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A Relational Improvement of a True Particular Case of Fierro's Maximality Theorem

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Abstract. In this paper, by using relational notations, we improve and supplement a true particular case of an inaccurate maximality theorem of Raúl Fierro from 2017, which has to be proved in addition to Zorn's lemma and a famous maximality principle of H. Brézis and F. Browder.

1. Introduction

In [7], by considering an ordered set X and using the notation

$$S(x) = \Big\{ y \in X : \quad y \geq x \Big\},$$

Brézis and Browder proved the following important maximality principle as an immediate consequence of a more general result.

Theorem 1.1. *Let $\phi : X \to \mathbb{R}$ be a function, bounded above, and satisfying*

(1) $x \leq y$ *and* $x \neq y$ *imply* $\phi(x) < \phi(y)$;

(2) *for any increasing sequence* $\{x_n\}$ *in* X, *there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n .

Then, for each $a \in X$ *there exists* $\bar{a} \in X$ *such that* $a \leq \bar{a}$ *and* $\bar{a}$ *is maximal (i. e.,* $S(\bar{a}) = \{\bar{a}\}$ *) .*

In [18], by considering a preordering $\preceq$ on a nonempty set X and using the notation

$$S(x, \preceq) = \Big\{ y \in X : \quad x \preceq y \Big\}$$

for all $x \in X$, Fierro tried to prove the following closely related maximality theorem by generalizing and supplementing a similar theorem of Park [31, Theorem 1]. (See also [28, 29] for some more general settings.)

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Theorem 1.2. *Let $x_0 \in X$. The following eight conditions are equivalent:*

(1) *there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$;*

(2) *there exists $x_1 \in S(x_0, \preceq)$ such that for each chain C in $S(x_1, \preceq)$, $\bigcap_{x \in C} S(x, \preceq) \neq \emptyset$;*

(3) *there exist $x_1 \in S(x_0, \preceq)$ and a maximal chain C^* in $S(x_1, \preceq)$ such that $\bigcap_{x \in C^*} S(x, \preceq) \neq \emptyset$;*

(4) *for each $T : S(x_0, \preceq) \to 2^X$ such that, for each $x \in S(x_0, \preceq) \setminus Tx$, there exists $y \in X \setminus \{x\}$ satisfying $x \preceq y$, there exists $z \in S(x_0, \preceq)$ such that $z \in Tz$;*

(5) *any function $f : S(x_0, \preceq) \to X$ such that $x \preceq f(x)$, for all $x \in S(x_0, \preceq)$, has a fixed point;*

(6) *for each $T : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ such that $x \preceq y$, for all $x \in S(x_0, \preceq)$ and $y \in Tx$, there exists $z \in S(x_0, \preceq)$ such that $Tz = \{z\}$;*

(7) *any family $\mathcal{F}$ of functions $f : S(x_0, \preceq) \to X$ such that $x \preceq f(x)$, for all $x \in S(x_0, \preceq)$, has a common fixed point;*

(8) *for any subset Y of X such that $S(x_0, \preceq) \cap Y = \emptyset$, there exists x in $S(x_0, \preceq) \setminus Y$ satisfying $S(x, \preceq) = \{x\}$.*

In [23], by considering a preorder relation S on a nonvoid set X and a function φ of X to $\mathbb{R}$, and using the notation

$$S(x) = \Big\{ y \in X : \quad (x, y) \in S \Big\}$$

for all $x \in X$, the second and the third authors have proved the following relational improvement and straightforward generalization of Theorem 1.1 of Brézis and Browder.

Theorem 1.3. *Suppose that $a \in X$ such that*

(1) *φ is bounded above on $S(a)$;*

(2) *φ is either strictly increasing or injective and increasing on $S(a)$;*

(3) *each increasing sequence $(x_n)_{n=1}^{\infty}$ in $X(S)$, with $x_1 = a$, is bounded above.*

Then, there exists $b \in S(a)$ such that b is a strongly maximal element of $X(S)$ in the sense that $S(b) = \{b\}$.

Now, knowing that the implication (3) $\Longrightarrow$ (4) in Theorem 1.2 of Fierro is not true without assuming the antisymmetry of $\preceq$ [6], and noticing that assertion (8) is also not very well formulated, we shall prove a similar improved and supplemented form of a true particular case of Theorem 1.2 which has to be treated after Zorn's lemma [27, p. 532] and Theorem 1.3.

2. Relations and Functions

A subset R of a product set $X \times Y$ is called a *relation on X to Y*. In particular, a relation R on X to itself is called a *relation on X*. And, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* of X.

If R is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $R(x) = \{y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the *images* or *neighbourhoods* of x and A under R, respectively.

If $(x, y) \in R$, then instead of $y \in R(x)$, we may also write $x R y$. However, instead of $R[A]$, we cannot write $R(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_R = \{x \in X : R(x) \neq \emptyset\}$ and $R[X]$ may be called the *domain* and *range* of R, respectively. And, if $D_R = X$, then we may say that R is a *relation of X to Y*, or that R is a *non-partial relation on X to Y*.

If R is a relation on X to Y and $U \subseteq D_R$, then the relation $R | U = R \cap (U \times Y)$ is called the *restriction* of R to U. Moreover, if R and S are relations on X to Y such that $D_R \subseteq D_S$ and $R = S | D_R$, then S is called an *extension* of R.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*(x, y)$, respectively.

If R is a relation on X to Y, then a function f of D_R to Y is called a *selection function of* R if $f(x) \in R(x)$ for all $x \in D_R$. Thus, by the Axiom of Choice [16], we can see that every relation is the union of its selection functions.

For a relation R on X to Y, we may naturally define two *set-valued functions* φ_R of X to $\mathcal{P}(Y)$ and Φ_R of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_R(x) = R(x)$ for all $x \in X$ and $\Phi_R(A) = R[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful tools than relations on X to Y. In [43], they were briefly called *corelations on X to Y*.

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation on X to Y* [34, 45]. Thus, closures (proximities) [47] are super (hyper) relations.

Note that a super relation on X to Y is an arbitrary subset of $\mathcal{P}(X)\times Y$. While, a corelation on X to Y is a particular subset of $\mathcal{P}(X)\times\mathcal{P}(Y)$. Thus, set inclusion is a natural partial order for super relations, but not for corelations.

For a relation R on X to Y, the relation, $R^c = (X\times Y) \setminus R$ is called the *complement* of R. Thus, it can be easily seen that $R^c(x) = R(x)^c = Y \setminus R(x)$ for all $x \in X$, and $R^c[A]^c = \bigcap_{a\in A} R(a)$ for all $A \subseteq X$.

Moreover, the relation $R^{-1} = \{(y, x) \in Y\times X : \ (x, y) \in R\}$ is called the *inverse* of F. Thus, it can be easily seen that $R^{-1}(y) = \{x \in X : \ y \in R(x)\}$ for all $y \in Y$, and $R^{-1}[B] = \{x \in X : \ R(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$.

If R is a relation on X to Y, then we have $R = \bigcup_{x\in X} (\{x\} \times R(x))$. Therefore, the values $R(x)$, where $x \in X$, uniquely determine R. Thus, a relation R on X to Y can also be naturally defined by specifying $R(x)$ for all $x \in X$.

For instance, if S is a relation on Y to Z, then the *composition* $S\circ R$ can be defined such that $(S\circ R)(x) = S[R(x)]$ for all $x \in X$. Thus, it can be easily seen that $(S \circ R)[A] = S\big[R[A]\big]$ also holds for $A \subseteq X$.

While, if S is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(R \boxtimes S)(x, z) = R(x) \times S(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(R \boxtimes S)[A] = S \circ A \circ R^{-1}$ for all $A \subseteq X\times Z$ [40].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

3. Preorder relations

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R \subseteq R^{-1}$, *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$, and *total* if $X^2 \subseteq R \cup R^{-1}$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For a relation R on X, we may now also naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [20].

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup (A^c \times X)$ is an important preorder on X [33]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r^d = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [49].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cup \left(A^c\right)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Now, as a straightforward generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = (A \times B) \cup (A^c \times Y)$ [22].

However, it is now more important to note that if $\mathcal{A} = (A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}(X)$, then the *Cantor relation* $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^{\infty} \left(A_n \times A_n^c\right)$ is also an important preorder on X [30].

Note that if R is only reflexive relation on X and $x \in X$, then $\mathcal{A}_R(x) = \left(R^n(x)\right)_{n=1}^{\infty}$ is already an increasing sequence in $\mathcal{P}(X)$. Thus, the preorder relation $R_{\mathcal{A}_R(x)}$ may also be naturally investigated.

Moreover, for a real function φ of X and a quasi-pseudo-metric d on X [17], the *Brøndsted relation* $R_{(\varphi,d)} = \{(x, y) \in X^2 : \ d(x, y) \leq \varphi(y) - \varphi(x)\}$ is also an important preorder on X [8] .

From this relation, by letting φ and d to be the zero functions, we can obtain the *specialization* and *preference relations* $R_d = \{(x, y) \in X^2 : \ d(x, y) = 0\}$ and $R_{\varphi} = \{(x, y) \in X^2 : \ \varphi(x) \leq \varphi(y)\}$, respectively. (See [14, 48].)

If R is a preorder relation on X, then having in mind the widely used abbreviation *poset* introduced by Birkhoff [3] for a partially ordered set, the ordered pair $X(R) = (X, R)$ may be called a *proset* (preordered set).

More generally, if R is a relation on X, then the pair $X(R)$ may be called a *goset* (generalized ordered set) [41]. While, if R is a relation on X to Y, then the pair $(X, Y)(R) = \big((X, Y), R\big)$ may be called a *formal context* [15, 19].

Instead of "formal context", the terms "relational space" or "simple relator space" may also be used. Namely, if $\mathcal{R}$ is a family of relations on X to Y, then the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ was called a *relator space* [36].

Several important notions used in posets, metric and topological spaces can be naturally generalized to relator spaces [36, 46]. However, instead of arbitrary relators it is frequently sufficient to consider only preorder relators [1, 38].

If R is a relation on X to Y, then for any $B \subseteq Y$, we may naturally define

$$\mathrm{Int}_R(B) = \left\{A \subseteq X : \ R[A] \subseteq B\right\} \qquad \text{and} \qquad \mathrm{int}_R(B) = \left\{x \in X : \ \{x\} \in \mathrm{Int}_R(B)\right\}.$$

Moreover, we may also naturally define $\mathcal{E}_R = \{B \subseteq Y : \ \mathrm{int}_R(B) \neq \emptyset\}$.

Furthermore, we may also naturally define

$$\mathrm{Lb}_R(B) = \left\{A \subseteq X : \ A \times B \subseteq R\right\} \qquad \text{and} \qquad \mathrm{lb}_R(B) = \left\{x \in X : \ \{x\} \in \mathrm{Lb}_R(B)\right\}.$$

However, these algebraic tools are not independent from the former topological ones. Namely, by [36], we have $\mathrm{Lb}_R = \mathrm{Int}_{R^c} \circ \mathcal{C}_Y$ and $\mathrm{lb}_R = \mathrm{int}_{R^c} \circ \mathcal{C}_Y$.

Now, if R is a relation on X, then for any $A \subseteq X$ we may also naturally define $\min_R(A) = A \cap \mathrm{lb}_R(A)$ and $\sup_R = \min_R\big(\mathrm{ub}_R(A)\big)$, where $\mathrm{ub}_R = \mathrm{lb}_{R^{-1}}$. Thus, if R is antisymmetric, then $\min_R(A)$ and $\sup_R(A)$ are at most singletons.

4. Maximal elements and fixed points

In the present paper, we shall only need very few basic notions in connection with prosets which can, in the same easy way, be defined even in a goset.

For instance, a subset A of a goset $X(R)$ will be called a *chain* if the restriction of the relation R to A is total. That is, either $x R y$ or $y R x$ for all $x, y \in A$.

Moreover, we may also naturally introduce the following

Definition 4.1. An element x of a goset $X(S)$ will be called

(1) *maximal* if $x\,S\,y$ implies $y\,S\,x$ for all $y \in X$;
(2) *strongly maximal* if $x\,S\,y$ implies $x = y$ for all $y \in X$.

Remark 4.2. Thus, x is maximal if and only if S is *symmetric at* x, i. e., $S(x) \subseteq S^{-1}(x)$. And, x is a strongly maximal if and only if $S(x) \subseteq \{x\}$. Note that S is *reflexive at* x, i. e., $x \in S(x)$, if and only if $\{x\} \subseteq S(x)$.

In addition to (1) and (2), x may also be naturally called *quasi-strongly maximal* if $x\,S\,y$ implies $S(x) = S(y)$ for all $y \in X$. Thus, it can be shown that x is quasi-strongly maximal if and only if $S(x) \subseteq S^{\circ}(x)$, with $S^{\circ} = \left(S^{-1} \circ S^{c}\right)^{c}$ [35].

Now, if S is reflexive and x is *quasi-strongly maximal*, then we can at once see that x is maximal. Moreover, if S is antisymmetric and x is maximal, then x is strongly maximal. Therefore, in a poset the three notions coincide.

In this respect, it is also worth noticing that S is symmetric if and only if every element of $X(S)$ is maximal. Moreover, x is a strongly maximal element of $X(S)$ if and only if $x\,S\,y$ does not hold for all $y \in X$ with $y \neq x$.

To check the latter statement, note that

$$S(x) \subseteq \{x\} \iff \{x\}^{c} \subseteq S(x)^{c} \iff S(x) \cap \{x\}^{c} = \emptyset .$$

Analogously to Definition 4.1, we may also naturally introduce the following

Definition 4.3. If T is a relation on X, then an element x of X will be called a

(1) *fixed point* of T if $x \in T(x)$;
(2) *strong fixed point* of T if $T(x) = \{x\}$.

Remark 4.4. Thus, x is fixed point of T if and only if T is reflexive at x. Moreover, x is a strong fixed point of T if and only if x is both a fixed point of T and a strongly maximal element of $X(T)$.

Our present terminology can only be motivated by the latter fact. Namely, in the existing literature, instead of "strong fixed point" usually the terms "stationary point", "invariant point" or "endpoint" are used [2, 24].

In addition to Definitions 4.1 and 4.3, we may also naturally introduce

Definition 4.5. A relation T on a goset $X(S)$ will be called

(1) *intensive* if for each $x \in X$ there exists $y \in T(x)$ such that $y\,S\,x$;
(2) *extensive* if for each $x \in X$ there exists $y \in T(x)$ such that $x\,S\,y$.

Remark 4.6. For a function f of poset $X(S)$ to itself, besides "extensive", the terms "expansive", "progressive", "inflationary" and "noncontractive" are also frequently used.

More curiously, Davey and Pristley [15, p. 186] would even call a function f of a goset $X(S)$ to itself to be "increasing" if each $x \in X$ is a "post-fixpoint" of f in the sense that $x\,S\,f(x)$.

Now, a function f of one goset $X(R)$ to another $Y(S)$ will, of course, be called *increasing* if $x\,R\,y$ implies $f(x)S\,f(y)$ for all $x,\ y \in X$. However, to define *strict increasingness*, we have at least two reasonable possibilities

Namely, for any relation R on X, instead of the usual strict relation $R \setminus \Delta_X$, we may also naturally consider the relation $R \setminus R^{-1}$ [32]. Note that $R \setminus R^{-1} \subseteq R \setminus \Delta_X$ if R is reflexive, and $R \setminus \Delta_X \subseteq R \setminus R^{-1}$ if R is antisymmetric.

A simple reformulations of property (2) in Definition 4.5 gives the following

Theorem 4.7. *For a relation T on a goset $X(S)$, the following assertions are equivalent:*

(1) *T is extensive;*

(2) *$S^{-1} \circ T$ is reflexive on X;*

(3) *$S(x) \cap T(x) \neq \emptyset$ for all $x \in X$.*

Remark 4.8. Thus, if T is extensive on $X(S)$, then T is in particular non-partial on X. Moreover, if both S and T are reflexive on X, then T is already extensive on $X(S)$.

The importance of strongly maximal elements is apparent from the following generalization of a simple, but important observation of Brøndsted [9, 10].

Theorem 4.9. *If T is an extensive relation on a goset $X(S)$, then each strongly maximal element x of $X(S)$ is a fixed point of T.*

Proof. Since T is extensive, there exists $y \in T(x)$ such that $x\,S\,y$. Hence, since x is strongly maximal, we can infer that $x = y$. Therefore, $x \in T(x)$, and thus x is a fixed point of T. □

Thus, in particular, we can also state the following

Corollary 4.10. *If f is an extensive function of a goset $X(S)$ to itself, then each strongly maximal element x of $X(S)$ is a fixed point of f.*

Now, by modifying an argument of Khamsi [26], we can also prove the following partial converse to the above theorem.

Theorem 4.11. *If every extensive function of a goset $X(S)$ to itself has a fixed point, then $X(S)$ has a strongly maximal element.*

Proof. Assume, on the contrary, that each $x \in X$ is not a strongly maximal element of $X(S)$. Then, by Definition 4.1, for each $x \in X$ there exists $y \in X$ such that $x\,S\,y$, but $x \neq y$. Hence, we can infer that $y \in S(x) \setminus \{x\}$. Define

$$T(x) = S(x) \setminus \{x\}$$

for all $x \in X$. Then, by the above observation, T is a non-partial relation on X. Thus, by the Axiom of Choice, there exists a function f of X to itself such that

$$f(x) \in T(x) = S(x) \setminus \{x\} \subseteq S(x)$$

for all $x \in X$. Hence, by Definition 4.5, we can see that f is an extensive function of $X(S)$. Thus, by the assumption of the theorem, there exists an $x \in X$ such that $f(x) = x$. Hence, we can infer that $x = f(x) \in T(x) = S(x) \setminus \{x\}$. Therefore, $x \notin \{x\}$, and thus $x \neq x$. This contradiction proves the assertion of the theorem. □

Thus, as a slight generalization of the dual of [26, Theorem 1] of Khamsi, we can also state

Corollary 4.12. *For a goset $X(S)$, the following assertions are equivalent:*

(1) *$X(S)$ has a strongly maximal element;*

(2) *every extensive relation on $X(S)$ has a fixed point;*

(3) *every extensive function of $X(S)$ to itself has a fixed point.*

Remark 4.13. To further clarify the importance of extensive relations, we can note that a *closure operation* on $X(S)$ is, by definition, an extensive function of $\mathcal{P}(X)$ to itself.

Moreover, a strictly increasing function f of a *well-ordered* set $X(S)$ to itself is extensive. For this, by [41, Theorem 79], we need only assume that S is antisymmetric and *min–complete* in the sense that $\min_S(A) \neq \emptyset$ if $\emptyset \neq A \subseteq X$.

5. Equivalent conditions for the existence of maximal elements

To have a true, improved particular case of Theorem 1.2 of Fierro, suggested by [31, Theorem 1] of Park, in addition to Theorem 1.3 we can now also prove

Theorem 5.1. *If S is a partial order on a nonvoid set X, then for each $a \in X$ the following assertions are equivalent:*

(1) *there exists $b \in S(a)$ such that b is a strongly maximal element of $X(S)$;*

(2) *there exists $b \in S(a)$ such that for each chain C in $S(b)$ we have $\bigcap_{x \in C} S(x) \neq \emptyset$;*

(3) *there exist $b \in S(a)$ and a maximal chain C in $S(b)$ such that $\bigcap_{x \in C} S(x) \neq \emptyset$;*

(4) *for every relation T on X such that $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in X$ with $x \in S(a) \setminus T(x)$, there exists $b \in S(a)$ such that b is a fixed point of T;*

(5) *if $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a)$, then for every relation T on X there exists $b \in S(a)$ such that b is a fixed point of T;*

(6) *for every extensive function f of X to itself there exists $b \in S(a)$ such that b is a fixed point of f;*

(7) *for every non-partial relation T on X such that, $y \in S(x)$ for all $x \in S(a)$ and $y \in T(x) \setminus \{x\}$, there exists $b \in S(a)$ such that b is a strong fixed point of T;*

(8) *for every nonvoid family $\mathcal{F}$ of extensive functions of X to itself there exists $b \in S(a)$ such that b is a fixed point of each element f of $\mathcal{F}$;*

(9) *for any $Y \subseteq X$, such that $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a) \setminus Y$, we have $S(a) \cap Y \neq \emptyset$.*

Proof. If assertion **(1)** holds, then by Remark 4.2 and the reflexivity of S, we have $S(b) = \{b\}$, Hence, we can at once see that $\emptyset$ and $\{b\}$ are the only chains in $S(b)$. Thus, since

$$\textstyle\bigcap_{x \in \emptyset} S(x) = \bigcap \emptyset = X \neq \emptyset \qquad \text{and} \qquad \bigcap_{x \in \{b\}} S(x) = S(b) = \{b\} \neq \emptyset,$$

it is clear that assertion **(2)** also holds.

Suppose now that assertion **(2)** holds. Then, there exists $b \in X$ such that $\bigcap_{x \in C} S(x) \neq \emptyset$ for every chain C in $S(b)$. Moreover, by a generalized Hausdorff maximal principle [27, p. 529], there exists a maximal chain C in $S(b)$ even if it is only assumed that S is a preorder on X. Furthermore, by the above intersection property, we now also have $\bigcap_{x \in C} S(x) \neq \emptyset$. Therefore, assertion **(3)** also holds.

Suppose next that assertion **(3)** holds and T is as in (4). Then, by assertion (3), there exist $c \in S(a)$ and a maximal chain C in $S(c)$ such that $\bigcap_{x \in C} S(x) \neq \emptyset$. Thus, because of the reflexivity of S, we necessarily have $C \neq \emptyset$. Namely, otherwise $\{c\}$ would be a bigger chain in $S(c)$. Moreover, by the above intersection property, there exists $b \in X$ such that

$$b \in S(x) \qquad \text{for all} \qquad x \in C.$$

Now, by taking $x \in C$, and using the inclusions $C \subseteq S(c)$ and $c \in S(a)$, and the transitivity of S, we can see that

$$x \in S(c) \subseteq S[\,S(a)\,] \subseteq S(a), \quad \text{and thus also} \quad S(x) \subseteq S[\,S(a)\,] \subseteq S(a).$$

Hence, since $x \in C$ implies $b \in S(x)$, we can see that $b \in S(a)$ also holds. Next, we show that b is a fixed point of T. For this, assume on the contrary that $b \notin T(b)$. Then, $b \in S(a) \setminus T(b)$. Thus, by our assumption on T, we have $S(b) \setminus \{b\} \neq \emptyset$. Therefore, there exists $y \in X$ such that

$$y \in S(b) \qquad \text{and} \qquad y \neq b.$$

Now, if $x \in C$, then by the inclusion $C \subseteq S(c)$ and the choice of b, we can see that $x \in S(c)$ and $b \in S(x)$. Hence, by using the transitivity of S, we can infer that

$$y \in S(b) \subseteq S[S(x)] \subseteq S(x), \quad \text{and thus also} \quad y \in S(x) \subseteq S[S(c)] \subseteq S(c).$$

This shows that $C \cup \{y\}$ is also a chain in $S(c)$. Hence, by using the maximality of C in $S(c)$, we can infer that $y \in C$, and thus $b \in S(y)$ by the choice of b. Now, because of $y \in S(b)$ and the antisymmetry of S, we can also state that $y = b$. This contradiction proves that $b \in T(b)$. Therefore, assertion **(4)** also holds.

It is clear that assertion **(4)** implies assertion **(5)**. Namely, if $S(x) \setminus \{x\} \neq \emptyset$ holds for all $x \in S(a)$, then for any relation T on X we have $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in X$ with $x \in S(a) \setminus T(x)$. Thus, assertion (4) can be applied to derive the existence of a fixed point b of T in $S(a)$.

Therefore, suppose now that assertion **(5)** holds and f is as in assertion (6). Then, by the extensivity of f, we have $f(x) \in S(x)$ for all $x \in X$. Moreover, if we assume, on the contrary, that f does not have a fixed point in $S(a)$, then we can also state that $f(x) \neq x$, and thus $f(x) \notin \{x\}$ for all $x \in S(a)$. Therefore, we actually have $f(x) \in S(x) \setminus \{x\}$, and thus $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a)$. Hence, by assertion (5), we can infer that every relation T on X has a fixed point b in $S(a)$. Thus, if in particular $T(x) = \{f(x)\}$ for all $x \in X$, then T also has a fixed point b in $S(a)$. That is, there exists $b \in S(a)$ such that $b \in T(b) = \{f(b)\}$, and thus $b = f(b)$. This contradiction shows that f has a fixed point in $S(a)$, and thus assertion **(6)** also holds.

Suppose next that assertion **(6)** holds and T is as in assertion (7). Moreover, assume on the contrary that T does not have a strong fixed point in $S(a)$. That is, $T(x) \neq \{x\}$ for all $x \in S(a)$. Then, since $T(x) \neq \emptyset$ for all $x \in X$, we necessarily have $T(x) \not\subseteq \{x\}$, and thus $T(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a)$. Hence, by the Axiom of Choice, we can see that there exists a function f of X such that

$$f(x) \in T(x) \setminus \{x\} \quad \text{for all} \quad x \in S(a) \qquad \text{and} \qquad f(x) = x \quad \text{for all} \quad x \in S(a)^c.$$

Now, by the assumed property of T, we can see that $f(x) \in S(x)$ for all $x \in S(a)$. Moreover, by the reflexivity of S, it is clear that we also have $f(x) \in S(x)$ for all $x \in S(a)^c$. Therefore, f is an extensive function of $X(S)$. Thus, by assertion (6), there exists $x \in S(a)$ such that $f(x) = x$. Hence, we can already infer that $x = f(x) \in T(x) \setminus \{x\} = T(x)$, and thus $x \neq x$. This contradiction proves that $T(b) = \{b\}$ for some $b \in S(a)$, and thus assertion **(7)** also holds.

Suppose now that assertion **(7)** holds and $\mathcal{F}$ is as in assertion (8). That is, each member f of $\mathcal{F}$ is a function of X to itself such that $f(x) \in S(x)$ for all $x \in X$. Define $T(x) = \left\{ f(x) : \quad f \in \mathcal{F} \right\}$ for all $x \in X$. Then, it is clear that T is a non-partial relation on X such that for any $x \in X$ and $y \in T(x)$, we have $y \in S(x)$. Therefore, by assertion (7), there exists $b \in S(a)$ such that b is a strong fixed point of T. That is, $T(b) = \{b\}$. Hence, we can already see that $f(b) = b$ for all $f \in \mathcal{F}$, and thus assertion **(8)** also holds.

Suppose now that assertion **(8)** holds. Moreover, assume on the contrary that assertion (9) does not hold. Then, there exists $Y \subseteq X$, such that

$$S(x) \setminus \{x\} \neq \emptyset \quad \text{for all} \quad x \in S(a) \setminus Y, \qquad \text{but still} \qquad S(a) \cap Y = \emptyset.$$

Thus, $S(a) \setminus Y = S(a)$. Therefore, we actually have $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a)$. Hence, by the Axiom of Choice, we can see that there exists a function f of X such that

$$f(x) \in S(x) \setminus \{x\} \quad \text{for all} \quad x \in S(a) \qquad \text{and} \qquad f(x) = x \quad \text{for all} \quad x \in S(a)^c.$$

Thus, we evidently have $f(x) \in S(x)$ for all $x \in S(a)$. Moreover, by the reflexivity of S, it is clear that we also have $f(x) \in S(x)$ for all $x \in S(a)^c$. Therefore, f is an extensive function of X. Thus, by the $\mathcal{F} = \{f\}$ particular case of assertion (8), there exists $b \in S(a)$ such that $f(b) = b$. Hence, we can already infer that $b = f(b) \in S(b) \setminus \{b\}$, and thus $b \neq b$. This contradiction shows that assertion **(9)** also holds.

Suppose now that assertion **(9)** holds. Define $Y = \left\{ x \in X : \quad S(x) = \{x\} \right\}$. Then, $Y \subseteq X$ such that for all $x \in S(a) \setminus Y$ we have $S(x) \neq \{x\}$. Hence, by using the reflexivity of S, we can infer that $S(x) \not\subseteq \{x\}$, and thus $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a) \setminus Y$. Therefore, by assertion (9), we have $S(a) \cap Y \neq \emptyset$. Thus, there exists $b \in S(a)$ such that $b \in Y$, and thus $S(b) = \{b\}$. Hence, it is clear that, $b\,S\,c \implies c \in S(b) \implies c \in \{b\} \implies c = b$. Therefore, b is a strongly maximal element of $X(S)$, and thus assertion **(1)** also holds. □

Remark 5.2. Note that some of the above implications do not require the assumed properties of the relation S. Moreover, only the implication $(3) \implies (4)$ requires the relation S to be antisymmetric.

In [6], we have shown that if the relation S is not supposed to be antisymmetric, then the implication $(3) \implies (4)$ need not be true. However, now having in mind Theorem 1.2 of Fierro, we can still prove the following two additional theorems.

Theorem 5.3. *If S is a preorder on a nonvoid set X, then the following assertions are true:*

(a) (1) *and* (6)—(9) *are equivalent;*

(b) (i) $\implies$ (i+1) *for all* $\mathrm{i} \in \{1, 2 \cdots, 8\} \setminus \{3\}$.

Proof. From the proof of Theorem 5.1, we can see that assertion (b) and the implication $(9) \implies (1)$ are true.

Therefore, to prove assertion (a), we need only show that (1) also implies (6). For this, assume that assertion (1) holds and f is as in assertion (6). Then, by assertion (1) and Remark 4.2, there exists $b \in S(a)$ such that $S(b) \subseteq \{b\}$. Moreover, by the extensivity of f, we have $f(b) \in S(b)$. Therefore, $f(b) \in \{b\}$, and thus $f(b) = b$. Thus, assertion (6) also holds. □

Theorem 5.4. *Suppose that S is a preorder on a nonvoid set X and, in addition to the assertions in Theorem 5.1, consider the following two assertions:*

(4*) *for every relation T on X such that $S(x) \setminus S^{-1}(x) \neq \emptyset$ for all $x \in X$ with $x \in S(a) \setminus T(x)$, there exists $b \in S(a)$ such that b is a fixed point of T;*

(5*) *if $S(x) \setminus S^{-1}(x) \neq \emptyset$ for all $x \in S(a)$, then for every relation T on X there exists $b \in S(a)$ such that b is a fixed point of T.*

Then, the following implications hold:

(a) $(3) \implies (4^*)$; (b) $(4) \implies (4^*) \implies (5^*)$.

Proof. Suppose that assertion (3) holds and T is as in (4*). Then, by assertion (3), there exist $c \in S(a)$ and a maximal chain C in $S(c)$ such that $\bigcap_{x \in C} S(x) \neq \emptyset$. Therefore, there exists $b \in X$ such that

$$b \in S(x) \qquad \text{for all} \qquad x \in C.$$

Hence, as in the proof of the implication $(3) \implies (4)$ in Theorem 5.1, we can infer that $b \in S(a)$. Therefore, if we assume on the contrary that T does not have fixed point in $S(a)$, then $b \notin T(b)$, and thus $b \in S(a) \setminus T(b)$. Hence, by our former assumption on T, we can infer that $S(b) \setminus S^{-1}(b) \neq \emptyset$. Therefore, there exists $y \in X$ such that

$$y \in S(b) \qquad \text{and} \qquad y \notin S^{-1}(b).$$

Hence, by using the maximality of C and the proof of the implication $(3) \implies (4)$ in Theorem 5.1, we can infer that $b \in S(y)$, and thus $y \in S^{-1}(b)$. This, contradiction proves that $b \in T(b)$, and thus assertion (4*) also holds.

The implication $(4^*) \implies (5^*)$ is again quite obvious. Namely, if $S(x) \setminus S^{-1}(x) \neq \emptyset$ holds for all $x \in S(a)$, then for any relation T on X we have $S(x) \setminus S^{-1}(x) \neq \emptyset$ for all $x \in X$ with $x \in S(a) \setminus T(x)$. Thus, assertion (4*) can be applied to derive the existence of a fixed point b of T in $S(a)$.

Now, to complete the proof, it remains only to show that assertion (4) implies (4*). For this, assume that assertion (4) is true and T is as in assertion (4*). Then, we have $S(x) \setminus S^{-1}(x) \neq \emptyset$ for all $x \in S(a) \setminus T(x)$.

Moreover, since $\underline{S \text{ is reflexive on } X}$, we can also see that $x \in S(x)$, $x \in S^{-1}(x)$ and thus $\{x\} \subseteq S^{-1}(x)$ for all $x \in X$. Hence, we can see that $S(x) \setminus S^{-1}(x) \subseteq S(x) \setminus \{x\}$, and thus $S(x) \setminus \{x\} \neq \emptyset$ for all $x \in S(a) \setminus T(x)$. Therefore, by assertion (4), the relation T has a fixed point b in $S(a)$, and thus assertion (4*) also holds. □

Remark 5.5. However, assertion (5*) seems not to be sufficiently strong enough to derive assertion (6).

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Example. If $X = \{1, 2\}$, $S = X^2$ and $T = X^2 \setminus \Delta_X$, then

(α) S is a total preorder, but not antisymmetric, relation on X;

(β) T is an involutive, extensive and intensive function of X onto itself without fixed points.

Moreover, for any $a \in X$, the following assertions hold:

(a) every $b \in S(a)$ is a maximal, but not strongly maximal element of $X(S)$;

(b) for any $b \in S(a)$ and any chain C in $S(b)$ we have $\bigcap_{x \in C} S(x) = X \neq \emptyset$;

(c) for each $b \in S(a)$, the set $C = S(b)$ is a maximal chain in $S(b)$ such that $\bigcap_{x \in C} S(x) = X \neq \emptyset$;

(d) $S(x) \setminus \{x\} = X \setminus \{x\} \neq \emptyset$ for all $x \in X$, but despite this the function T fails to have a fixed point.

The above assertions show, for instance, that the implication (3) $\Longrightarrow$ (4) in Theorem 5.1, and thus also in Theorem 1.2, does not hold without assuming the antisymmetry of the corresponding preorders. This statement was also proved in our former paper [6] by using a more complicated and instructive example.

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