(i_s^* \cdots i_2^* i_1^*)(i_1^* i_2 \cdots i_s) \\ &= i_1(i_1^* i_2 \cdots i_s) = i_1 i_2 \cdots i_s.\end{aligned}$$

(2) To show that $\langle I \rangle$ is a band, we first notice that $V_{E^o}(a) \neq \emptyset$ for every $a \in \langle I \rangle$ by (1). Let $a = i_1 i_2 \cdots i_s \in \langle I \rangle$ and $b = i_{s+1} \cdots i_t \in \langle I \rangle$. It follows from (1) and Lemma 1.8 that $V_{S^o}(x) = V_{E^o}(x)$ for every $x \in \langle I \rangle$. Denoting $e = i_s^* \cdots i_1^*$ and $f = i_t^* \cdots i_{s+1}^*$. For each $h \in V_{S^o}(a)$, notice that $e \in V_{S^o}(a)$, we have $ah \mathcal{R}\, ae \mathcal{L}\, e \in E^o$, and so by Lemma 1.1, $ah \mathcal{L}\, eah \mathcal{R}\, e$. Similarly $ha \mathcal{L}\, ea \mathcal{R}\, e \in E^o$ and $h \mathcal{R}\, hae \mathcal{L}\, e$. It follows from $h^o hae \in E^o I \subseteq E$ that

$$hae \cdot h^o \cdot hae = h \cdot h^o hae \cdot h^o hae = h \cdot h^o hae = hae.$$

Thus hae is regular and so $hae = (ha)e \in \Lambda E^o \subseteq E$ by Lemma 1.7. That $eah = e(ah) \in E^o I \subseteq E$ is obvious. Consequently,

$$ehe = e \cdot haeah \cdot e = e \cdot hae \cdot ahe = eah \cdot e = e$$

and

$$heh = h \cdot ahe \cdot e \cdot eha \cdot h = h \cdot ae \cdot e \cdot ea \cdot h = h \cdot aea \cdot h = hah = h,$$

that is, $e \in V_{S^o}(h)$. Certainly $e \in V_{S^o}(e)$ and so $V_{S^o}(h) = V_{S^o}(e)$ since the regular elements of S^o form an orthodox semigroup. Hence $h \mathcal{D}^{E^o} e$ and therefore $V_{E^o}(a) \subseteq E^o(e)$ with $E^o(e)$ denoting the $\mathcal{D}$-class of the band E^o containing e. It is a routine matter to show that $E^o(e) \subseteq V_{E^o}(a)$ and so $V_{E^o}(a) = E^o(e)$. Similarly, $V_{E^o}(b) = E^o(f), V_{E^o}(ba) = E^o(fe)$. Therefore $V_{E^o}(a)V_{E^o}(b) \subseteq V_{E^o}(ba)$ and so $\langle I \rangle$ is indeed an orthodox semigroup. Thus $E(\langle I \rangle)$ is a band and it follows from $\langle I \rangle \supseteq E(\langle I \rangle) \supseteq I$ and $\langle I \rangle$ is the smallest subsemigroup containing I that $\langle I \rangle = E(\langle I \rangle)$ is a band. □

Definition 2.2 Let S^o be an RGQA transversal of the abundant semigroup S. Then we define S^o to be *left weakly simplistic* if $S^o I S^o \subseteq R$ and $E^o I \subseteq RegS$; *right weakly simplistic* if $S^o \Lambda S^o \subseteq L$ and $\Lambda E^o \subseteq RegS$; and *weakly simplistic* if S^o is left weakly simplistic and right weakly simplistic together.

Theorem 2.6 *Let S^o be an RGQA transversal of the abundant semigroup S. Then the following conditions are equivalent:*

(1) *S^o is left weakly simplistic;*
(2) *$E^o I \subseteq I$;*
(3) *$E^o I \subseteq RegR$;*
(4) *$S^o I \subseteq R$ and $E^o I \subseteq RegS$;*
(5) *$S^o R \subseteq R$ and $E^o I \subseteq RegS$;*
(6) *$S^o R S^o \subseteq R$ and $E^o I \subseteq RegS$;*
(7) *R is a subsemigroup of S and $E^o I \subseteq RegS$;*
(8) *I is a subband of S.*

Proof. (1) $\Longrightarrow$ (2). For any $i \in I$, there exists $i^* \in E^o$ such that $i^* \mathcal{L}\, i$. Hence for any $e^o \in E^o$, $e^o i = e^o i i^* \in S^o I S^o \subseteq R$. It follows from $E^o I \subseteq RegS$ and Lemma 1.7 that $e^o i \in E$ and thus $e^o i \in R \cap E = I$.

(2) $\Longrightarrow$ (3). This is clear.

(3) $\Longrightarrow$ (4). For any $i \in I, s^o \in S^o$, it is that $s^o i \mathcal{L}^* s^{o*} i \in E^o I \subseteq R$. Hence $s^{o*} i \mathcal{L}^* l$ for some $l \in E^o$ and consequently $s^o i \mathcal{L}^* l$. Therefore, $s^o i \in R$ by Lemma 1.5.

(4) $\Longrightarrow$ (5). For any $s^o \in S^o, x \in R$, clearly $s^o x = s^o e_x \overline{x} \lambda_x \in S^o I \cdot S^o E^o \subseteq R S^o E^o \subseteq R$.

(5) $\Longrightarrow$ (6). This is obvious.

(6) $\Longrightarrow$ (7). For any $x, y \in R, l \in E^o$ with $x \mathcal{L}^* l$, we have $xy \mathcal{L}^* ly = li_y\overline{y}\lambda_y \in S^oRS^oE^o \subseteq RE^o \subseteq R$. Hence $ly \mathcal{L}^* e^o$ for some $e^o \in E^o$ and so $xy \mathcal{L}^* ly\mathcal{L}^*e^o$. Thus $xy \in R$ by Lemma 1.5.

(7) $\Longrightarrow$ (8). Let $e, f \in I$. Then $e, f \in R$ and so $ef \in R$ by the assumption that R is a subsemigroup. It follows from Theorem 2.5 that $ef \in E$ and so $ef \in R \cap E = I$, that is, I is a band.

(8) $\Longrightarrow$ (1). Clearly, if I is a band, $E^oI \subseteq RegS$. For any $s^o, t^o \in S^o, i \in I$, there exists $s^{o*} \in E^o$ such that $s^o \mathcal{L}^* s^{o*}$ and $s^{o*}i \in E^oI \subseteq II \subseteq I$. Thus there exists $(s^{o*}i)^* \in E^o$ such that $(s^{o*}i)^* \mathcal{L}^* s^{o*}i$. Consequently,

$$s^oit^o \mathcal{L}^* s^{o*}it^o \mathcal{L}^* (s^{o*}i)^*t^o \mathcal{L}^* ((s^{o*}i)^*t^o)^* \in E^o.$$

Thus $s^oit^o \in R$ and so $S^oIS^o \subseteq R$. By Definition 2.2, S^o is left weakly simplistic. □

Combining Theorem 2.6 with its dual, we have the following result.

Theorem 2.7 *Let S^o be an RGQA transversal of the abundant semigroup S. Then the following statements are equivalent:*

(1) *S^o is weakly simplistic, that is, $S^oIS^o \subseteq R$, $S^o\Lambda S^o \subseteq L$ and $E^oI, \Lambda E^o \subseteq RegS$;*
(2) *$E^oI \subseteq I$ and $\Lambda E^o \subseteq \Lambda$;*
(3) *$E^oI \subseteq R$, $\Lambda E^o \subseteq L$ and $E^oI, \Lambda E^o \subseteq RegS$;*
(4) *$S^oI \subseteq R$, $\Lambda S^o \subseteq L$ and $E^oI, \Lambda E^o \subseteq RegS$;*
(5) *$S^oR \subseteq R$, $LS^o \subseteq L$ and $E^oI, \Lambda E^o \subseteq RegS$;*
(6) *$S^oRS^o \subseteq R$, $S^oLS^o \subseteq L$ and $E^oI, \Lambda E^o \subseteq RegS$;*
(7) *R and L are both subsemigroups of S and $E^oI, \Lambda E^o \subseteq RegS$;*
(8) *I and Λ are both subbands of S.*

Theorem 2.8 *Let S^o be an RGQA transversal of the abundant semigroup S. Then the following conditions are equivalent:*

(1) *$S^oSS^o \subseteq R$;*
(2) *$\Lambda I \subseteq R$;*
(3) *$SS^o \subseteq R$;*
(4) *$SR \subseteq R$;*
(5) *$LR \subseteq R$;*
(6) *$LI \subseteq R$;*
(7) *$LIS^o \subseteq R$;*
(8) *$LRS^o \subseteq R$;*
(9) *R is a subsemigroup of S and $LS^o \subseteq R$.*

Proof. (1) $\Longrightarrow$ (2). For any $e \in i, f \in \Lambda$, there exists $e^*, f^+ \in E^o$ such that $e \mathcal{L} e^*, f \mathcal{R} f^+$ and so $fe = e^*eff^+ \in S^oSS^o \subseteq R$.

(2) $\Longrightarrow$ (3). For any $a \in S$ and $s^o \in S^o$, it follows from $\mathcal{L}^*$ is a right congruence that

$$as^o \mathcal{L}^* \lambda_as^o = \lambda_as^{o+} \cdot s^o \in \Lambda E^os^o \subseteq \Lambda Is^o \subseteq Rs^o \subseteq R.$$

(3) $\Longrightarrow$ (4). For any $a \in S$ and $x \in R$, it is clear that $ax = axl \in SS^o \subseteq R$, where $l \in E^o$ with $x\mathcal{L}^*l$.y

(4) $\Longrightarrow$ (5). This is clear.

(5) $\Longrightarrow$ (6). This is clear.

(6) $\Longrightarrow$ (7). It follows from $\mathcal{L}^*$ is a right congruence that $RS^o \subseteq R$ and so (7) valids.

(7) $\Longrightarrow$ (8). This is clear.

(8) $\Longrightarrow$ (9).For any $x, y \in R, l \in E^o$ with $x \mathcal{L}^* l$, we have $xy \mathcal{L}^* ly = li_y\overline{y}\lambda_y \in S^oRS^oE^o \subseteq RE^o \subseteq R$. Hence $ly \mathcal{L}^* e^o$ for some $e^o \in E^o$ and so $xy \mathcal{L}^* ly\mathcal{L}^*e^o$. Thus $xy \in R$ by Lemma 1.5. It is obvious that $LS^o = LS^oS^o \subseteq LRS^o \subseteq R$.

(9) $\Longrightarrow$ (1). For any $s^o, t^o \in S^o$ and $x \in R$, it follows from R is a subsemigroup that

$$s^oxt^o = s^o \cdot i_x\overline{x}\lambda_x \cdot t^o \in s^o \cdot I \cdot \overline{x} \cdot LS^o \subseteq s^oR\,\overline{x}\,R \subseteq R,$$

and so $S^oSS^o \subseteq R$. □

Definition 2.3 We say that S^o is a
- *left quasi-ideal* of S if any one of the equivalent properties of Theorem 2.8 holds;
- *right quasi-ideal* of S if any one of the dual properties of Theorem 2.8 holds.

It is easy to see that S^o is both a left quasi-ideal and a right quasi-ideal if and only if S^o is a quasi-ideal.

Theorem 2.9 *The following statements are equivalent:*
(1) *S^o is left simplistic;*
(2) *$S^oIS^o \subseteq R$ and S^o is a right quasi-ideal;*
(3) *$S^oIS^o \subseteq R$, $S^o\Lambda S^o \subseteq L$ and $S^oR \subseteq L$.*

Proof. (1) $\Longrightarrow$ (2). For any $a^o, b^o \in S^o$ and $c \in S$, $a^ocb^o = a^o(cb^o) \in S^oS \subseteq L$ by Theorem 2.6 and S^o is a right quasi-ideal. That $S^oIS^o \subseteq S^o \subseteq R$ is obvious.

(2) $\Longrightarrow$ (3). We only need notice that $S^oR \subseteq S^oRE^o \subseteq S^oSS^o \subseteq L$.

(3) $\Longrightarrow$ (1). For any $a^o, b^o \in S^o$ and $i \in I$, there exists $a^{o*} \in E^o$ such that $a^{o*} \ \mathcal{L}^* \ a^o$, we have $a^{o*}i \in E^oI \subseteq S^oR \subseteq L$, and so $a^{o*}i \in E(L) = \Lambda$. Thus $a^oib^o = a^o(a^{o*}i)b^o \in a^o\Lambda b^o \subseteq L$ and therefore $a^oib^o \in S^oIS^o \cap L \subseteq R \cap L \subseteq S^o$, that is, S^o is left simplistic. □

Theorem 2.10 *Let S be an abundant semigroup with an RGQA transversal S^o. Then the following statements are equivalent:*
(1) *S^o is a left quasi-ideal and weakly multiplicative;*
(2) *$\Lambda I \subseteq I$.*

Proof. (1) $\Longrightarrow$ (2). If S^o is a left quasi-ideal, it follows from Theorem 2.8 that, $\Lambda I \subseteq R$. Thus for any $\lambda \in \Lambda, i \in I, \lambda i = \lambda i(\lambda i)^o(\lambda i)^{oo}$ for some $(\lambda i)^o \in V_{S^o}(\lambda i)$, $(\lambda i)^{oo} \in V_{S^o}((\lambda i)^o)$ by λi is regular. It follows from S^o is weakly multiplicative that $(\lambda i)^o \in E^o$. Thus $(li)^{oo} \in E^o$ and $\lambda i = \lambda i(\lambda i)^o(\lambda i)^{oo} \in IE^o \subseteq E$. Therefore $\lambda i \in I$.

(2) $\Longrightarrow$ (1). If $\Lambda I \subseteq I$, then $V_{S^o}(\Lambda I) \subseteq V_{S^o}(I) \subseteq E^o$. Thus S^o is weakly multiplicative. Clearly $\Lambda I \subseteq I \subseteq R$ and so S^o is a left quasi-ideal by Theorem 2.8. □

Let S be an abundant semigroup and S^o a weakly simplistic RGQA transversal of S. Then by Theorem 2.7, R is an abundant semigroup with an RGQA transversal S^o with $E(R) = I$ is a band. Consequently, R is quasi-adequate and for every $x \in R$ and $\lambda_x \in \Lambda_x$, there exists $\overline{x}^* \in E^o$ such that $\lambda_x = \overline{x}^*$. For $a \in R$, the $\mathcal{R}^*$-class of R containing a will be denoted by R^*_a and we define $K(a) = K(b)$ if $R^*_a = R^*_b$ and $C_{S^o}(a) = C_{S^o}(b)$ for $a, b \in R$. The relation $\mathcal{K}$, defined on R by $(a, b) \in \mathcal{K}$ if and only if $K(a) = K(b)$, is an equivalence relation on R. By Theorem 2.7, Λ is a band with an RGQA transversal $E^o = E(S^o)$ and each element in Λ is $\mathcal{R}$-related to some element in E^o. For each $e \in \Lambda$, let $\phi_e : R \to R$ be a mapping defined by $\phi_e x = ex\overline{ex}^*$ for a given $\overline{ex}^*$, where $\overline{ex} \in C_{S^o}(ex)$. For each $y \in R$, let $\psi_y : \Lambda \to \Lambda$ be a mapping defined by $f\psi_y = \lambda_{fy}$ for a given $\lambda_{fy} \in \Lambda_{fy}$. For $e \in \Lambda$ and $x \in R$, let $LP(E^o) = \{(e, x) \in \Lambda \times R : e \ \mathcal{R} \ e^+ \ \mathcal{L}^* \ x$ for some $e^+ \in E^o\}$. Then we have the following properties associated with ϕ and ψ.

Proposition 2.1 *Let S be an abundant semigroup with a weakly simplistic RGQA transversal S^o and R, Λ, ϕ and ψ be defined as above. Then for any $e, f \in \Lambda$ and $x, y \in R$:*
(1) *$C_{S^o}(\phi_e y) = C_{S^o}(ey)$;*
(2) *$x(\phi_e y) \ \mathcal{R}^* \ xey$ and $(e\psi_y)f \ \mathcal{L}^* \ eyf$;*
(3) there exists $\lambda_{(\phi_e x)} \in E^o$ such that $e\psi_x \ \mathcal{R} \ \lambda_{(\phi_e x)}$;
(4) $(\phi_e x)(\phi_{(e\psi_x)f} y) \ \mathcal{K} \ \phi_e(x(\phi_f y))$ and $(e\psi_{x(\phi_f y)})(f\psi_y) \ \mathcal{L}^* \ ((e\psi_x)f)\psi_y$;
(5) $e^+(\phi_e x) = \phi_e x$ and $(f\psi_y)\lambda_y = f\psi_y$ for any $e^+ \in E^o$ with $e^+ \ \mathcal{R} \ e$ and $\lambda_y \in \Lambda_y$;
(6) *if $e' \in \Lambda$, $y' \in R$ with $e' \ \mathcal{L} \ e \ \mathcal{R} \ e^+ \in E^o$ and $y \ \mathcal{K} \ y'$, then $C_{S^o}(\phi_e y) = C_{S^o}(\phi_{e'} y')$;*
(7) *if $e', f' \in \Lambda$ and $x', y' \in R$ with $f' \ \mathcal{L} \ f, x \ \mathcal{L} \ e^+$ and $x \ \mathcal{K} \ x'$ such that $(e', x'), (f, y), (f', y') \in LP(E^o)$, then $x(\phi_e y) \ \mathcal{R} \ x'(\phi_{e'} y')$ and $(e\psi_y)f \ \mathcal{L} \ (e'\psi_{y'})f'$;*
(8) for $\phi_g x = gx\overline{gx}^*$, if $g \in E^o$ or $x \ \mathcal{L}^* \ g^+ \ \mathcal{R} \ g$ for some $g^+ \in E^o$ with $x(\phi_g x) \ \mathcal{K} \ x$, then $(\phi_g x)\lambda_x = \phi_g x$;
(9) if $h\psi_x \ \mathcal{L} \ r\psi_x$ and $y(\phi_h x) = z(\phi_r x)$, then $h\psi_{i_x} \ \mathcal{L} \ r\psi_{i_x}$ and $y(\phi_h i_x) = z(\phi_r i_x)$.

Proof. (1). Since $\phi_e y = ey\overline{ey}^* = i_{ey}\overline{ey}\lambda_{ey}\overline{ey}^* = i_{ey}\overline{ey}\ \overline{ey}^*$, we have $\overline{ey} \in C_{S^o}(\phi_e y)$. Consequently, $C_{S^o}(\phi_e y) \cap C_{S^o}(ey) \neq \emptyset$ and $C_{S^o}(\phi_e y) = C_{S^o}(ey)$.

(2). It follows from $xey\overline{ey}^*\lambda_{ey} = xey\lambda_{ey} = xey$ that $xey\overline{ey}^*\ \mathcal{R}^*\ xey$ and so $x(\phi_e y) = xey\overline{ey}^*\ \mathcal{R}^*\ xey$. Similarly, $(e\psi_y)f\ \mathcal{L}^*\ eyf$ since $ey\ \mathcal{L}^*\ \lambda_{ey} = e\psi_y$ and $\mathcal{L}^*$ is a right congruence.

(3). Since $\phi_e x = ex\overline{ex}^* = i_{ex}\overline{ex}\ \overline{ex}^*$ and $\overline{ex}^*\ \mathcal{R}^*\ \lambda_{ex} \in \Lambda_{ex}$, take $e\psi_x = \lambda_{ex}$ and $\lambda_{(\phi_e x)} = \overline{ex}^* \in E^o$, then $e\psi_x = \lambda_{ex}\ \mathcal{R}^*\ \overline{ex}^* = \lambda_{(\phi_e x)}$.

(4). It follows from (2) that

$$(\phi_e x)(\phi_{(e\psi_x)f}y)\ \mathcal{R}^*\ (\phi_e x)((e\psi_x)fy) = ex\overline{ex}^*\lambda_{ex}fy = ex\lambda_{ex}fy = exfy$$

and $\phi_e(x(\phi_f y))\ \mathcal{R}^*\ ex(\phi_f y)\ \mathcal{R}^*\ exfy$. Thus $(\phi_e x)(\phi_{(e\psi_x)f}y)\ \mathcal{R}^*\ \phi_e(x(\phi_f y))$.

By the definition of ϕ and ψ, we have

$$\begin{aligned} &(\phi_e x)(\phi_{(e\psi_x)f}y) = ex\overline{ex}^*(\phi_{(\lambda_{ex})f}y)\\ = \;& ex\overline{ex}^*\cdot\lambda_{ex}fy\cdot\overline{\lambda_{ex}fy}^* = ex\overline{ex}^*\lambda_{ex}\cdot fy\cdot\overline{\lambda_{ex}fy}^*\\ = \;& exfy\cdot\overline{\lambda_{ex}fy}^* = i_{exfy}\overline{exfy}\lambda_{exfy}\cdot\overline{\lambda_{ex}fy}^*. \end{aligned}$$

It follows from Λ is a band that $\lambda_{exfy}\cdot\overline{\lambda_{ex}fy}^* \in \Lambda E^o \subseteq \Lambda$. Since $\mathcal{L}^*$ is a right congruence, we have $\lambda_{exfy}\ \mathcal{L}^*\ exfy\ \mathcal{L}^*\ \lambda_{ex}fy\ \mathcal{L}^*\ \lambda_{(\lambda_{ex}fy)}$. Thus $\overline{exfy}^*\ \mathcal{R}^*\ \lambda_{exfy}\ \mathcal{L}^*\ \lambda_{(\lambda_{ex}fy)}\ \mathcal{R}^*\ \overline{\lambda_{ex}fy}^*$ and by Lemma 1.1, $\lambda_{exfy}\ \mathcal{R}^*\ \lambda_{exfy}\overline{\lambda_{ex}fy}^*\ \mathcal{L}^*\ \overline{\lambda_{ex}fy}^*$. Consequently, $\lambda_{exfy}\overline{\lambda_{ex}fy}^*\ \mathcal{R}^*\ \overline{exfy}^*$ and so $\overline{exfy} \in C_{S^o}((\phi_e x)(\phi_{(e\psi_x)f}y))$. For a similar proof, we have $\overline{exfy} \in C_{S^o}(\phi_e(x(\phi_f y)))$. Therefore $(\phi_e x)(\phi_{(e\psi_x)f}y)\ \mathcal{K}\ \phi_e(x(\phi_f y))$. Similarly,

$$(e\psi_{x(\phi_f y)})(f\psi_y)\ \mathcal{L}^*\ (ex(\phi_f y))\lambda_{fy} = exfy\overline{fy}^*\lambda_{fy} = exfy\ \mathcal{L}^*\ (e\psi_x)fy\ \mathcal{L}^*\ ((e\psi_x)f)\psi_y.$$

(5). For any $e^+ \in E^o$ with $e^+\ \mathcal{R}\ e$ and $\lambda_y \in \Lambda_y$, we have $e^+(\phi_e x) = e^+ex\overline{ex}^* = ex\overline{ex}^* = \phi_e x$ and $(f\psi_y)\lambda_y = \lambda_{fy}\lambda_y = \lambda_{fy} = (f\psi_y)$.

(6). Since $y\ \mathcal{K}\ y'$, we have $y = y'(\overline{y}^{*\prime}\lambda_y)$, where $\overline{y}^{*\prime}\lambda_y \in E^oE^o \subseteq E^o$. It follows from $e'\ \mathcal{L}\ e\ \mathcal{R}\ e^+ \in E^o$ and Lemma 1.1 that $e'\ \mathcal{R}\ e'e^+\ \mathcal{L}\ e^+$ with $e'e^+ \in \Lambda E^o \subseteq \Lambda$ and so $e = e^+e'$. Thus for any $s \in C_{S^o}(e'y') = C_{S^o}(\phi_{e'}y')$, we have

$$ey = e^+e'\cdot y'\overline{y}^{*\prime}\lambda_y = e^+(i_{e'y'}s\lambda_{e'y'})(\overline{y}^{*\prime}\lambda_y) = (e^+i_{e'y'})s(\lambda_{e'y'}\overline{y}^{*\prime}\lambda_y).$$

Since $(e'y'\cdot\overline{y}^{*\prime}\lambda_y)\lambda_{y'} = e'y'(\overline{y}^{*\prime}\lambda_y\lambda_{y'}) = e'y'\lambda_{y'} = e'y'$, it follows from $e'y'\ \mathcal{L}^*\ \lambda_{e'y'}$ that $\lambda_{e'y'}(\overline{y}^{*\prime}\lambda_y)\lambda_{y'} = \lambda_{e'y'}$. Thus $\lambda_{e'y'}\ \mathcal{R}\ \lambda_{e'y'}(\overline{y}^{*\prime}\lambda_y)$ and $\lambda_{e'y'}(\overline{y}^{*\prime}\lambda_y)\ \mathcal{R}\ \lambda_{e'y'}\ \mathcal{R}\ s^*\ \mathcal{L}\ s$. Similarly, $e'\cdot e^+i_{e'y'} = i_{e'y'}$ and $e^+i_{e'y'}\ \mathcal{L}\ i_{e'y'}\ \mathcal{L}\ s^+\ \mathcal{R}\ s$. Therefore $s \in C_{S^o}(ey) = C_{S^o}(\phi_e y)$ and $C_{S^o}(\phi_e y) = C_{S^o}(\phi_{e'}y')$.

(7). Since $x\ \mathcal{K}\ x'$, we have $x = i_x\overline{x}\lambda_x, x' = i_{x'}\overline{x}\lambda_{x'}$ with $i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^*, i_{x'}\ \mathcal{L}\ \overline{x}^{+\prime}, \lambda_{x'}\ \mathcal{R}\ \overline{x}^{*\prime}$ and $i_x\ \mathcal{R}^*\ x\ \mathcal{R}\ x'\ \mathcal{R}^*\ i_{x'}$, $\overline{x}^+\ \mathcal{R}^*\ \overline{x}\ \mathcal{R}^*\ \overline{x}^{+\prime}$. It follows from $i_{x'}\ \mathcal{R}\ i_x\ \mathcal{L}\ \overline{x}^+\ \mathcal{R}\ \overline{x}^{+\prime}\ \mathcal{L}\ i_{x'}$ and Lemma 1.1 that $i_x\overline{x}^{+\prime} = i_{x'}$. Thus

$$x = i_x\overline{x}\lambda_x = i_x(\overline{x}^{+\prime}x'\overline{x}^{*\prime}\lambda_x) = (i_x\overline{x}^{+\prime})x'(\overline{x}^{*\prime}\lambda_x) = i_{x'}x'(\overline{x}^{*\prime}\lambda_x) = x'(\overline{x}^{*\prime}\lambda_x).$$

It follows from $x\ \mathcal{L}^*\ e^+\ \mathcal{R}\ e\ \mathcal{L}\ e'\ \mathcal{R}\ e'^+\ \mathcal{L}^*\ x'$ and Lemma 1.1 that $x\ \mathcal{L}^*\ e'e^+\ \mathcal{R}\ e'$. Thus $x'e', xe' = x'\overline{x}^{*\prime}\lambda_x e' \in R^*_{x'} \cap L_{e'}$. It is easy to see

$$x'e' = i_{x'}\cdot\overline{x}\lambda_{x'}e' \ \text{ with }\ \lambda_{x'}e' \in E^o\Lambda \subseteq \Lambda \ \text{ and }\ \lambda_{x'}e'\ \mathcal{R}\ \lambda_{x'}\ \mathcal{R}\ \overline{x}^{*\prime}$$

$$xe' = i_x\cdot\overline{x}\lambda_x e' \ \text{ with }\ \lambda_x e' \in E^o\Lambda \subseteq \Lambda \ \text{ and }\ \lambda_x e'\ \mathcal{R}\ \lambda_x\ \mathcal{R}\ \overline{x}^*.$$

Thus $\overline{x} \in C_{S^o}(x'e') \cap C_{S^o}(xe')$ and so $xe' = x'\overline{x}^{*\prime}\lambda_x e' = x'\lambda_{x'}e' = x'e'$.

From (2) and the proof of (6) we deduce that $x(\phi_e y)\ \mathcal{R}^*\ xey\ \mathcal{R}^*\ x\cdot e^+i_{e'y'} = (xe^+)i_{e'y'} = xi_{e'y'}\ \mathcal{R}^*\ x(e'y') = (xe')y' = (x'e')y'\ \mathcal{R}^*\ x'(\phi_{e'}y')$. Similarly, $(e\psi_y)f\ \mathcal{L}\ (e'\psi_{y'})f'$.

(8). If $g \in E^o$, then $gx \in E^oR \subseteq S^oR \subseteq R$ and so $gx\ \mathcal{L}^*\ \overline{gx}\ \mathcal{L}^*\ \overline{gx}^*$ for some $\overline{gx} \in C_{S^o}(gx)$. Thus $\phi_g x = gx\overline{gx}^* = gx$ and $(\phi_g x)\lambda_x = gx\lambda_x = gx = \phi_g x$.

If $x(\phi_g x)\ \mathcal{K}\ x$ then $x(\phi_g x) = x(\overline{x}^*\lambda_\Delta)$, where $\lambda_\Delta \in \Lambda_{x(\phi_g x)}$. Since $x \in R$, we may assume $x\ \mathcal{L}^*\ \overline{x}$ and so $x(\phi_g x)\ \mathcal{L}^*\ \lambda_\Delta\ \mathcal{R}\ \overline{x}^{*\prime}\ \mathcal{L}^*\ \overline{x}^*\ \mathcal{L}^*\ \overline{x}\ \mathcal{L}^*\ x\ \mathcal{L}^*\ g^+$. It follows from Lemma 1.1 that $\lambda_\Delta\ \mathcal{L}\ \overline{x}^*\lambda_\Delta\ \mathcal{R}\ \overline{x}^*$ and

$\lambda_\Delta \ \mathcal{L}\ g^+\overline{x}^*\lambda_\Delta \ \mathcal{R}\ g^+$ with $g^+\overline{x}^*\lambda_\Delta \in E^oE^oE^o \subseteq E^o$ since $x(\phi_g x) \in RR \subseteq R$. From $x \ \mathcal{L}^*\ g^+$ we deduce that $\phi_g x = g^+(\phi_g x) = g^+\overline{x}^*\lambda_\Delta \in E^o$ and so $gx \ \mathcal{R}^*\ gx\overline{gx}^* = \phi_g x = g^+\overline{x}^*\lambda_\Delta \ \mathcal{R}\ g^+$. Since $gx = (gx\overline{gx}^*)\lambda_{gx} = \phi_g x\lambda_{gx} \in E^o\Lambda \subseteq \Lambda$, then g^+ is an inverse of $gx \in \Lambda$ in E^o. Thus we may take $\overline{gx} = \overline{gx}^* = g^+$. Consequently

$$(\phi_g x)\lambda_x = (gx\overline{gx}^*)\lambda_x = (gxg^+)\lambda_x = (gx)\lambda_x = g(x\lambda_x) = gx = gxg^+ = gx\overline{gx}^* = \phi_g x.$$

(9). If $h\psi_x \ \mathcal{L}\ r\psi_x$, that is $\lambda_{hx} \ \mathcal{L}\ \lambda_{rx}$, it follows from $\mathcal{L}^*$ is a right congruence that $\lambda_{hx} \cdot \overline{x}^* \ \mathcal{L}\ \lambda_{rx} \cdot \overline{x}^*$. Consequently, $hx\overline{x}^* \ \mathcal{L}^*\ rx\overline{x}^*$ and so $\lambda_{(hx\overline{x}^*)} \ \mathcal{L}\ \lambda_{(rx\overline{x}^*)}$. That is $\lambda_{(hi_x\overline{x})} \ \mathcal{L}\ \lambda_{(ri_x\overline{x})}$ since $x\overline{x}^* = i_x\overline{x}\lambda_x\overline{x}^* = i_x\overline{x}$. Thus $\lambda_{(hi_x)}\overline{x} \ \mathcal{L}^*\ \lambda_{(ri_x)}\overline{x}$ and so $\lambda_{(hi_x)}\overline{x}^+ \ \mathcal{L}^*\ \lambda_{(ri_x)}\overline{x}^+$. Hence $hi_x\overline{x}^+ \ \mathcal{L}^*\ ri_x\overline{x}^+$, that is $hi_x \ \mathcal{L}^*\ ri_x$. Therefore $\lambda_{(hi_x)} \ \mathcal{L}\ \lambda_{(ri_x)}$ and $h\psi_{ix} \ \mathcal{L}\ r\psi_{ix}$.

Since $\lambda_{hx} \ \mathcal{L}\ \lambda_{rx}$, we have $hx \ \mathcal{L}^*\ rx$. It follows from $x \ \mathcal{L}^*\ y \Leftrightarrow \Lambda_x = \Lambda_y$ that $\Lambda_{hx} = \Lambda_{rx}$ and so for any $\lambda_{hx} \in \Lambda_{hx}$, we have $\lambda_{hx} \in \Lambda_{rx}$. Let $\phi_h x = hx\overline{hx}^*$, $\lambda_{hx} \ \mathcal{R}\ \overline{hx}^*$ and $\phi_r x = rx\overline{rx}^*$ with $\overline{rx}^* \ \mathcal{R}\ \lambda_{hx}$. Since $y(\phi_h x) = z(\phi_r x)$, that is $yhx\overline{hx}^* = zrx\overline{rx}^*$, postmultiplying by λ_{hx}, we acquire

$$yhx = yhx\overline{hx}^*\lambda_{hx} = zrx\overline{rx}^*\lambda_{hx} = zrx.$$

Thus $yhi_x = zri_x$ since $x \ \mathcal{R}^*\ i_x$. Since $hi_x \ \mathcal{L}^*\ ri_x$, similar to the above proof, we can choose an element $\overline{hi_x}^* = \overline{ri_x}^*$ satisfying $yhi_x\overline{hi_x}^* = zri_x\overline{ri_x}^*$. Therefore $y(\phi_h i_x) = z(\phi_r i_x)$. □

3. A Structure Theorem

In this section we will establish a structure theorem for abundant semigroups with weakly simplistic RGQA transversals.

In what follows Λ denotes a band with an RGQA (in fact, a band) transversal B and each element in Λ is $\mathcal{R}$-related to some element in B and R denotes a quasi-adequate semigroup with an RGQA transversal S^o. Suppose that the band of idempotents of S^o is isomorphic to B. For the sake of simplicity, we coincide $E(S^o)$ with B, and denote it by E^o.

For each $e \in \Lambda$, let $\phi_e : R \to R$ be a mapping given by $x \to \phi_e x$ and for each $y \in R$, let $\psi_y : \Lambda \to \Lambda$ be a mapping given by $f \to f\psi_y$. Then the pair of mappings (ϕ, ψ) is said to be *normal* if for any $e, f \in \Lambda$ and $x, y \in R$, the following conditions are satisfied:

(1) there exists $\lambda_{\phi_e x} \in E^o$ such that $e\psi_x \ \mathcal{R}\ \lambda_{\phi_e x}$;

(2) $(\phi_e x)(\phi_{(e\psi_x)f} y) \ \mathcal{K}\ \phi_e(x(\phi_f y))$ and $(e\psi_{x(\phi_f y)})(f\psi_y) \ \mathcal{L}\ ((e\psi_x)f)\psi_y$;

(3) $e^+(\phi_e x) = \phi_e x$ and $(f\psi_y)\lambda_y = f\psi_y$ for some $e^+ \in E^o$ with $e^+ \ \mathcal{R}\ e$ and $y^o \in V_{S^o}(y)$;

(4) if $e' \in \Lambda$, $y' \in R$ with $e' \ \mathcal{L}\ e \ \mathcal{R}\ e^+ \in E^o$ and $y \ \mathcal{K}\ y'$, then $C_{S^o}(\phi_e y) = C_{S^o}(\phi_{e'} y')$;

(5) if $e', f' \in \Lambda$ and $x', y' \in R$ with $f' \ \mathcal{L}\ f, x \ \mathcal{L}\ e^+$ and $x \ \mathcal{K}\ x'$ such that $(e', x'), (f, y), (f', y') \in LP(E^o)$, then $x(\phi_e y) \ \mathcal{R}\ x'(\phi_{e'} y')$ and $(e\psi_y)f \ \mathcal{L}\ (e'\psi_{y'})f'$;

(6) for $\phi_g x = gx\overline{gx}^*$, if $g \in E^o$ or $x \ \mathcal{L}^*\ g^+ \ \mathcal{R}\ g$ for some $g^+ \in E^o$ with $x(\phi_g x) \ \mathcal{K}\ x$, then $(\phi_g x)\lambda_x = \phi_g x$;

(7) if $h\psi_x \ \mathcal{L}\ r\psi_x$ and $y(\phi_h x) = z(\phi_r x)$, then $h\psi_{i_x} \ \mathcal{L}\ r\psi_{i_x}$ and $y(\phi_h i_x) = z(\phi_r i_x)$.

The quadruple $(R, \Lambda; \phi, \psi)$ is said to be *permissible* if the pair of mappings (ϕ, ψ) is normal.

Theorem 3.1 *Let $(R, \Lambda; \phi, \psi)$ be a permissible quadruple. Define a multiplication on the set*

$$\Gamma = R/\mathcal{K}\ |\times|\Lambda/\mathcal{L} = \{(K(x), L_e) \in R/\mathcal{K} \times \Lambda/\mathcal{L} : (\exists e^+ \in E^o)\ e \ \mathcal{R}\ e^+ \ \mathcal{L}^*\ x\}$$

by

$$(K(x), L_e)(K(y), L_f) = (K(x(\phi_e y)), L_{(e\psi_y)f}).$$

Then Γ is an abundant semigroup with a weakly simplistic RGQA transversal isomorphic to S^o.

Conversely, every abundant semigroup with a weakly simplistic RGQA transversal can be constructed in this way.

We first notice a simple but useful result that, if $x \in R$, then $x \ \mathcal{L}^*\ \overline{x}'$ for some $\overline{x}' \in C_{S^o}(x)$. For $x \in R$, we have $x = i_x\overline{x}\lambda_x, i_x \ \mathcal{L}\ \overline{x}^+, \lambda_x \ \mathcal{R}\ \overline{x}^*$ for some $\overline{x}^+, \overline{x}^* \in E^o$ with $\lambda_x \in E^o$. Then $i_x \cdot \overline{x}^+x = i_x\overline{x}^+ \cdot x = i_x x = x$ and

$\overline{x}^+x\ \mathcal{L}^*\ x$. Clearly, $\overline{x}^+x = \overline{x}^+i_x\overline{x}\lambda_x = \overline{x}\lambda_x$ and $\overline{x}\lambda_x \cdot \overline{x}^* = \overline{x} \cdot \lambda_x\overline{x}^* = \overline{x}\overline{x}^* = \overline{x}$. Thus $x\ \mathcal{L}^*\ \overline{x}^+x = \overline{x}\lambda_x\ \mathcal{R}^*\ \overline{x}\ \mathcal{R}^*\ \overline{x}^+$ and $\overline{x}\lambda_x \in S^oE^o \subseteq S^o$. It follows from $x = i_x\overline{x}\lambda_x = i_x(\overline{x}\lambda_x)\lambda_x$ and $i_x\ \mathcal{L}\ \overline{x}^+\ \mathcal{R}^*\ \overline{x}\lambda_x$, $\lambda_x \in E^o$ that $\overline{x}\lambda_x \in C_{S^o}(x)$ and $x\ \mathcal{L}^*\ \overline{x}\lambda_x$. In the following, we identity $\overline{x}' = \overline{x}\lambda_x$ with $\overline{x}$ for $x \in R$ and in fact, we assume $x\ \mathcal{L}^*\ \overline{x}$.

Obviously the definition of Γ is not dependent on the choice of x and e. In fact, if $e_1\ \mathcal{L}\ e$ and $x_1 \in K(x)$, then by Remark 1 in [20], $x_1 = xh$ for some $h \in E^o$ and $\overline{x}^*\ \mathcal{R}\ h\ \mathcal{L}^*\ x_1$. It follows from $e_1e^+\ \mathcal{L}\ e^+\ \mathcal{L}^*\ x\ \mathcal{L}^*\ \overline{x}\ \mathcal{L}^*\ \overline{x}^*\ \mathcal{R}\ h$ and Lemma 1.1 that $e_1e^+\ \mathcal{R}\ e_1e^+h\ \mathcal{L}\ h$ with $e_1e^+h \in \Lambda E^oE^o \subseteq \Lambda$ since Λ is a band. Furthermore $e_1e^+h\ \mathcal{L}\ h \in E^o$ implies that $e_1e^+h \in I$ and so $e_1e^+h \in I \cap \Lambda = E^o$. Notice that $e_1\ \mathcal{R}\ e_1e^+h\ \mathcal{L}^*\ x_1$ and so $(K(x_1), L_{e_1}) \in \Gamma$ with $(K(x), L_e) = (K(x_1), L_{e_1})$.

Lemma 3.1 Γ *is a semigroup.*

Proof. Let $(K(x), L_e),\ (K(y), L_h) \in \Gamma$. Then there exist $e^+, h^+ \in E^o$ such that $e\ \mathcal{R}\ e^+\ \mathcal{L}^*\ x$ and $h\ \mathcal{R}\ h^+\ \mathcal{L}^*\ y$. We first prove that $(K(x(\phi_e y)), L_{(e\psi_y)f}) \in \Gamma$. By conditions (1) and (3), there exists $\lambda_{(\phi_e x)} \in E^o$ and so

$$x(\phi_e y)\ \mathcal{L}^*\ e^+(\phi_e y) = \phi_e y\ \mathcal{L}^*\ \lambda_{(\phi_e x)} \in E^o$$

and

$$(e\psi_y)h\ \mathcal{R}^*\ (e\psi_y)h^+ = (e\psi_y)\ \mathcal{R}^*\ \lambda_{(\phi_e x)} \in E^o.$$

Hence $(K(x(\phi_e y)), L_{(e\psi_y)f}) \in \Gamma$.

Now we prove that the multiplication on Γ is not dependent on the choice of x, e, y and h. If

$$(K(x), L_e) = (K(x'), L_{e'}) \quad \text{and} \quad (K(y), L_h) = (K(y'), L_{h'}),$$

then

$$(K(x), L_e)\ (K(y), L_h) = (K(x(\phi_e y)), L_{(e\psi_y)h})$$

and

$$(K(x'), L_{e'})\ (K(y'), L_{h'})\ = (K(x'(\phi_{e'} y')), L_{(e'\psi_{y'})h'}).$$

We shall prove that $C_{S^o}(x(\phi_e y)) = C_{S^o}(x'(\phi_{e'} y'))$. Since $(K(x), L_e) = (K(x'), L_{e'})$, $x\ \mathcal{K}\ x'$ and $e\ \mathcal{R}\ e^+\ \mathcal{L}^*\ x\ \mathcal{R}^*\ x'\ \mathcal{L}^*\ e'^+\ \mathcal{R}\ e'\ \mathcal{L}\ e$. Then $e'e^+ \in L_{e^+} \cap R_{e'}$ and $e^+e' = e$. It is easy to see that $x = x'e'e^+$ and $e'e^+e = e'$. Similarly, $y = y'h'h^+$. Thus $x(\phi_e y) = (x'e'e^+)(\phi_e(y'h'h^+)) = x'(\phi_{(e'e^+e)}y')(h'h^+) = x'(\phi_{e'} y')(h'h^+)$ and so $C_{S^o}(x(\phi_e y)) \cap C_{S^o}(x'(\phi_{e'} y')) \neq \emptyset$. Consequently, $C_{S^o}(x(\phi_e y)) = C_{S^o}(x'(\phi_{e'} y'))$. It follows from (4),(5) that $x(\phi_e y)\ \mathcal{R}\ x'(\phi_{e'} y')$ and $(e\psi_y)f\ \mathcal{L}\ (e'\psi_{y'})f'$. Hence

$$(K(x(\phi_e y)), L_{(e\psi_y)h}) = (K(x'(\phi_{e'} y')), L_{(e'\psi_{y'})h'})$$

and the multiplication on Γ is well-defined.

For any $a = (K(x), L_e),\ b = (K(x_1), L_{e_1}),\ c = (K(x_2), L_{e_2}) \in \Gamma$, then by (2)

$$\begin{aligned}(ab)c &= (K(x(\phi_e x_1)), L_{(e\psi_{x_1})e_1})(K(x_2), L_{e_2})\\ &= (K(x(\phi_e x_1)(\phi_{(e\psi_{x_1})e_1} x_2)), L_{(((e\psi_{x_1})e_1)\psi_{x_2})e_2})\\ &= (K(x\phi_e(x_1(\phi_{e_1} x_2))), L_{(((e\psi_{x_1})e_1)\psi_{x_2})e_2})\end{aligned}$$

and

$$\begin{aligned}a(bc) &= (K(x), L_e)(K(x_1(\phi_{e_1} x_2)), L_{(e_1\psi_{x_2})e_2})\\ &= (K(x\phi_e(x_1(\phi_{e_1} x_2))), L_{(e\psi_{x_1(\phi_{e_1} x_2)})((e_1\psi_{x_2})e_2)})\\ &= (K(x\phi_e(x_1(\phi_{e_1} x_2))), L_{(((e\psi_{x_1})e_1)\psi_{x_2})e_2}).\end{aligned}$$

Therefore $(ab)c = a(bc)$ and Γ is a semigroup. □

Lemma 3.2 Γ *is an abundant semigroup.*

Proof. Let $(K(x), L_e) \in \Gamma$. We first show that $(K(x), L_e) \in E(\Gamma)$ if and only if $\phi_e x = e^+ \mathcal{L}\, e\psi_x$, where $e^+ \in E^o$ and $e\, \mathcal{R}\, e^+ \mathcal{L}^*\, x$. If $(K(x), L_e) \in E(\Gamma)$, then

$$(K(x), L_e)(K(x), L_e) = (K(x(\phi_e x)), L_{(e\psi_x)e}) = (K(x), L_e).$$

Hence $K(x(\phi_e x)) = K(x), L_{(e\psi_x)e} = L_e$ and so $x(\phi_e x)\, \mathcal{K}\, x$. Consequently, $x(\phi_e x) = x(\overline{x}^* \lambda_\Delta)$, where $\lambda_\Delta \in \Lambda_{x(\phi_e x)}$. Thus $x = x(\phi_e x)\lambda_x = x(\overline{x}^* \lambda_\Delta)\lambda_x$ and $x\, \mathcal{L}^*\, e^+$ implies that $e^+(\phi_e x) = \phi_e x = e^+(\overline{x}^* \lambda_\Delta) \in E^o E^o E^o \subseteq E^o$. Therefore by (6), $e^+ = e^+(\phi_e x)\lambda_x = e^+(\phi_e x) = \phi_e x$. Since $\mathcal{L}^*$ is a right congruence, $(e\psi_x)e\, \mathcal{L}\, e$ implies that $(e\psi_x)ee^+\, \mathcal{L}\, ee^+$, that is $(e\psi_x)e^+\, \mathcal{L}\, e^+$. It follows from $x\, \mathcal{L}^*\, e^+$ and $e^+ \in E^o$ that $(e\psi_x)e^+ = e\psi_x$. Therefore $e\psi_x\, \mathcal{L}\, e^+$.

Conversely, for $(K(x), L_e) \in \Gamma$, if $\phi_e x = e^+ \mathcal{L}\, e\psi_x$, then

$$\begin{aligned}(K(x), L_e)(K(x), L_e) &= (K(x(\phi_e x)), L_{(e\psi_x)e}) \\ &= (K(xe^+), L_{e^+e}) = (K(x), L_e).\end{aligned}$$

Denoting $u = (K(i_x), L_{\overline{x}^+}), v = (K(\lambda_x), L_e)$, where $x = i_x \overline{x} \lambda_x, i_x\, \mathcal{L}\, \overline{x}^+$ and $\lambda_x\, \mathcal{R}\, \overline{x}^*$. Then certainly $u, v \in E(\Gamma)$ and we shall show that $u\, \mathcal{R}^*\, (K(x), L_e)\, \mathcal{L}^*\, v$. Computing

$$\begin{aligned}(K(i_x), L_{\overline{x}^+})(K(x), L_e) &= (K(i_x \overline{x}^+ x), L_{(\overline{x}^+\psi_x)e}) \\ &= (K(x), L_{\overline{x}^+ xe}) \\ &= (K(x), L_{\overline{x}\lambda_x e}) \\ &= (K(x), L_{\lambda_x e}) \quad (\text{since } \overline{x}^* \lambda_x e = \lambda_x e \text{ and } \overline{x}\, \mathcal{L}^*\, \overline{x}^*) \\ &= (K(x), L_e). \qquad (\text{since } \lambda_x\, \mathcal{L}\, e^+\, \mathcal{R}\, e)\end{aligned}$$

Suppose that $(K(y), L_h), (K(z), L_r) \in \Gamma^1$ are such that

$$(K(y), L_h)(K(x), L_e) = (K(z), L_r)(K(x), L_e).$$

This implies that

$$(K(y(\phi_h x)), L_{(h\psi_x)e}) = (K(z(\phi_r x)), L_{(r\psi_x)e}),$$

that is, $y(\phi_h x)\, \mathcal{K}\, z(\phi_r x)$ and $(h\psi_x)e\, \mathcal{L}\, (r\psi_x)e$. From $(h\psi_x)e\, \mathcal{L}\, (r\psi_x)e$, we have $(h\psi_x)ee^+\, \mathcal{L}\, (r\psi_x)ee^+$, where $e^+ \in E^o$ and $e\, \mathcal{R}\, e^+ \mathcal{L}^*\, x$, this implies that $(h\psi_x)e^+\, \mathcal{L}\, (r\psi_x)e^+$. It follows that $(hx)e^+\, \mathcal{L}^*\, (rx)e^+$ and thus $hx\, \mathcal{L}^*\, rx$ since $x\, \mathcal{L}^*\, e^+$. By $\phi_h x = h^+(\phi_h x), \phi_r x = r^+(\phi_r x)$ and $y\, \mathcal{L}^*\, h^+, z\, \mathcal{L}^*\, r^+$, we have $y(\phi_h x)\, \mathcal{L}^*\, z(\phi_r x)$. Hence by Lemma 1.9

$$(y(\phi_h x), z(\phi_r x)) \in \mathcal{K} \cap \mathcal{L}^* = \iota.$$

That is $y(\phi_h x) = z(\phi_r x)$. From $(h\psi_x)\, \mathcal{L}\, (r\psi_x)$ and (7) we deduce that $y(\phi_h i_x) = z(\phi_r i_x)$ and $(h\psi_{i_x})\, \mathcal{L}\, (r\psi_{i_x})$. Therefore

$$\begin{aligned}(K(y), L_h)(K(i_x), L_{\overline{x}^+}) &= (K(y(\phi_h i_x)), L_{(h\psi_{i_x})}) \quad (\text{since } i_x\, \mathcal{L}\, \overline{x}^+) \\ &= (K(z(\phi_r i_x)), L_{(r\psi_{i_x})}) \\ &= (K(z), L_r)(K(i_x), L_{\overline{x}^+}). \quad (\text{since } i_x\, \mathcal{L}\, \overline{x}^+)\end{aligned}$$

By Lemma 1.3, $u\, \mathcal{R}^*\, (K(x), L_e)$.

On the other hand, we have

$$\begin{aligned}(K(x), L_e)v &= (K(x), L_e)(K(\lambda_x), L_e) \\ &= (K(x(e\lambda_x)), L_{(e\lambda_x)e}) \qquad (\lambda_x \in E^o \text{ since } x \in R) \\ &= (K(xe^+), L_{e^+e}) \qquad (\text{since } e\lambda_x = e^+ \in E^o) \\ &= (K(x), L_e). \qquad (\text{since } x\, \mathcal{L}^*\, e^+\, \mathcal{R}\, e)\end{aligned}$$

If $(K(x), L_e)(K(y), L_h) = (K(x), L_e)(K(z), L_r)$ for any $(K(y), L_h), (K(z), L_r) \in \Gamma^1$, then

$$(K(x(\phi_e y)), L_{(e\psi_y)h}) = (K(x(\phi_e z)), L_{(e\psi_z)r}).$$

That is

$$x(\phi_e y)\ \mathcal{K}\ x(\phi_e z)\ \text{ and }\ (e\psi_y)h\ \mathcal{L}\ (e\psi_z)r.$$

By $x(\phi_e y)\ \mathcal{K}\ x(\phi_e z)$, there exists $l \in E^o$ with $\overline{x(\phi_e z)}^{*}\ \mathcal{R}\ l\ \mathcal{L}^*\ x(\phi_e y)$ such that $x(\phi_e y) = x(\phi_e z)l$. From $x\ \mathcal{L}^*\ \lambda_x$ and Lemma 1.1 we deduce that $\lambda_x(\phi_e y) = \lambda_x(\phi_e z)l$. Moreover l is $\mathcal{R}$ related to some $\overline{\lambda_x(\phi_e z)}^{*}$ and $l\ \mathcal{L}^*\ \lambda_x(\phi_e y)$ since $x\ \mathcal{L}^*\ \lambda_x \in E^o$, whence $C_{S^o}(\lambda_x(\phi_e y)) \cap C_{S^o}(\lambda_x(\phi_e z)) \neq \emptyset$. Similarly we have $\lambda_x(\phi_e y)h = \lambda_x(\phi_e z)$ for some $h \in E^o$. Thus $\lambda_x(\phi_e y)\ \mathcal{R}^*\ \lambda_x(\phi_e z)$ and therefore $K(\lambda_x(\phi_e y)) = K(\lambda_x(\phi_e z))$. Consequently,

$$\begin{aligned}(K(\lambda_x), L_e)(K(y), L_h) &= (K(\lambda_x(\phi_e y)), L_{(e\psi_y)h}) \\ &= (K(\lambda_x(\phi_e z)), L_{(e\psi_z)r}) \\ &= (K(\lambda_x), L_e), (K(z), L_r).\end{aligned}$$

By Lemma 1.3, $v\ \mathcal{L}^*\ (K(x), L_e)$ and Γ is an abundant semigroup. □

Lemma 3.3 *Let*

$$W = \{(K(x), L_{x^*}) :\ x \in S^o, (x^* \in E^o)\ x^*\ \mathcal{L}^*\ x\}.$$

Then W is isomorphic to S^o and is a quasi-adequate ∗-subsemigroup of Γ, moreover, $E(W) = \{(K(x), L_x) :\ x \in E^o\}$.

Proof. Obviously $W \subseteq \Gamma$. Define $\sigma :\ S^o \longrightarrow W$ by $s\sigma = (K(s), L_{s^*})$, where $s \in S^o, s^* \in E^o$ and $s^*\ \mathcal{L}^*\ s$, then σ is well-defined. For any $s, t \in S^o$, we have

$$s\sigma \cdot t\sigma = (K(s), L_{s^*})(K(t), L_{t^*}) = (K(s(\phi_{s^*}t)), L_{(s^*\psi_t)t^*}) = (K(ss^*t), L_{\lambda_{s^*t}t^*}).$$

It follows from $\mathcal{L}^*$ is a right congruence that $\lambda_{s^*t}t^*\ \mathcal{L}^*\ s^*tt^* = s^*t\ \mathcal{L}^*\ st\ \mathcal{L}^*\ (st)^*$ and so $(K(ss^*t), L_{\lambda_{s^*t}t^*}) = (K(st), L_{(st)^*})$. Thus $s\sigma \cdot t\sigma = (st)\sigma$ and σ is a homomorphism.

If $s\sigma = t\sigma$, then $C_{S^o}(s) \cap C_{S^o}(t) \neq \emptyset, R^*_s = R^*_t$ and $L_{s^*} = L_{t^*}$. Thus $s = t$ and so σ is injective. Obviously σ is surjective. Therefore σ is an isomorphism.

To prove that W is a ∗−subsemigroup, let $(K(x), L_{x^*}) \in W$. By the proof of Lemma 3.2, $u = (K(x^+), L_{x^+}) \in E(W)$ and $v = (K(x^*), L_{x^*}) \in E(W)$. In the following we will prove that

$$v\ \mathcal{L}^*(\Gamma)\ (K(x), L_{x^*})\ \mathcal{R}^*(\Gamma)\ u.$$

It is readily that

$$(K(x), L_{x^*})v = K(xx^*x^*, L_{\lambda_{(x^*x^*)x^*}}) = (K(x), L_{x^*})$$

and

$$u(K(x), L_{x^*}) = (K(x^+x^+x), L_{\lambda_{(x^+x)^*x^*}}) = (K(x), L_{x^*}).$$

For all $(K(y_1), L_{h_1}), (K(y_2), L_{h_2}) \in \Gamma^1$, if

$$(K(x), L_{x^*})(K(y_1), L_{h_1}) = (K(x), L_{x^*})(K(y_2), L_{h_2}),$$

then

$$(K(xx^*y_1), L_{(x^*\psi_{y_1})h_1}) = (K(xx^*y_2), L_{(x^*\psi_{y_2})h_2}).$$

Thus $xx^*y_1\ \mathcal{K}\ xx^*y_2$ and $(x^*y_1)h_1\ \mathcal{L}\ (x^*y_2)h_2$. Similar to the proof of Lemma 3.2, we can show that $x^*x^*y_1\ \mathcal{K}\ x^*x^*y_2$. Therefore

$$(K(x^*), L_{x^*})(K(y_1), L_{h_1}) = (K(x^*), L_{x^*})(K(y_2), L_{h_2}).$$

By Lemma 1.3, $v\ \mathcal{L}^*(\Gamma)\ (K(x), L_{x^*})$. Similarly, $u\ \mathcal{R}^*(\Gamma)\ (K(x), L_{x^*})$.

By Lemma 3.2, it is easy to see that $E(W) = \{(K(x), L_x) :\ x \in E^o\}$. □

Lemma 3.4 *W is a generalised quasi-adequate transversal of Γ.*

Proof. Let $(K(x_1), L_{e_1}), (K(x_2), L_{e_2}) \in \Gamma$. We first show that $(K(x_1), L_{e_1})\ \mathcal{R}^*\ (K(x_2), L_{e_2})$ if and only if $x_1\ \mathcal{R}^*\ x_2$. By Lemma 3.3, it is equivalent to show that

$$(K(i_{x_1}), L_{\overline{x}_1^+})\ \mathcal{R}^*\ (K(i_{x_2}), L_{\overline{x}_2^+}) \text{ if and only if } x_1\ \mathcal{R}^*\ x_2.$$

Now $u_1 = (K(i_{x_1}), L_{\overline{x}_1^+})\ \mathcal{R}^*\ (K(i_{x_2}), L_{\overline{x}_2^+}) = u_2$

$\Longleftrightarrow u_1u_2 = u_2$ and $u_2u_1 = u_1$, that is $(K(i_{x_1}\overline{x}_1^+ i_{x_2}), L_{\lambda_{(\overline{x}_1^+ i_{x_2})}\overline{x}_2^+}) = (K(i_{x_2}), L_{\overline{x}_2^+})$ and $(K(i_{x_2}\overline{x}_2^+ i_{x_1}), L_{\lambda_{(\overline{x}_2^+ i_{x_1})}\overline{x}_1^+}) = (K(i_{x_1}), L_{\overline{x}_1^+})$

$\Longleftrightarrow (K(i_{x_1}i_{x_2}), L_{\overline{x}_1^+ i_{x_2}}) = (K(i_{x_2}), L_{\overline{x}_2^+})$ and $(K(i_{x_2}i_{x_1}), L_{\overline{x}_2^+ i_{x_1}}) = (K(i_{x_1}), L_{\overline{x}_1^+})$ since $i_{x_1}\ \mathcal{L}\ \overline{x}_1^+, i_{x_2}\ \mathcal{L}\ \overline{x}_2^+$.

$\Longleftrightarrow i_{x_1}i_{x_2}\ \mathcal{K}\ i_{x_2}, \overline{x}_2^+\ \mathcal{L}\ \lambda_{\overline{x}_1^+ i_{x_2}}\ \mathcal{L}\ \overline{x}_1^+ i_{x_2}$ and $i_{x_2}i_{x_1}\ \mathcal{K}\ i_{x_1}, \overline{x}_1^+\ \mathcal{L}\ \lambda_{\overline{x}_2^+ i_{x_1}}\ \mathcal{L}\ \overline{x}_2^+ i_{x_1}$. It follows from $i_{x_1}\ \mathcal{L}\ \overline{x}_1^+$ that $i_{x_1}i_{x_2}\ \mathcal{L}\ \overline{x}_1^+ i_{x_2}\ \mathcal{L}\ \overline{x}_2^+\ \mathcal{L}\ i_{x_2}$ and dually, $i_{x_2}i_{x_1}\ \mathcal{L}\ i_{x_1}$. Thus it is equivalent to

$\Longleftrightarrow i_{x_1}i_{x_2} = i_{x_2},\ i_{x_2}i_{x_1} = i_{x_1}$

$\Longleftrightarrow x_1\ \mathcal{R}^*\ x_2$ since $x_1\ \mathcal{R}^*\ i_{x_1}, x_2\ \mathcal{R}^*\ i_{x_2}$.

Similarly we may show that $(K(x_1), L_{e_1})\ \mathcal{L}^*\ (K(x_2), L_{e_2})$ if and only if $e_1\ \mathcal{L}\ e_2$.

Let $a = (K(x), L_e) \in \Gamma$, $V = \{(K(y), L_{y^*}) \in W : y \in C_{S^o}(x)\}$ and $(K(y), L_{y^*}) \in V$. Since $y \in C_{S^o}(x)$, there exist $i, \lambda \in E(R)$ such that $x = iy\lambda$, where $i\ \mathcal{L}^*\ y^+, \lambda\ \mathcal{R}^*\ y^*$ for some $y^+, y^* \in E^o$. It follows that

$$(K(x), L_e) = (K(i), L_{y^+})(K(y), L_{y^*})(K(\lambda), L_e).$$

Furthermore, we have

$$(K(i), L_{y^+})\ \mathcal{L}\ (K(y^+), L_{y^+})\ \mathcal{R}^*\ (K(y), L_{y^*})$$

and

$$(K(\lambda), L_e)\ \mathcal{R}\ (K(y^*), L_{y^*})\ \mathcal{L}^*\ (K(y), L_{y^*}).$$

Hence $(K(y), L_{y^*}) \in C_W(a)$ and $V \subseteq C_W(a)$.

Conversely, let $(K(y), L_{y^*}) \in C_W(a)$. Then there exist $(K(y_1), L_{h_1}), (K(y_2), L_{h_2}) \in E(\Gamma)$ such that

$$(K(x), L_e) = (K(y_1), L_{h_1})(K(y), L_{y^*})(K(y_2), L_{h_2}),$$

and

$$(K(y_1), L_{h_1})\ \mathcal{L}\ (K(y), L_{y^*})^+ \text{ for some } (K(y), L_{y^*})^+ \in E(W),$$

$$(K(y_2), L_{h_2})\ \mathcal{R}\ (K(y), L_{y^*})^* \text{ for some } (K(y), L_{y^*})^* \in E(W).$$

By Lemma 1.3, $(K(y_1), L_{h_1})\ \mathcal{R}^*\ a\ \mathcal{L}^*\ (K(y_2), L_{h_2})$. Hence, $y_1\ \mathcal{R}^*\ x$ and $e\ \mathcal{L}\ h_2$.

On the other hand, by Lemma 3.3 there exist $x', x'' \in E^o$ such that

$$(K(y), L_{y^*})^+ = (K(x'), L_{x'}) \text{ with } x'\ \mathcal{R}^*\ y,$$

and

$$(K(y), L_{y^*})^* = (K(x''), L_{x''}) \text{ with } x''\ \mathcal{L}\ y^*.$$

It follows that

$$(K(x'), L_{x'})(K(x), L_e)(K(x''), L_{x''}) = (K(y), L_{y^*}),$$

and so that $x'x'xex''\ \mathcal{K}\ y$ and $(x'xex'')\psi_{x''}\ \mathcal{L}\ y^*$. Thus $x'xex''\ \mathcal{L}^*\ y$ and so $y = x'xex''$ since $\mathcal{K} \cap \mathcal{L}^* = \iota$.

Since $(K(y_1), L_{h_1})\ \mathcal{L}\ (K(y), L_{y^*})^+ = (K(x'), L_{x'})$, we have $h_1\ \mathcal{L}\ x'$. Hence $(K(y_1), L_{x'}) = (K(y_1), L_{h_1}) \in E(\Gamma)$ and there exists $x'^+ \in E^o$ such that $y_1\ \mathcal{L}^*\ x'^+\ \mathcal{R}\ x'$. And from $(K(y_1), L_{x'}) \in E(\Gamma)$ by Lemma 3.3, $\phi_{x'}y_1 = x'y_1 = x'^+$, and so $y_1x'y_1 = y_1x'^+ = y_1$ since $y_1\ \mathcal{L}^*\ x'^+$. Thus y_1 is regular. Since $x'y_1 = x'^+$ and y_1, x'^+, x' are all regular, from $y_1\ \mathcal{L}\ x'^+\ \mathcal{R}\ x'$, we deduce that there exists an idempotent e in $R_{y_1} \cap L_{x'}$. Consequently $y_1 = ex'^+ \in E(R)E^o = IE^o \subseteq I = E(R)$ and $x'\ \mathcal{L}\ e\ \mathcal{R}\ y_1\ \mathcal{R}^*\ x$.

Since $(K(y_2), L_{h_2})\ \mathcal{R}\ (K(y), L_{y^*})^* = (K(x''), L_{x''})$, we have $y_2\ \mathcal{R}^*\ x''$. Also

$$(K(y_2), L_{h_2})(K(x''), L_{x''}) = (K(x''), L_{x''}),$$

and

$$(K(x''), L_{x''})(K(y_2), L_{h_2}) = (K(y_2), L_{h_2}).$$

That is

$$(K(y_2\phi_{h_2}x''), L_{(h_2x'')x''}) = (K(x''), L_{x''}),$$

and

$$(K(x''y_2), L_{(x''\psi_{y_2})h_2}) = (K(y_2), L_{h_2}).$$

From $y_2 \mathcal{R}^* x''$ we have $y_2 \in L$ and so $y_2 \in L \cap R = S^o$. Thus $h_2^+ = \phi_{h_2}y_2 = h_2y_2$ since $(K(y_2), L_{h_2}) \in E(\Gamma)$. This implies that $y_2h_2y_2 = y_2h_2^+ = y_2$ and so y_2 is regular. It follows from $y_2 \mathcal{R}^* x''$, $y_2 \mathcal{L}^* h_2^+ \mathcal{R}\, h_2$ with $x'', h_2^+ \in E^o$ that $y_2 = x''h_2^+ \in E^oE^o \subseteq E^o$. From $x \mathcal{L}^* e^+ \mathcal{R}\, e \mathcal{L}\, h_2$ we deduce that $ex'' \in \Lambda E^o \subseteq \Lambda$ and $ex'' \mathcal{L}\, x''$ implies that $ex'' \in \Lambda \cap I = E^o$. It follows from $y = x' \cdot x \cdot ex''$ that $eye^+ = ex'xex''e^+ = exe^+ = x$ and $e \mathcal{L}\, x' \mathcal{R}^* y$, $e^+ \mathcal{R}\, ex'' \mathcal{L}^* y$, and so $y \in C_{S^o}(x)$. Therefore $C_W(a) \subseteq V$ and W is a generalised quasi-adequate transversal of Γ. □

Lemma 3.5 *W is a weakly simplistic RGQA transversal of Γ.*

Proof. It follows from W is a generalised quasi-adequate transversal of Γ that

$$I(\Gamma) = \{(K(x), L_e) \in E(\Gamma) : (K(x), L_e)\ \mathcal{L}\ (K(y), L_y) \in E(W)\}$$

$$\Lambda(\Gamma) = \{(K(x), L_e) \in E(\Gamma) : (K(x), L_e)\ \mathcal{R}\ (K(z), L_z) \in E(W)\}.$$

We give a useful description of $I(\Gamma)$ and $\Lambda(\Gamma)$. Denoting

$$P = \{(K(i_x), L_{\overline{x}^+}) \in \Gamma : x \in R, x = i_x\overline{x}\lambda_x,\ i_x, \lambda_x \in E, i_x\ \mathcal{L}\ \overline{x}^+, \lambda_x\ \mathcal{R}\ \overline{x}^*\},$$

$$Q = \{(K(x), L_e) \in \Gamma : x \in E^o\}.$$

It is obvious that by Lemma 3.2 and Lemma 3.4, $P \subseteq I(\Gamma)$ and $Q \subseteq \Lambda(\Gamma)$. For any $(K(x), L_e) \in E(\Gamma)$ with $(K(x), L_e)\ \mathcal{L}\ (K(y), L_y) \in E(W)$ then by Lemma 3.4, $e\ \mathcal{L}\ y \in E^o$ and so $e \in I \cap \Lambda = E^o$. Thus $\phi_e x = ex = e^+$ by Lemma 3.2 and there exists $e \in E$ such that $x\ \mathcal{R}^*\ e\ \mathcal{L}\ e$. From $e \in E^o$ we deduce that $e \in I$ and so $x = ee^+ \in IE^o \subseteq I$. It is easy to see $e = xe$ and $K(x) = K(xe)$. Consequently $(K(x), L_e) = (K(xe), L_e) \in P$ and $I(\Gamma) = P$.

If $(K(x), L_e) \in E(\Gamma)$ with $(K(x), L_e)\ \mathcal{R}\ (K(z), L_z) \in E(W)$, it follows from Lemma 3.5 that $x\ \mathcal{R}^*\ z \in E^o$ and so $x \in R \cap L = S^o$. From $(K(x), L_e) \in E(\Gamma)$ and Lemma 3.2 we deduce that $e^+ = \phi_e x = ex$. Thus $(xe)^2 = xe \cdot xe = x(ex)e = x(e^+e) = xe$ and xe is idempotent. It follows from $xe \cdot e^+ = x(ee^+) = xe^+ = x$ that $x\ \mathcal{R}^*\ xe$ and so $xe \in L$ with $xe \in L \cap E = \Lambda$. Therefore $x = xee^+ = (xe)e^+ \in \Lambda E^o \subseteq \Lambda$ and so $x \in \Lambda \cap S^o = E^o$. Consequently $Q = \Lambda(\Gamma)$.

For any $(K(i_x), L_{\overline{x}^+}) \in I(\Gamma), (K(y), L_y) \in E(W)$, where $y \in E^o$, we have

$$(K(i_x), L_{\overline{x}^+})(K(y), L_y) = (K(i_x\overline{x}^+y), L_{\overline{x}^+yy}) = K(i_xy), L_{\overline{x}^+y}) \in I(\Gamma)$$

since $i_xy \in IE^o \subseteq I, \overline{x}^+y \in E^oE^o \subseteq E^o$ and $i_xy\ \mathcal{L}\ \overline{x}^+y$. On the other hand,

$$(K(y), L_y)(K(i_x, L_{\overline{x}^+}) = (K(yyi_x), L_{(y\psi i_x)\overline{x}^+}) = (K(yi_x), L_{\lambda_{yi_x}}) \in I(\Gamma)$$

since $(y\psi i_x)\overline{x}^+ = \overline{x}^+yi_x\overline{x}^+ = (\overline{x}^+y)(yi_x)\ \mathcal{L}\ yi_x\ \mathcal{L}\ \lambda_{yi_x}$ with $yi_x \in E^oI \subseteq I$ and $\overline{x}^+y \in V_{S^o}(yi_x)$. Similarly, for any $(K(x), L_e) \in \Lambda$ with $x \in E^o$ and $(K(y), L_y) \in E^o$, we have

$$(K(x), L_e)(K(y), L_e) = (K(x\phi_e y), L_{eyy}) \in \Lambda(\Gamma)$$

since $x(\phi_e y) \in E^oE^o \subseteq E^o$ $(y \in E^o, \phi_e y \in E^o)$ and

$$(K(y), L_y)(K(x), L_e) = (K(yyx), L_{y\psi_x e}) \in \Lambda(\Gamma)$$

since $yyx = yx \in E^oE^o \subseteq E^o$. Therefore the generalised quasi-adequate transversal W is refined and W is weakly simplistic. □

To prove the converse part of Theorem 3.1, suppose that S is an abundant semigroup with a weakly simplistic RGQA transversal S^o. It follows from Theorem 2.7 that R is an abundant semigroup with an RGQA transversal S^o and $E(R) = I$ is a band. Consequently, R is quasi-adequate and for every $x \in R$ and $\lambda_x \in \Lambda_x$, there exists $\overline{x}^* \in E^o$ such that $\lambda_x = \overline{x}^*$. By Theorem 2.7, Λ is a band with an RGQA transversal $E^o = E(S^o)$ and each element in Λ is $\mathcal{R}$-related to some element in E^o. For each $e \in \Lambda$, let $\phi_e : R \to R$ be a mapping defined by $\phi_e x = ex\overline{ex}^*$ for a given $\overline{ex}^* \in E^o$, where $\overline{ex} \in C_{S^o}(ex)$ and $\overline{ex}^* \ \mathcal{L}^* \ \overline{ex}$. For each $y \in R$, let $\psi_y : \Lambda \to \Lambda$ be a mapping defined by $f\psi_y = \lambda_{fy}$ for a given $\lambda_{fy} \in \Lambda_{fy}$. It follows from Proposition 2.1 that conditions (1) $\sim$ (7) are satisfied and so the quadruple $(R, \Lambda; \phi, \psi)$ is permissible.

Thus we may construct a semigroup Γ in the method of the direct part of Theorem 3.1 with the multiplication is

$$(K(x), L_e)(K(y), L_f) = (K(xey\overline{ey}^*), L_{\lambda_{ey}f}).$$

In the following we will prove that Γ is isomorphic to S.

For any $(K(x), L_e) \in \Gamma$, define a mapping $\sigma : \Gamma \longrightarrow S$ given by

$$(K(x), L_e)\sigma = xe,$$

then σ is well-defined. In fact, if $y \in K(x)$ and $h \ \mathcal{L} \ e$, we have $xee^+ = xe^+ = x$ since $e \ \mathcal{R} \ e^+ \ \mathcal{L}^* \ x$ and similarly $yhh^+ = y$. Thus $xe \ \mathcal{R}^* \ x \ \mathcal{R}^* \ y \ \mathcal{R}^* \ yh$. For any $a, b \in S^1$, if $xea = xeb$ then $e^+ea = e^+eb$ since $x \ \mathcal{L}^* \ e^+$, that is $ea = eb$, thus $xe \ \mathcal{L}^* \ e$. Similarly we have $yh \ \mathcal{L}^* \ h$. Hence $xe \ \mathcal{L}^* \ e \ \mathcal{L} \ h \ \mathcal{L}^* \ yh$ and consequently xe and yh in the same $\mathcal{H}^*$-class. Let $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(y)$. Then $x = i_x\overline{x}\lambda_x, y = i_y\overline{x}\lambda_y$ with $i_x \ \mathcal{L} \ \overline{x}^+, \lambda_x \ \mathcal{R} \ \overline{x}^*, i_y \ \mathcal{L} \ \overline{x}^{+\prime}, \lambda_y \ \mathcal{R} \ \overline{x}^{*\prime}$. Thus $xe = i_x\overline{x}\lambda_x e$ and it follows from $e \ \mathcal{R} \ e^+ \ \mathcal{L}^* \ x \ \mathcal{L}^* \ \lambda_x$ and Lemma 1.1 that $e \ \mathcal{L} \ \lambda_x e \ \mathcal{R} \ \lambda_x$ with $\lambda_x e \in E^o\Lambda \subseteq \Lambda$. Consequently $\lambda_x e \ \mathcal{R} \ \lambda_x \ \mathcal{R} \ \overline{x}^*$ and $\overline{x} \in C_{S^o}(xe)$. Similarly $yh = i_y\overline{x}\lambda_y h, \lambda_y h \in E^o\Lambda \subseteq \Lambda$ with $\lambda_y h \ \mathcal{R} \ \lambda_y \ \mathcal{R} \ \overline{x}^{*\prime}$ and $\overline{x} \in C_{S^o}(yh)$. Therefore $C_{S^o}(xe) \cap C_{S^o}(yh) \neq \emptyset$ and consequently $xe = yh$ by Lemma 1.9.

For every $x \in S$, it follows from $x\overline{x}^* \ \mathcal{L}^* \ \lambda_x\overline{x}^* = \overline{x}^* \ \mathcal{R} \ \lambda_x$ that $(K(x\overline{x}^*), L_{\lambda_x}) \in \Gamma$. Hence $(K(x\overline{x}^*), L_{\lambda_x})\sigma = x\overline{x}^*\lambda_x = x\lambda_x = x$ and σ is surjective.

For any $(K(x), L_e),\ (K(y), L_f) \in \Gamma$, we have

$$\begin{aligned}
[(K(x), L_e)(K(y), L_f)]\sigma &= (K(xey\overline{ey}^*), L_{\lambda_{ey}f})\sigma = (xey\overline{ey}^*)\lambda_{ey}f \\
&= xey(\overline{ey}^*\lambda_{ey})f = x(ey\lambda_{ey})f \\
&= x(ey)f = xe \cdot yf \\
&= (K(x), L_e)\sigma \cdot (K(y), L_f)\sigma.
\end{aligned}$$

Thus σ is a homomorphism.

If $(K(x), L_e),\ (K(y), L_f) \in \Gamma$ with the property that $(K(x), L_e)\sigma = (K(y), L_f)\sigma$, that is $xe = yf$. Since $e \ \mathcal{R} \ e^+ \ \mathcal{L}^* \ x$ and $f \ \mathcal{R} \ f^+ \ \mathcal{L}^* \ y$ for some $e^+, f^+ \in E^o$, it is easy to see that

$$xee^+ = xe^+ = x \ \text{ and } \ yff^+ = yf^+ = y.$$

Hence $x \ \mathcal{R}^* \ xe = yf \ \mathcal{R}^* \ y$ and $R^*_x = R^*_y$. Similarly, $e = e^+e \ \mathcal{L}^* \ xe = yf \ \mathcal{L}^* \ f^+f = f$ and $L_e = L_f$. Similar to the proof of σ is well-defined, we have $\overline{x} \in C_{S^o}(x) \cap C_{S^o}(xe)$ and $C_{S^o}(x) = C_{S^o}(xe)$. Similarly $C_{S^o}(y) = C_{S^o}(yf)$. From $xe = yf$ we deduce that $C_{S^o}(xe) = C_{S^o}(yf)$ and so $C_{S^o}(x) = C_{S^o}(y)$. Hence $K(x) = K(y)$. Combining with $L_e = L_f$ implies that σ is injective and so σ is an isomorphism.

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