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Construction of a new modification of Baskakov operators on $(0, \infty)$

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Abstract. In this manuscript, we construct new modification of Baskakov operators on $(0, \infty)$ using the second central moment of the classical Baskakov operators. And the moments and the central moments computation formulas and their quantitative properties are computed. Then, rate of convergence, point-wise estimates, weighted approximation and Voronovskaya type theorem for the new operators are established. Also, Kantorovich and Durrmeyer type generalizations are discussed. Finally, some graphs and numerical examples are showed by using Matlab algorithms.

1. Introduction

The classical Baskakov operators $B_m(g)$ see [7] and are defined as

$$B_m(g;t) = \sum_{k=0}^{\infty} b_{m,k}(t) g\left(\frac{k}{m}\right),\ t \in [0,\infty),\ m \in \mathbb{N}_+ := \{1, 2, \cdots\},$$

where $b_{m,k}(t) = \binom{m+k-1}{k}\frac{t^k}{(1+t)^{m+k}}$, $k = 0, 1, \cdots$. Later, many different generalized modifications of the Baskakov operators have been constructed and studied by many researches(see (p, q)-Baskakov-Durrmeyer-Stancu operators [3], generalized-Baskakov-Durrmeyer operators [12], (p, q)-Baskakov-Durrmeyer operators [21], Apostol-Genocchi-Baskakov-Durrmeyer operators [10], q-Baskakov operators [6, 13], generalized q-Baskakov operators [5] and so on). Recently, it is a hot topic how to construct new Baskakov type operators based on new Bernstein type operators. For instance: In [9], Deniz et al. constructed Baskakov-Durrmeyer-Kantorovich operators in terms of the method of Stan constructed Bernstein-Durrmeyer-Kantorovich operators in [29]. In [4], A, Aral et al. defined α-Baskakov operators in term of the results of X. Y. Chen et al. in [8]. In [30], F. Usta constructed new modification of Bersntein operators on $(0, 1)$ by means of the second

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order central moments of the classical Bernstein operators. Motivated from the works of Usta, we construct a new modification of Baskakov operators as:

$$\mathfrak{B}_m(g;t) = \frac{1}{m}\sum_{k=0}^{\infty}\binom{m+k-1}{k}\frac{t^{k-1}}{(1+t)^{m+k+1}}(mt-k)^2 g\left(\frac{k}{m}\right),\ t\in(0,\infty),\ m\in\mathbb{N}_+. \tag{1}$$

where $g \in C[0,\infty)$. We regard the operators (1) as a mapping from $[0,\infty)$ to $(0,\infty)$ and emphasize that the domain of our operators are $(0,\infty)$ but the classical Baskakov operators are $[0,\infty)$.

2. Auxiliary Results

Lemma 2.1. *[15, Lemma 2, $p = q = 1$] If we define $T_{m,i}(t) = B_m(u^i;t)$ and $i = 0,1,2,\cdots$, then there holds the following relations:*

$$T_{m,0}(t) = 1; T_{m,1}(t) = t; T_{m,2}(t) = t^2 + \frac{t(1+t)}{m};$$

$$mT_{m,i+1}(t) = t(1+t)(T_{m,i}(t))' + mtT_{m,i}(t).$$

Lemma 2.2. *If we define $\mathfrak{T}_{m,i}(t) = \mathfrak{B}_m(u^i;t)$, $i = 0,1,2,\cdots$, then there holds the following relations:*

$$\frac{t(1+t)}{m}\mathfrak{T}_{m,i}(t) = T_{m,i+2}(t) - 2tT_{m,i+1}(t) + t^2T_{m,i}(t); \tag{2}$$

$$\mathfrak{T}_{m,i}(t) = T'_{m,i+1}(t) - tT'_{m,i}(t); \tag{3}$$

$$\mathfrak{T}_{m,i}(t) = \frac{1}{m}\left(t(1+t)T'_{m,i}(t)\right)' + T_{m,i}(t). \tag{4}$$

Proof. By the definitions of B_m and $\mathfrak{B}_m$, we can write

$$\begin{aligned}\frac{t(1+t)}{m}\mathfrak{T}_{m,i}(t) &= \sum_{k=0}^{\infty}\binom{m+k-1}{k}\frac{t^k}{(1+t)^{m+k}}\left(t-\frac{k}{m}\right)^2\left(\frac{k}{m}\right)^i \\ &= T_{m,i+2}(t) - 2tT_{m,i+1}(t) + t^2T_{m,i}(t).\end{aligned}$$

Next, applying Lemma 2.1 and (2), we can obtain

$$\begin{aligned}t(1+t)\mathfrak{T}_{m,i}(t) &= mT_{m,i+2}(t) - 2mtT_{m,i+1}(t) + mt^2T_{m,i}(t) \\ &= t(1+t)(T_{m,i+1}(t))' - mtT_{m,i+1}(t) + mt^2T_{m,i}(t) \\ &= t(1+t)(T_{m,i+1}(t))' - t\left(t(1+t)(T_{m,i}(t))' + mtT_{m,i}(t)\right) + mt^2T_{m,i}(t) \\ &= t(1+t)(T_{m,i+1}(t))' - t^2(1+t)(T_{m,i}(t))'.\end{aligned}$$

Thus, $\mathfrak{T}_{m,i}(t) = T'_{m,i+1}(t) - tT'_{m,i}(t)$. Finally,

$$\begin{aligned}\mathfrak{T}_{m,i}(t) &= T'_{m,i+1}(t) - tT'_{m,i}(t) \\ &= \left(\frac{t(1+t)}{m}(T_{m,i}(t))' + tT_{m,i}(t)\right)' - tT'_{m,i}(t) \\ &= \frac{1}{m}\left(t(1+t)(T_{m,i}(t))'\right)' + T_{m,i}(t).\end{aligned}$$

We complete the proof of the Lemma 2.2. □

Then, the following lemma can be obtain immediately:

Lemma 2.3. *For $m \in \mathbb{N}$ and $t \in (0,\infty)$, the new Baskakov operators have the following equalities:*

$$\mathfrak{T}_{m,0}(t) = 1;\ \mathfrak{T}_{m,1}(t) = \left(1+\frac{2}{m}\right)t + \frac{1}{m};$$
$$\mathfrak{T}_{m,2}(t) = \left(1+\frac{7}{m}+\frac{6}{m^2}\right)t^2 + \left(\frac{5}{m}+\frac{6}{m^2}\right)t + \frac{1}{m^2};$$
$$\mathfrak{T}_{m,3}(t) = \left(1+\frac{15}{m}+\frac{38}{m^2}+\frac{24}{m^3}\right)t^3 + \left(\frac{12}{m}+\frac{48}{m^2}+\frac{36}{m^3}\right)t^2 + \left(\frac{13}{m^2}+\frac{14}{m^3}\right)t + \frac{1}{m^3};$$
$$\mathfrak{T}_{m,4}(t) = \left(1+\frac{26}{m}+\frac{131}{m^2}+\frac{226}{m^3}+\frac{120}{m^4}\right)t^4 + \left(\frac{22}{m}+\frac{186}{m^2}+\frac{404}{m^3}+\frac{240}{m^4}\right)t^3$$
$$+\left(\frac{61}{m^2}+\frac{211}{m^3}+\frac{150}{m^4}\right)t^2 + \left(\frac{29}{m^3}+\frac{30}{m^4}\right)t + \frac{1}{m^4}.$$

Lemma 2.4. *For any $t \in (0,\infty)$. Then, we can prove the following central moments properties:*

$$A_m(t) := \mathfrak{B}_m(u-t;t) = \frac{2t+1}{m}; \tag{5}$$
$$B_m(t) := \mathfrak{B}_m\left((u-t)^2;t\right) = \left(\frac{3}{m}+\frac{6}{m^2}\right)(t^2+t) + \frac{1}{m^2}; \tag{6}$$
$$\lim_{m\to\infty} mA_m(t) = 2t+1; \tag{7}$$
$$\lim_{m\to\infty} mB_m(t) = 3(t^2+t); \tag{8}$$
$$\lim_{m\to\infty} m^2\mathfrak{B}_m\left((u-t)^4;t\right) = 15t^2(1+t)^2; \tag{9}$$
$$m^3\mathfrak{B}_m\left((u-t)^6;t\right) = O(1),\ m\to\infty. \tag{10}$$

Proof. (5) to (8) can be easily obtained. We only prove (9). Applying Lemma 2.3, we can write

$$\begin{aligned}\mathfrak{B}_m\left((u-t)^4;t\right) &= \sum_{i=0}^{4}\binom{4}{i}\mathfrak{B}_m(u^i;t)t^{4-i}\\ &= \left(\frac{15}{m^2}+\frac{130}{m^3}+\frac{120}{m^4}\right)t^4 + \left(\frac{30}{m^2}+\frac{260}{m^3}+\frac{240}{m^4}\right)t^3\\ &+\left(\frac{15}{m^2}+\frac{155}{m^3}+\frac{150}{m^4}\right)t^2 + \left(\frac{29}{m^3}+\frac{30}{m^4}\right)t + \frac{1}{m^4}.\end{aligned} \tag{11}$$

Thus, (9) is proved. Similarly, we also obtain (10). □

Using the definition of the operators, Lemma 2.3 and the well-known Korovkin theorem, we can obtain the two following lemmas:

Lemma 2.5. *For $g \in S_B(0,\infty)$, then we have*

$$\|\mathfrak{B}_m(g)\| \le \|g\|,$$

where $S_B(0,\infty) := \{g \in C[0,\infty) : g$ be bounded function on $(0,\infty)\}$ endowed with the norm $\|g\| = \sup\{|g(t)| : t \in (0,\infty)\}$.

Lemma 2.6. *For $g \in C[0,\infty)$ and any finite interval $I \subset (0,\infty)$, then the sequence $\{\mathfrak{B}_m(g;t)\}$ converges to g uniformly on I, where $C(0,\infty)$ be space of all real valued continuous functions on $(0,\infty)$.*

3. Rate of Convergence

In this section, we will focus on the rate of convergence of the new Baskakov operators in terms of the modulus of smoothness, **K**-functional and Steklov mean respectively. For any $g \in S_B(0,\infty)$, we recall the following **K**-functional:

$$\mathbf{K}(g;\delta) = \inf_{h\in\mathbf{W}^2} \{\|g-h\| + \delta\|h''\|\},$$

where $\delta \in (0,\infty)$ and $\mathbf{W}^2 = \{h \in S_B(0,\infty) : h', h'' \in S_B(0,\infty)\}$. The usual modulus of smoothness and the second-order modulus of smoothness of g can be defined by

$$\omega(g;\delta) = \sup_{0<u<\delta}\sup_{t\in(0,\infty)} |g(t+u) - g(t)|$$

and

$$\omega_2(g;\delta) = \sup_{0<u<\delta}\sup_{t\in(0,\infty)} |g(t+2u) - g(t+u) + g(t)|.$$

Applying [11, p.177, Theorem 2.4], there exists an absolute positive constant M such that

$$\mathbf{K}(g;\delta) \le M\omega_2(g;\sqrt{\delta}), \delta > 0. \tag{12}$$

Meantime, for $g \in S_B(0,\infty)$ and $z > 0$, the Steklov mean is defined as

$$g_z(t) = \frac{4}{z^2}\int_0^{\frac{z}{2}}\int_0^{\frac{z}{2}} [2g(t+u+v) - g(t+2(u+v))]\mathrm{d}u\mathrm{d}v.$$

Thus, $g \in S_B(0,\infty)$, we can write

$$g_z(t) - g(t) = \frac{4}{z^2}\int_0^{\frac{z}{2}}\int_0^{\frac{z}{2}} [2g(t+u+v) - g(t+2(u+v)) - g(t)]\mathrm{d}u\mathrm{d}v.$$

It is obvious that $|g_z(t) - g(t)| \le \omega_2(g;z)$ and $\|g_z - g\| \le \omega_2(g;z)$. If g is continuous, then $g_z', g_z'' \in S_B(0,\infty)$ and

$$g_z'(t) = \frac{4}{z^2}\left[2\int_0^{\frac{z}{2}}\left(g\left(t+u+\frac{z}{2}\right) - g(t+u)\right)\mathrm{d}u - \frac{1}{2}\int_0^{\frac{z}{2}}(g(t+z+2u) - g(t+2u))\mathrm{d}u\right].$$

Thus, we have $\|g_z'\| \le \frac{5}{z}\omega(g;z)$. Similarly, $\|g_z''\| \le \frac{9}{z^2}\omega_2(g;z)$.

Theorem 3.1. *For $g \in S_B(0,\infty)$, $m \in \mathbb{N}_+$ and $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \le 2\omega(g;\sqrt{B_m(t)}).$$

Proof. Applying [11, p. 41, (6.5)], for $g \in S_B(0,\infty)$ and any $\delta > 0$, we have

$$|g(u) - g(t)| \le \omega\left(g : |u-t|\right) \le \omega(g;\delta)\left(1 + \frac{|u-t|}{\delta}\right).$$

Combining the monotonicity and the linearity of the operators (1) and Cauchy-Schwarz inequality, for any $\delta > 0$, we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &\le \mathfrak{B}_m(\left|g(u) - g(t)\right|;t) \le \mathfrak{B}_m\left(\omega\left(g : |u-t|\right);t\right)\\ &\le \mathfrak{B}_m\left(\omega(g;\delta)\left(1 + \frac{|u-t|}{\delta}\right);t\right) \le \omega(g;\delta)\left(1 + \frac{\mathfrak{B}_m\left(|u-t|;t\right)}{\delta}\right)\\ &\le \omega(g;\delta)\left(1 + \frac{\sqrt{\mathfrak{B}_m\left((u-t)^2;t\right)}}{\delta}\right)\\ &\le \omega(g;\delta)\left(1 + \frac{\sqrt{B_m(t)}}{\delta}\right),\end{aligned}$$

by choosing $\delta = \sqrt{B_m(t)}$. We complete the proof of Theorem 3.1. □

Theorem 3.2. *For $g' \in S_B(0,\infty)$, $m \in \mathbb{N}_+$ and $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq |A_m(t)||g'(t)| + 2\sqrt{B_m(t)}\omega\left(g';\sqrt{B_m(t)}\right).$$

Proof. Applying $\mathfrak{B}_m$ to both sides of $g(u) = g(t) + g'(t)(u-t) + g(u) - g(t) - g'(t)(u-t)$, we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &\leq |g'(t)||\mathfrak{B}_m(u-t;t)| + \mathfrak{B}_m\left(\left|g(u) - g(t) - g'(t)(u-t)\right|;t\right)\\ &\leq |A_m(t)||g'(t)| + \mathfrak{B}_m\left(|u-t|\left(1+\frac{|u-t|}{\delta}\right);t\right)\omega\left(g';\delta\right)\\ &\leq |A_m(t)||g'(t)| + \sqrt{\mathfrak{B}_m\left((u-t)^2;t\right)}\left(1+\frac{\sqrt{\mathfrak{B}_m\left((u-t)^2;t\right)}}{\delta}\right)\omega\left(g';\delta\right)\end{aligned}$$

with the help of Cauchy-Schwartz inequality and mean value theorem. Choosing $\delta = \sqrt{B_m(t)}$, we can get the desired result. □

Theorem 3.3. *For $g \in S_B(0,\infty)$, $m \in \mathbb{N}_+$ and $t \in (0,\infty)$, there exists an absolute positive constant $M_1 = 4M$ such that*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq M_1\omega_2\left(g;\sqrt{A_m^2(t) + B_m(t)}\right) + \omega(g;|A_m(t)|).$$

Proof. For $g \in S_B(0,\infty)$, we consider the following new operators by:

$$\widetilde{\mathfrak{B}}_m(g;t) = \mathfrak{B}_m(g;t) + g(t) - g\left(A_m(t)+t\right),\ z \in (0,\infty), m \in \mathbb{N}_+.$$

Applying Lemma 2.3, we can obtain

$$\widetilde{\mathfrak{B}}_m(1;t) = \mathfrak{B}_m(1;t) = \mathfrak{T}_{m,0}(t) = 1,$$

and

$$\widetilde{\mathfrak{B}}_m(u;t) = \mathfrak{B}_m(u;t) + t - (A_m(t)+t) = \mathfrak{T}_{m,1}(t) - A_m(t) = t.$$

For any $h \in \mathbf{W}^2$, applying the Taylors expansion formula, we can obtain

$$h(u) = h(t) + h'(t)(u-t) + \int_t^u (u-v)h''(v)\mathrm{d}v.$$

Using the operators $\widetilde{\mathfrak{B}}_m$ to both sides of the above equality, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_m(h;t) &= h(t) + \widetilde{\mathfrak{B}}_m\left(\int_t^u (u-v)h''(v)\mathrm{d}v;t\right)\\ &= g(t) + \mathfrak{B}_m\left(\int_t^u (u-v)h''(v)\mathrm{d}v;t\right) - \int_t^{A_m(t)+t} (A_m(t)+t-v)h''(v)\mathrm{d}v.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\widetilde{\mathfrak{B}}_m(h;t) - h(t)\right| &= \left|\widetilde{\mathfrak{B}}_m\left(\int_t^u (u-v)h''(v)\mathrm{d}v;t\right)\right|\\ &\leq \left|\mathfrak{B}_m\left(\left|\int_t^u (u-v)h''(v)\mathrm{d}v\right|;t\right)\right| + \left|\int_t^{A_m(t)+t} |A_m(t)+t-v|h''(v)\mathrm{d}v\right| \qquad (13)\\ &\leq \left(A_m^2(t) + B_m(t)\right)\|h''\|.\end{aligned}$$

Using Lemma 2.5, we have

$$|\widetilde{\mathfrak{B}}_m(g;t)| \leq |\mathfrak{B}_m(g;t)| + |g(t)| + \left|g(t + A_m(t))\right| \leq 3\|g\|, \forall g \in S_B(0,\infty). \tag{14}$$

For $g \in S_B(0,\infty)$ and any $h \in \mathbf{W}^2$, combining (13) and (14), we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &= \left|\widetilde{\mathfrak{B}}_m(g;t) - g(t) + g(A_m(t) + t) - g(t)\right| \\ &\leq |\widetilde{\mathfrak{B}}_m\,(g - h;t)\,| + \left|\widetilde{\mathfrak{B}}_m\,(h;t) - h(t)\right| + |g(t) - h(t)| + \left|g(A_m(t) + t) - g(t)\right| \\ &\leq 4\|g - h\| + \left\{A_m^2(t) + B_m(t)\right\}\|h''\| + \omega\,(g;|A_m(t)|)\,.\end{aligned}$$

Taking infinum on the right hand side over all $h \in \mathbf{W}^2$, using (12), we complete the proof of Theorem 3.3. □

Theorem 3.4. *For $g \in S_B(0,\infty)$, $m \in \mathbb{N}_+$ and $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq 5\sqrt{m}|A_m(t)|\omega\left(g;\frac{1}{\sqrt{m}}\right) + \left(\frac{9}{2}mB_m(t) + 2\right)\omega_2\left(g;\frac{1}{\sqrt{m}}\right).$$

Proof. For $z, t \in (0,\infty)$, by the definition of the Steklov mean, we can obtain

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq \mathfrak{B}_m(\left|g_z - g\right|;t) + \left|\mathfrak{B}_m(g_z;t) - g_z(t)\right| + \left|g_z(t) - g(t)\right|.$$

Applying Lemma 2.5, we have

$$\mathfrak{B}_m(\left|g_z - g\right|;t) \leq \|\mathfrak{B}_m(\left|g_z - g\right|;t)\| \leq \|g_z - g\| \leq \omega_2(g;z).$$

By the Taylor's expansion formula for g_z, we have

$$g_z(u) = g_z(t) + g_z'(t)(u - t) + \int_t^u (u - v)g_z''(v)\mathrm{d}v.$$

and

$$\begin{aligned}\left|\mathfrak{B}_m(g_z;t) - g_z(t)\right| &\leq \left|\mathfrak{B}_m(g_z'(t)(u - t);t)\right| + \left|\mathfrak{B}_m\left(\int_t^u (u - v)g_z''(v)\mathrm{d}v;t\right)\right| \\ &\leq \|g_z'\|\,|\mathfrak{B}_m((u - t);t)| + \|g_z''\|\left|\mathfrak{B}_m\left(\int_t^u |u - v|\mathrm{d}v;t\right)\right| \\ &\leq \|g_z'\||A_m(t)| + \frac{1}{2}\|g_z''\|B_m(t) \\ &\leq \frac{5}{z}|A_m(t)|\omega(g;z) + \frac{9}{2z^2}B_m(t)\omega_2(g;z).\end{aligned}$$

Hence,

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq \frac{5}{z}|A_m(t)|\omega(g;z) + \left(\frac{9}{2z^2}B_m(t) + 2\right)\omega_2(g;z)$$

for $t \in (0,\infty)$. Choosing $z = \frac{1}{\sqrt{m}}$, we obtain the desired result. □

4. Point-Wise Estimates

In this section, we establish three point-wise estimates of the operators (1). First, we denote that a function g belongs to $\mathrm{Lip}_M(\gamma, D)$, $\gamma \in (0,1]$, $D \subset (0,\infty)$ if it satisfies the following condition:

$$|g(u) - g(t)| \leq M|u - t|^{\gamma},\ u \in D,\ t \in (0,\infty),$$

where M is a positive constant depending only on γ and λ.

Theorem 4.1. *For $g \in S_B(0,\infty) \bigcap \mathrm{Lip}_M(\gamma, D)$, then for any $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \le M\left((B_m(t))^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(t;D)\right),$$

where $\mathrm{d}(t;D) = \inf\{|u-t| : u \in D\}$ denotes the distance between t and D.

Proof. Let $\overline{D}$ be the closure of D. Applying the properties of infimum, there is at least a point $u_0 \in \overline{D}$ such that $\mathrm{d}(t;D) = |t - u_0|$. By the triangle inequality

$$|g(u) - g(t)| \le |g(u) - g(u_0)| + |g(t) - g(u_0)|,$$

we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &\le \mathfrak{B}_m\left(|g(u) - g(u_0)|;t\right) + \mathfrak{B}_m\left(|g(t) - g(u_0)|;t\right)\\ &\le M\left\{\mathfrak{B}_m\left(|u-u_0|^\gamma;t\right) + |t-u_0|^\gamma\right\}\\ &\le M\left\{\mathfrak{B}_m\left(|u-t|^\gamma + |t-u_0|^\gamma;t\right) + |t-u_0|^\gamma\right\}\\ &\le M\left\{\mathfrak{B}_m\left(|u-t|^\gamma;t\right) + 2|t-u_0|^\gamma\right\}.\end{aligned}$$

Choosing $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$ and using the well-known Hölder inequality, we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &\le M\left\{\left(\mathfrak{B}_m\left(|u-t|^{p\gamma};t\right)\right)^{\frac{1}{p}}\left(\mathfrak{B}_m\left(1^q;t\right)\right)^{\frac{1}{q}} + 2\mathrm{d}^\gamma(t;D)\right\}\\ &\le M\{\left(\mathfrak{B}_m\left(|u-t|^2;t\right)\right)^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(t;D)\}\\ &\le M\left((B_m(t))^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(t;D)\right).\end{aligned}$$

we complete the proof. □

Next, we obtain the local direct estimate of the operators (1), using the Lipcshitz type maximal function of the order γ introduced by Lenze [19] as

$$\tilde{\omega}_\gamma(g;t) = \sup_{u,t\in(0,\infty),u\neq t} \frac{|g(u)-g(t)|}{|u-t|^\gamma}, \quad \gamma \in (0,1]. \tag{15}$$

Theorem 4.2. *For $g \in S_B(0,\infty)$, then for any $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \le \tilde{\omega}_\gamma(g;t)\,(B_m(t))^{\frac{\gamma}{2}}.$$

Proof. From the equation (15), we have

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \le \tilde{\omega}_\gamma(g;t)\mathfrak{B}_m\left(|u-t|^\gamma;t\right).$$

Applying the well-known Hölder inequality, we have

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \le \tilde{\omega}_\gamma(g;t)\left(\mathfrak{B}_m\left((u-t)^2;t\right)\right)^{\frac{\gamma}{2}} = \tilde{\omega}_\gamma(g;t)\,(B_m(t))^{\frac{\gamma}{2}},$$

we yield the desired result. □

Finally, we establish pointwise estimate of the operators (1) in the following Lipschitz-type space (see [26]) with two distinct parameters $k_1, k_2 \in (0,\infty)$:

$$\mathrm{Lip}_M^{(k_1,k_2)}(\gamma) := \left\{g \in C(0,\infty) : |g(u) - g(t)| \le M\frac{|u-t|^\gamma}{(u + k_1t^2 + k_2t)^{\frac{\gamma}{2}}}\right\}, u, t \in (0,\infty),$$

where $\gamma \in (0,1]$, M is a positive constant depending only on γ, k_1, k_2 and g.

Theorem 4.3. *For $g \in \mathrm{Lip}_M^{(k_1,k_2)}(\gamma)$, then for any $t \in (0,\infty)$, we have*

$$\left|\mathfrak{B}_m(g;t) - g(t)\right| \leq M\left(\frac{B_m(t)}{k_1t^2 + k_2t}\right)^{\frac{\gamma}{2}}.$$

Proof. Applying the well-known Hölder inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}\left|\mathfrak{B}_m(g;t) - g(t)\right| &\leq \mathfrak{B}_m\left(|g(u) - g(t)|;t\right)\\ &\leq \mathfrak{B}_m\left(M\frac{|u-t|^\gamma}{(u + k_1t^2 + k_2t)^{\frac{\gamma}{2}}};t\right)\\ &\leq \frac{M}{(k_1t^2 + k_2t)^{\frac{\gamma}{2}}}\mathfrak{B}_m\left(|u-t|^\gamma;t\right)\\ &\leq \frac{M}{(k_1t^2 + k_2t)^{\frac{\gamma}{2}}}\left(\mathfrak{B}_m\left(|u-t|^{p\gamma};t\right)\right)^{\frac{1}{p}}\left(\mathfrak{B}_m\left(1^q;t\right)\right)^{\frac{1}{q}}\\ &= M\left(\frac{B_m(t)}{k_1t^2 + k_2t}\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Thus, the proof is completed. □

5. Weighted Approximation

Let $B_{t^2}(0,\infty)$ denote the function space of all functions g such that $|g(t)| \leq M_g(1 + t^2)$, where M_g is a positive constant depending only on g. By $C_{t^2}(0,\infty)$ we denote the subspace of all continuous functions in the function space $B_{t^2}(0,\infty)$. By $C^0_{t^2}(0,\infty)$ we denote the subspace of all functions $f \in C_{t^2}(0,\infty)$ for which $\lim\limits_{t\to\infty}\frac{|g(t)|}{1+t^2}$ exists finitely. Norm of the subspace $C^0_{t^2}(0,\infty)$ is defined by

$$||g||_2 = \sup_{t\in(0,\infty)}\frac{|g(t)|}{1+t^2}.$$

Meanwhile, we denote the modulus of continuity of g on the interval $(0,\mathbf{a}]$, $\mathbf{a} \in (0,\infty)$, by

$$\omega_{\mathbf{a}}(g;\delta) = \sup_{|u-t|\leq\delta}\ \sup_{u,t\in(0,\mathbf{a}]}|g(u) - g(t)|.$$

As is known, if $g \in C(0,\infty)$ is not uniform, the limit $\lim\limits_{\delta\to0^+}\omega(g;\delta) = 0$ may be not true. In [32], Yüksel and Ispir defined the following weighted modulus of continuity:

$$\Omega(g;\delta) = \sup_{t\in(0,\infty),0<h\leq\delta}\frac{\left|g(t+h) - g(t)\right|}{(1+t^2)(1+h^2)} \text{ for } g \in C^0_{t^2}(0,\infty), \tag{16}$$

and proved the properties of monotone increasing about $\Omega(g;\delta)$ as $\delta > 0$, $\lim\limits_{\delta\to0^+}\Omega(g;\delta) = 0$ and the inequality

$$\Omega(g;\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(g;\delta), \lambda > 0. \tag{17}$$

For any $g \in C^0_{t^2}(0,\infty)$, it follows from (16) and (17) that

$$\begin{aligned}\left|g(u) - g(t)\right| &\leq \left(1 + (u-t)^2\right)(1+t^2)\Omega(g;|u-t|) \qquad (18)\\ &\leq 2\left(1 + \frac{|u-t|}{\delta}\right)(1+\delta^2)\Omega(g;\delta)\left(1 + (u-t)^2\right)(1+t^2).\end{aligned}$$

Theorem 5.1. *For $g \in C_{t^2}(0,\infty)$, $m \in \mathbb{N}_+$ and $\mathbf{a} > 0$, we have*

$$\|\mathfrak{B}_m(g;t) - g\|_{(0,\mathbf{a}]} \leq 4M_g(1+\mathbf{a}^2)B_m(\mathbf{a}) + 2\omega_{\mathbf{a}+1}(g;\sqrt{B_m(\mathbf{a})}).$$

Proof. For any $t \in (0,\mathbf{a}]$ and $u > \mathbf{a}+1$, we easily obtain $1 \leq (u-\mathbf{a})^2 \leq (u-t)^2$. Hence,

$$\begin{aligned} |g(u) - g(t)| &\leq |g(u)| + |g(t)| \leq M_g\left(2 + u^2 + t^2\right) \\ &\leq M_g\left(2 + t^2 + (u-t+t)^2\right) \leq M_g\left(2 + 3t^2 + 2(u-t)^2\right) \\ &\leq M_g(4+3t^2)(u-t)^2 \leq 4M_g(1+\mathbf{a}^2)(u-t)^2, \end{aligned} \tag{19}$$

and for all $t \in (0,\mathbf{a}]$, $u \in (0,\mathbf{a}+1]$ and $\delta > 0$, we have

$$|g(u) - g(t)| \leq \omega_{\mathbf{a}+1}(g;|u-t|) \leq \left(1 + \frac{|u-t|}{\delta}\right)\omega_{\mathbf{a}+1}(g;\delta). \tag{20}$$

Combining (19) and (20), we have

$$|g(u) - g(t)| \leq 4M_g(1+\mathbf{a}^2)(u-t)^2 + \left(1 + \frac{|u-t|}{\delta}\right)\omega_{\mathbf{a}+1}(g;\delta).$$

Using the operators $\mathfrak{B}_m$ both sides of the above inequality, we obtain

$$\begin{aligned} |\mathfrak{B}_m(g;t) - g(t)| &\leq \mathfrak{B}_m(|g(u) - g(t)|;t) \\ &\leq 4M_g(1+\mathbf{a}^2)\mathfrak{B}_m\left((u-t)^2;t\right) + \mathfrak{B}_m\left(\left(1 + \frac{|u-t|}{\delta}\right);t\right)\omega_{\mathbf{a}+1}(g;\delta) \\ &\leq 4M_g(1+\mathbf{a}^2)B_m(t) + \omega_{\mathbf{a}+1}(g;\delta)\left(1 + \frac{\sqrt{\mathfrak{B}_m((u-t)^2;t)}}{\delta}\right) \\ &\leq 4M_g(1+\mathbf{a}^2)B_m(t) + \omega_{\mathbf{a}+1}(g;\delta)\left(1 + \frac{1}{\delta}\sqrt{B_m(t)}\right) \\ &\leq 4M_g(1+\mathbf{a}^2)B_m(\mathbf{a}) + \omega_{\mathbf{a}+1}(g;\delta)\left(1 + \frac{1}{\delta}\sqrt{B_m(\mathbf{a})}\right). \end{aligned}$$

By choosing $\delta = \sqrt{B_m(\mathbf{a})}$ and supremum over all $t \in (0,\mathbf{a}]$, we accomplish the proof of Theorem 5.1. □

Theorem 5.2. *For $g \in C^0_{t^2}(0,\infty)$, we have*

$$\lim_{m\to\infty} \|\mathfrak{B}_m(g;.) - g\|_2 = 0.$$

Proof. By the Korovkin theorem [14], we only need to prove the following three conditions:

$$\lim_{m\to\infty} \|\mathfrak{B}_m(u^i;.) - t^i\|_2 = 0, \ i = 0,1,2. \tag{21}$$

Since $\mathfrak{B}_m(1;t) = \mathfrak{T}_{m,0}(t) = 1$, the condition (21) holds for $i = 0$. From Lemma 2.3, we have

$$\|\mathfrak{B}_m(u;.) - t\|_2 = \sup_{t\in(0,\infty)} \frac{1}{1+t^2}\left|\left(1 + \frac{2}{m}\right)t + \frac{1}{m} - t\right| \leq \frac{2}{m}.$$

Hence, $\lim_{m\to\infty} \|\mathfrak{B}_m(u;.) - t\|_2 = 0$. Finally, we have

$$\begin{aligned} \|\mathfrak{B}_m(u^2;.) - t^2\|_2 &= \sup_{t\in(0,\infty)} \frac{1}{1+t^2}\left|\left(1 + \frac{7}{m} + \frac{6}{m^2}\right)t^2 + \left(\frac{5}{m} + \frac{6}{m^2}\right)t + \frac{1}{m^2} - t^2\right| \\ &\leq \frac{9.5}{m} + \frac{10}{m^2} \to 0, \text{ as } m \to \infty. \end{aligned}$$

Thus the proof is completed. □

Theorem 5.3. *For each $g \in C^0_{t^2}(0,\infty)$ and $s \in (0,\infty)$, we have*

$$\lim_{m\to\infty} \sup_{t\in(0,\infty)} \frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^{1+s}} = 0.$$

Proof. Let $t_0 \in (0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned} &\sup_{t\in(0,\infty)} \frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^{1+s}} \\ &\le \sup_{t\in(0,t_0)} \frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^{1+s}} + \sup_{t\in[t_0,\infty)} \frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^{1+s}} \\ &\le \|\mathfrak{B}_m(g;t)-g\|_{(0,t_0)} + M_g \sup_{t\in[t_0,\infty)} \frac{|\mathfrak{B}_m((1+u^2);t)|}{(1+t^2)^{1+s}} + \sup_{t\in[t_0,\infty)} \frac{|g(t)|}{(1+t^2)^{1+s}} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{22}$$

By $|g(t)| \le \|g\|_2(1+t^2)$, we have

$$I_3 = \sup_{t\in[t_0,\infty)} \frac{|g(t)|}{(1+t^2)^{1+s}} \le \sup_{t\in[t_0,\infty)} \frac{\|g\|_2(1+t^2)}{(1+t^2)^{1+s}} \le \frac{\|g\|_2}{(1+t_0^2)^s}.$$

$\forall \epsilon > 0$. Since $\lim_{m\to\infty} \sup_{t\in[t_0,\infty)} \frac{|\mathfrak{B}_m((1+u^2);t)|}{(1+t^2)} = 1$, there exists $M_1 \in \mathbb{N}$, such that for all $m > M_1$,

$$\frac{\|g\|_2 \mathfrak{B}_m((1+u^2);t)}{(1+t^2)^{1+s}} \le \frac{\|g\|_2}{(1+t^2)^{1+s}} \left((1+t^2) + \frac{\epsilon}{3\|g\|_2}\right) \le \frac{\|g\|_2}{(1+t^2)^s} + \frac{\epsilon}{3}. \tag{23}$$

Thus,

$$\|g\|_2 \sup_{t\in[t_0,\infty)} \frac{\mathfrak{B}_m((1+u^2);t)}{(1+t^2)^{1+s}} \le \frac{\|g\|_2}{(1+t_0^2)^s} + \frac{\epsilon}{3}, \ \forall m \ge M_1.$$

Hence

$$I_2 + I_3 < \frac{2\|g\|_2}{(1+t_0^2)^s} + \frac{\epsilon}{3}, \ \forall m \ge M_1. \tag{24}$$

Next, for sufficiently large t_0 such that $\frac{\|g\|_2}{(1+t_0^2)^s} < \frac{\epsilon}{6}$. Then, $I_2 + I_3 < \frac{2\epsilon}{3}, \forall m \ge M_1$. Applying Theorem 5.1, there exists $M_2 \in \mathbb{N}$, such that for all $m > M_2$,

$$\|\mathfrak{B}_m(g;t) - g\|_{C(0,t_0)} < \frac{\epsilon}{3}. \tag{25}$$

Let $M = \max\{M_1, M_2\}$. Combining (22), (23), (24) and (25), we have

$$\sup_{t\in(0,\infty)} \frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^{1+s}} < \epsilon, \quad \forall m \ge M.$$

Hence, the proof of Theorem 5.3 is completed. □

Theorem 5.4. *For $g \in C^0_{t^2}(0,\infty)$, then, for large enough m, we have*

$$\frac{|\mathfrak{B}_m(g;t)-g(t)|}{(1+t^2)^3} \le 279\Omega\left(g; \frac{1}{\sqrt{m}}\right).$$

Proof. Applying inequality (21), we have

$$|g(u)-g(t)| \leq \begin{cases} 4(1+\delta^2)^2(1+t^2)\Omega\,(g;\delta)\,, & |u-t|\leq \delta, \\ 4(1+\delta^2)^2(1+t^2)\frac{(u-t)^4}{\delta^4}\Omega\,(g;\delta)\,, & |u-t|>\delta. \end{cases}$$

By taking $\delta \in \left(0,\frac{1}{\sqrt{2}}\right)$, we have

$$\begin{aligned} |g(u)-g(t)| &\leq 4(1+\delta^2)^2(1+t^2)\left(1+\frac{(u-t)^4}{\delta^4}\right)\Omega\,(g;\delta) \\ &\leq 9(1+t^2)\left(1+\frac{(u-t)^4}{\delta^4}\right)\Omega\,(g;\delta)\,. \end{aligned} \tag{26}$$

Hence,

$$|\mathfrak{B}_m(g;t)-g(t)| \leq 9(1+t^2)\left(1+\frac{\mathfrak{B}_m\left((u-t)^4;t\right)}{\delta^4}\right)\Omega\,(g;\delta)\,. \tag{27}$$

Using (11), we easily obtain

$$\lim_{m\to\infty} m^2\mathfrak{B}_m\left((u-t)^4;t\right) = 15t^2(1+t)^2 < 30(1+t^2)^2.$$

Thus, for large enough $m>2$, $m^2\mathfrak{B}_m\left((u-t)^4;t\right) < 30(1+t^2)^2$. Choosing $\delta=\frac{1}{\sqrt{m}}$ in (27), we can yield the desired result. □

Theorem 5.5. *For $g\in C^0_{t^2}(0,\infty)$ satisfies $g',g''\in C^0_{t^2}(0,\infty)$, then, for large enough m, we have*

$$m\left|\mathfrak{B}_m(g;t)-g(t)-g'(t)A_m(t)-\frac{g''(t)}{2!}B_m(t)\right| \leq O(1)\Omega\left(g'';\frac{1}{\sqrt{m}}\right).$$

Proof. By Taylor's expansion formula for $g\in C^0_{t^2}(0,\infty)$, we have

$$\begin{aligned} g(u) &= g(t)+g'(t)(u-t)+\frac{g''(s)}{2!}(u-t)^2 \\ &= g(t)+g'(t)(u-t)+\frac{g''(t)}{2!}(u-t)^2+\Phi(u,t), \end{aligned} \tag{28}$$

where $|s-t|\leq|u-t|$ and

$$\Phi(u,t)=\frac{g''(s)-g''(t)}{2!}(u-t)^2.$$

Using (18) and choosing $\delta\in(0,1)$, we have

$$|\Phi(u,t)| = 2(1+\delta^2)^2(1+t^2)\Omega\,(g'';\delta)\left(1+\frac{(u-t)^4}{\delta^4}\right)(t-z)^2. \tag{29}$$

Combining (10), (28) and (29) and using the operators $\mathfrak{B}_m$ both sides of the above equality, we have

$$\begin{aligned} \mathfrak{B}_m\left(|\Phi(u,t)|;t\right) &\leq 2(1+\delta^2)^2(1+t^2)\Omega\,(g'';\delta)\left(\mathfrak{B}_m\left((u-t)^2;t\right)+\frac{\mathfrak{B}_m\left((u-t)^6;t\right)}{\delta^4}\right) \\ &\leq 2(1+\delta^2)^2(1+t^2)\Omega\,(g'';\delta)\left(B_m(t)+\frac{\mathfrak{B}_m\left((u-t)^6;t\right)}{\delta^4}\right) \\ &\leq 2(1+\delta^2)^2(1+t^2)\Omega\,(g'';\delta)\left(O\left(\frac{1}{m}\right)+\frac{1}{\delta^4}O\left(\frac{1}{m^3}\right)\right). \end{aligned}$$

Taking $\delta=\frac{1}{\sqrt{m}}$, we complete the proof of Theorem 5.5. □

From Theorem 5.5, we obtain the following immediate corollary:

Corollary 5.6. *For $g, h \in C^0_{t^2}(0,\infty)$ satisfy gh, g', h', $(gh)'$, g'', h'' and $(gh)'' \in C^0_{t^2}(0,\infty)$. Then, for any $t \in (0,\infty)$, we have*

$$\lim_{m\to\infty} m\left(\mathfrak{B}_m(gh;t) - \mathfrak{B}_m(g;t)\cdot\mathfrak{B}_m(h;t)\right) = 3(t+t^2)g'(t)h'(t).$$

6. Voronovskaya type Theorem

Theorem 6.1. *For $g \in C[0,\infty)$ and g'' exists at a point $t \in (0,\infty)$, then.*

$$\lim_{m\to\infty} m\left(\mathfrak{B}_m(g;t) - g(t)\right) = (1+2t)g'(t) + \frac{3}{2}(t+t^2)g''(t).$$

Proof. By the Taylor's expansion formula for g, we have

$$g(u) = g(t) + g'(t)(u-t) + \frac{1}{2}g''(t)(u-t)^2 + R_2(u;t)(u-t)^2,$$

where

$$R_2(u;t) = \begin{cases} \dfrac{g(u) - g(t) - g'(u)(u-t) - \frac{1}{2}g''(u)(u-t)^2}{(u-t)^2}, & u \neq t; \\ 0, & u = t. \end{cases}$$

Applying the L'Hospital's Rule,

$$\lim_{u\to t} R_2(u;t) = \frac{1}{2}\lim_{u\to t}\frac{g'(u) - g'(t)}{u-t} - \frac{1}{2}g''(t) = 0.$$

Thus, $R(.;t) \in C(0,\infty)$. Consequently, we can write

$$\mathfrak{B}_m(g;t) - g(t) = A_m(t)g'(t) + \frac{1}{2}B_m(t)g''(t) + \mathfrak{B}_m\left((R_2(u;t)(u-t)^2;t\right).$$

By the Cauchy-Schwarz inequality, we have

$$\left|m\mathfrak{B}_m\left((R_2(u;t)(u-t)^2;t\right)\right| \leq \sqrt{\mathfrak{B}_m\left(R_2^2(u;t);t\right)}\sqrt{m^2\mathfrak{B}_m\left((u-t)^4;t\right)}.$$

We observe that $R_2^2(u;t) = 0$ and $R_2^2(u;t) \in C(0,\infty)$. Then, it follows in Lemma 2.6 that

$$\lim_{m\to\infty} \mathfrak{B}_m\left((R_2^2(u;t);t\right) = R_2^2(t;t) = 0.$$

Hence, from (9), we can obtain

$$\lim_{l\to\infty} m\mathfrak{B}_m\left((R_2(u;t)(u-t)^2;t\right) = 0.$$

Combining (7) and (8), we complete the proof of Theorem 6.1. □

Corollary 6.2. *For $g'' \in C[0,\infty)$ then we have*

$$\lim_{m\to\infty} m\left(\mathfrak{B}_m(g;t) - g(t)\right) = (1+2t)g'(t) + \frac{3}{2}(t+t^2)g''(t),$$

uniformly with respect to any finite interval $I \subset (0,\infty)$.

7. Kantorovich and Durrmeyer Type Generalizations

Kantorovich and Durrmeyer Type Generalizations of classical probability operators and their modifications have always been a hot topic since the 1930s, we mention some of them:[1, 2, 16–18, 20, 22–25, 27, 28, 31]. Let $\mathfrak{b}_{m,k}(t) = \frac{1}{m}\binom{m+k-1}{k}\frac{t^{k-1}}{(1+t)^{m+k+1}}(mt-k)^2$, $k = 0, 1, \cdots$ denote the new modified Baskakov basis function, then the operators (1) can be re-written as:

$$\mathfrak{B}_m(g;t) = \sum_{k=0}^{\infty} \mathfrak{b}_{m,k}(t) g\left(\frac{k}{m}\right), \ t \in (0,\infty), \ m \in \mathbb{N}_+.$$

The Kantorovich type generalization of the operators (1) is defined by

$$\mathfrak{K}_m(g;t) = m \sum_{k=0}^{\infty} \mathfrak{b}_{m,k}(t) \int_{\frac{k}{m}}^{\frac{k+1}{m}} g(u)\mathrm{d}u, \ t \in (0,\infty), \ m \in \mathbb{N}_+. \tag{30}$$

Applying the property of the Beta function: $B(n,m) = \int_0^\infty \frac{u^{n-1}}{(1+u)^{n+m}}\mathrm{d}u$, $n, m \in \mathbb{N}_+$, for any $i \in \mathbb{N}_+$, we can obtain

$$\int_0^\infty \mathfrak{b}_{m,k}(u)u^i \mathrm{d}i = \frac{(m-i-2)!(i+k-1)!}{m!k!}\left(i(i+1)(m+k)+mk\right), m \geq i+2. \tag{31}$$

Hence, the Durrmeyer type generalization of the operators (1) can be defined by

$$\mathfrak{D}_m(g;t) = (m-1)\sum_{k=0}^{\infty} \mathfrak{b}_{m,k}(t) \int_0^\infty \mathfrak{b}_{m,k}(u)g(u)\mathrm{d}u, \ t \in (0,\infty), \ m \geq 2. \tag{32}$$

Then, we directly compute the first and second order moments for the operators (30) and the operators (32):

$$\mathfrak{K}_m(1;t) = 1; \ \mathfrak{K}_m(u;t) = \left(1+\frac{2}{m}\right)t + \frac{3}{2m};$$
$$\mathfrak{K}_m(u^2;t) = \left(1+\frac{7}{m}+\frac{6}{m^2}\right)t^2 + \left(\frac{6}{m}+\frac{8}{m^2}\right)t + \frac{7}{3m^2};$$
$$\mathfrak{D}_m(1;t) = 1; \ \mathfrak{D}_m(u;t) = \frac{(m+2)^2}{m(m-2)}t + \frac{3m+2}{m(m-2)}, \ m \geq 3;$$
$$\mathfrak{D}_m(u^2;t) = \frac{(m+1)(m+6)^2}{m(m-2)(m-3)}t^2 + \frac{12m^2+56m+48}{m(m-2)(m-3)}t + \frac{14m+12}{m(m-2)(m-3)}, \ m \geq 4.$$

Now, Theorem 6.1 can be modified as following:

Theorem 7.1. *For $g \in C[0,\infty)$ and g'' exists at a point $t \in (0,\infty)$, then.*

$$\lim_{m\to\infty} m\left(\mathfrak{K}_m(g;t) - g(t)\right) = \left(\frac{3}{2}+2t\right)g'(t) + \frac{3}{2}(t+t^2)g''(t),$$
$$\lim_{m\to\infty} m\left(\mathfrak{D}_m(g;t) - g(t)\right) = (3+6t)g'(t) + 3(t+t^2)g''(t).$$

The proving methods are similar, hence, we omit the details.

8. Numerical Examples

In this section we will analyze the theoretical results presented in the previous sections by numerical examples.

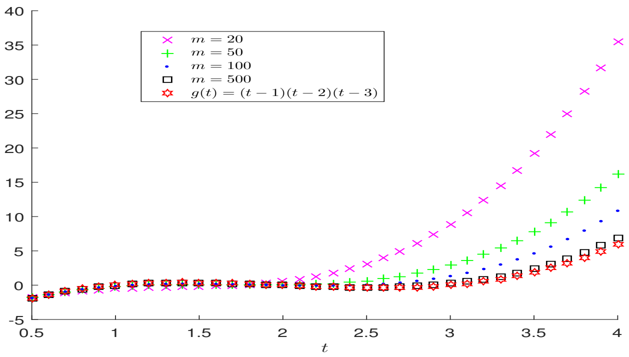

Figure 1: Approximation process by $\mathfrak{B}_m$

Table 1: Error of approximation $E_{m,1}(g;t)$ for $m = 20, 50, 100, 500$

t	$E_{20,1}(g;t)$	$E_{50,1}(g;t)$	$E_{100,1}(g;t)$	$E_{500,1}(g;t)$
0.5	0.0469	0.0238	0.0128	0.0027
1.0	0.5381	0.2310	0.1178	0.0239
1.5	0.5839	0.3101	0.1671	0.0353
2.0	0.5456	0.0229	0.0196	0.0088
2.5	3.4864	1.0047	0.4399	0.0782
3.0	8.8744	2.8717	1.3269	0.2482
3.5	17.3456	5.8606	2.7567	0.5240
4.0	29.5361	10.2078	4.8449	0.9281

Let $E_{m,1}(g;t) = \left|\mathfrak{B}_m(g;t) - g(t)\right|$, $E_{m,2}(g;t) = \left|\mathfrak{K}_m(g;t) - g(t)\right|$, $E_{m,3}(g;t) = \left|\mathfrak{D}_m(g;t) - g(t)\right|$ denote respectively the error function of approximation by the operators $\mathfrak{B}_m$, $\mathfrak{K}_m$, $\mathfrak{D}_m$.

First, we consider the function $g(t) = (t-1)(t-2)(t-3)$, $t \in [0.5, 4]$. The convergence of the operators $\mathfrak{B}_m$ to function g is showed in Figure 1. Meantime, we compute the error of approximation for $E_{20,1}$, $E_{50,1}$, $E_{100,1}$, $E_{500,1}$ at points $\{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0\}$ in Table 1.

Second, we consider the function $g(t) = (t-2)(t-3)(t-4)$, $t \in [1, 8]$. The convergence of the operators $\mathfrak{K}_m$ to function g is showed in Figure 2. Meantime, we compute the error of approximation for $E_{15,2}$, $E_{30,2}$, $E_{60,2}$, $E_{120,2}$ at points $\{1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0\}$ in Table 2.

Table 2: Error of approximation $E_{m,2}(g;t)$ for $m = 15, 30, 60, 120$

t	$E_{15,2}(g;t)$	$E_{30,2}(g;t)$	$E_{60,2}(g;t)$	$E_{120,2}(g;t)$
1.0	1.4479	0.8002	0.4204	0.2155
2.0	3.8171	2.0366	1.0444	0.5280
3.0	4.4635	3.0827	1.7384	0.9165
4.0	6.5648	0.9206	0.0974	0.1841
5.0	36.3238	13.2319	5.4425	2.4352
6.0	91.8694	37.1098	16.4453	7.7073
7.0	180.2577	75.8130	34.4750	16.3980
8.0	308.5446	132.6003	61.0956	29.2733

Finally, we consider the function $g(t) = (t-2)(t-9)$, $t \in [1, 10]$. The convergence of the operators $\mathfrak{D}_m$ to function g is showed in Figure 3. Meantime, we compute the error of approximation for $E_{25,3}$, $E_{75,3}$, $E_{150,3}$, $E_{300,3}$, $E_{500,3}$ at points $\{1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0\}$ in Table 3.

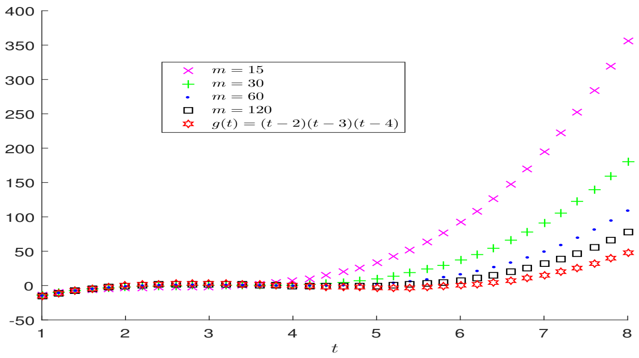

Figure 2: Approximation process by $\mathfrak{K}_m$

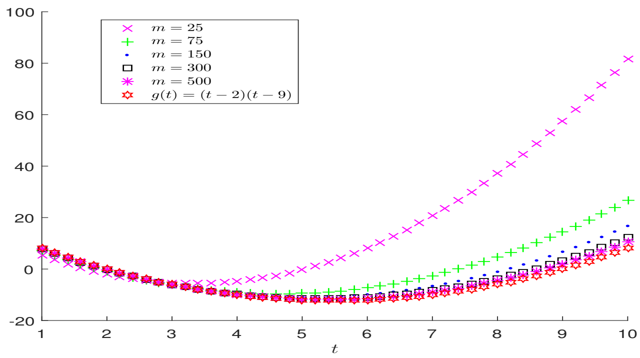

Figure 3: Approximation process by $\mathfrak{D}_m$

Table 3: Error of approximation $E_{m,3}(g;t)$ for $m = 25, 75, 150, 300, 500$

t	$E_{25,3}(g;t)$	$E_{75,3}(g;t)$	$E_{150,3}(g;t)$	$E_{300,3}(g;t)$	$E_{500,3}(g;t)$
1.0	2.7080	0.9186	0.4599	0.2300	0.1380
2.0	2.0212	0.8539	0.4443	0.2262	0.1366
3.0	0.6160	0.2594	0.1767	0.0994	0.0622
4.0	5.2035	0.8650	0.3430	0.1504	0.0853
5.0	11.7413	2.5192	1.1148	0.5231	0.3058
6.0	20.2296	4.7033	2.1387	1.0188	0.5995
7.0	30.6681	7.4173	3.4147	1.6375	0.9661
8.0	43.0571	10.6611	4.9428	2.3792	1.4059
9.0	57.3964	14.4348	6.7229	3.2438	1.9187
10.0	73.6860	18.7384	8.7552	4.2315	2.5046

9. Conclusion

In the paper, we construct new modification of Baskakov operators on $(0,\infty)$ using the second central moment of the classical Baskakov operators. And the moments and the central moments computation formulas and their quantitative properties are computed. Then, rate of convergence, point-wise estimates, weighted approximation and Voronovskaya type theorem for the new operators are established. Also, Kantorovich and Durrmeyer type generalizations are discussed. Finally, some graphs and numerical examples are showed by using Matlab algorithms.

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