Filomat 36:20 (2022), 7067–7081
https://doi.org/10.2298/FIL2220067U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Topology on Categorical Algebras

Erdal Ulualan[a], **Koray Yılmaz**[a]

[a] *Department of Mathematics, Science and Arts Faculty, Kütahya Dumlupınar University, Kütahya, Turkey*

Abstract. On categorical algebras, we define an idempotent filter as an analogous version of Gabriel's. We also construct topologies on 2-crossed modules of algebras and categorical algebras.

1. Introduction

Before Poincare, the only relevant topological concept was the Euler characteristic of surfaces whose name comes from Euler's article [10] on what is now known as the Euler polyhedron formula. Poincare [16] revealed his ambition of developing an n-dimensional geometry in his first major topology paper, the Analysis situs which was followed by five additional papers between 1899 and 1904 [17–21]. These publications established the area of algebraic topology by providing the first systematic study of topology and revolutionizing the topic by utilizing algebraic structures to identify between non-homeomorphic topological spaces. That is Poincare's methods gave rise to algebraic topology.

Topology is an abstraction of the concept of all coverings, which is a "Grothendieck topology" at one level and a "topology on a topos" at a higher degree of abstraction. It is known that there is a bijection between left exact radicals and hereditary torsion theories in the classical case of Grothendieck categories, which is a certain form of localization. Both classes are in bijection with right Gabriel topologies in the case of categories of modules. One of the goals of this work is to develop a categorical algebra equivalent of this bijection.

Crossed modules [25] were introduced by Whitehead as models for the connected homotopy 2-types. Categorical algebras are well known at least as an analogue of categorical groups in another category. A description of categorical algebras in Shammu's Ph.D. thesis [24] is implicit in more general expositions of categorical objects by Ellis[9] and Porter [23]. Some light on homotopy 2-types was also shed by Baez, Crans and Lauda [5, 6] which categorifies 2-groups and Lie 2-algebras.

In his work [24] Shammu constructed a topology on $\mathbf{XMod}/_{\mathbf{R}}$, crossed R-modules, and proved that given a localization on $\mathbf{XMod}/_{\mathbf{R}}$ a topology can be constructed as defined by Barr-Wells [7]. Similarly to Shammu, we construct a topology on categorical algebras and 2-crossed $(C \to R)$-modules of algebras. 2-crossed modules introduced by Conduche [8] as a model for homotopy 3-types. Later commutative algebra case for 2-crossed modules was defined by Ellis [9]. In his studies, Arvasi showed connections between 2-crossed modules and homotopy 3-types by simplicial methods. For more information on homotopy 3-types see [1, 4].

2020 *Mathematics Subject Classification*. Primary 54B30; Secondary 18F10, 18B25
Keywords. Categorical algebra, crossed module, topology
Received: 31 January 2022; Revised: 22 April 2022; Accepted: 16 June 2022
Communicated by Ljubiša D.R. Kočinac
Email addresses: e.ulualan@dpu.edu.tr (Erdal Ulualan), koray.yilmaz@dpu.edu.tr (Koray Yılmaz)

In this paper, we construct a topology on $\mathbf{X_2Mod}/_{\mathbf{C\to R}}$ and a topology on categorical algebras analogous to that given by Shammu for $\mathbf{XMod}/_{\mathbf{R}}$. We also show that crossed filters in categorical algebras implies any localization in $\mathbf{Cat}/_{\mathbf{C_0}}$ gives a topology.

2. Preliminaries

Let R be a commutative ring with identity. By an R-algebra we mean a unitary R-bimodule A endowed with an R-bilinear associative multiplication $A \times A \longrightarrow A$, $(a, a') \mapsto aa'$. In this work, we call an algebra as a commutative algebra over a ring $\mathbf{k}$ such that a multiplicative identity is not required.

As an algebraic model of homotopy 2-types, the notion of crossed module was introduced by Whitehead in [25]. Porter investigated the commutative algebra analogue of crossed modules in [23]. We refer to an action of $r \in R$ on $c \in C$ by $r \cdot c$ throughout this text. Let R be a $\mathbf{k}$-algebra with identity and C be a k-algebra such that R acts on C. A pre-crossed module of algebras is a k-algebra homomorphism $\partial : C \longrightarrow R$, $\partial(r \cdot c) = r\partial c$ for all $c \in C, r \in R$ and a crossed module if in addition, for all $c, c_0 \in C, \partial c \cdot c_0 = cc_0$. The second condition is called the *Peiffer identity*. The triple (C, R, ∂) is used to denote a crossed module.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of $\mathbf{k}$-algebra morphisms, $\Phi : C \longrightarrow C'$ and $\Psi : R \longrightarrow R'$ such that $\Phi(r \cdot c) = \Psi(r) \cdot \Phi(c)$ and $\partial'\Phi(c) = \Psi\partial(c)$.

A subcrossed module of a crossed module (C, R, ∂) is a crossed module (C', R, ∂') such that C' is an subalgebra of C and $\partial' = \partial_{|C'} : C' \longrightarrow R$, the restriction of ∂ to C'. We denote the category of crossed modules by **XMod**. Note that in the case of a morphism (Φ,Ψ) between crossed modules with the same base R where Ψ is identity morphism on R, we say that Φ is a morphism of crossed R-modules such that $\Phi \circ \partial' = \partial$. Thus we get a subcategory $\mathbf{XMod}/_{\mathbf{R}}$ of **XMod**.

Next, we recall some basic definitions from (cf.[7, 11]).

Definition 2.1. Let C be a small category with pullbacks. Given an object X of C, a subobject is an equivalence class of monomorphisms $i : S \hookrightarrow X$ where

$$i \cong i' \Leftrightarrow \exists \varphi : S \to S'$$

is an isomorphism such that $i = i'\varphi$.

Let$SubX$ be the set of all subobjects of X. It is possible to extend functor $Sub : X \to Sub(X)$ to $Sub : C^{op} \to Set$. That is, if $f : X \to X'$ is a morphism and if $i' : S' \to X'$ is a subobject of X' then in a pullback diagram

$$\begin{array}{ccc} S & \dashrightarrow & S' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

the morphism i is also a monomorphism. Namely when $i' : S' \to X'$ is an element of $Sub(X')$ then the pullback of i' along f, $i : S \to X$, is an element of $Sub(X)$ which can be seen as $Subf$ maps $f : S' \to X'$ to $i : S \to X$.

Definition 2.2. Let R be a ring. A nonempty set E of right ideals of R will be called an idempotent filter on R [11] if the following conditions hold.

F1 If I a right ideal in R contains some $J \in E$ then $I \in E$,

F2 If $I \in E$, then for all $a \in R$, $(I.a) = \{x \in R | ax \in I\}$ is in E,

F3 If I is a right ideal of R and there is an element J of E such that $(I.J) \in E$ for all $j \in J$, then $I \in E$.

Definition 2.3. A topology on a category $\mathfrak{C}$ with pullbacks is a natural endomorphism τ of the contravariant subobject functor, Sub, which is

1. idempotent: $\tau \circ \tau = \tau$,
2. inflationary: $A' \subseteq \tau(A')$, for any subobject A' of an object A (where $\tau(A')$ means $\tau_A(A')$),
3. order-preserving: If A_0 and A_1 are subobjects of A and $A_0 \subseteq A_1$, then $\tau(A_0) \subseteq \tau(A_1)$.

A subobject A_0 of A will be called $\tau -$ *closed* [7] in A if $\tau_A(A_0) = A_0$.

A closure operator on a topological space is very similar to a topology on a category. However, a topology on a category does not preserve finite unions. In this manner, the terms dense and closed are therefore deceptive. However, because it is common in the literature, we keep these terms.

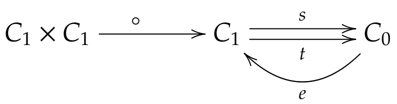
The simplest object of a categorical structure is known as the trivial object. The singleton set $\{*\}$ is an algebra over any ring $\mathbf{k}$ with respect to the multiplication defined as $k* = k$ for every $k \in \mathbf{k}$ and the addition $* + * = *$ which makes it a trivial algebra. 0 is commonly used to represent trivial algebras (instead of $\{0\}$).

We recall the following result from Shammu [24] for a topology over crossed R-modules of algebras.

Proposition 2.4. ([24]) *Given an idempotent filter F on R there exists a topology τ on the category* $\mathbf{XMod}_{/\mathbf{R}}$ *such that $\tau-$dense subobjects of* $(R, 0)$ *are the ideals in F.*

3. A topology on categorical algebras

In this section, we give a construction of a topology on categorical algebras. Recall that the *categorical algebra* is a category object (internal category [13]) in the category of R-algebras which consists of an object of objects C_0, an object of morphisms C_1 together with source and target morphisms $s, t : C_1 \to C_0$, a morphism $e : C_0 \to C_1$ that assigns identities and a composition morphism $\circ : C_1 \times C_1 \to C_1$ satisfying the standard axioms for categories. In particular, a categorical algebra is a diagram



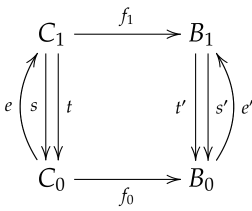
of R-algebras in which C_1 is a small category over C_0 with source, target and identity maps s,t,e respectively, such that $se = te = I_{C_0}$. If $x, y \in C_0$, for an arrow from x to y is represented by $a : x \to y$, then $s(a) = x$ and $t(a) = y$. A composition of two arrows a and b is denoted by $a \circ b$, if $s(a) = t(b)$. Then, we have $t(a \circ b) = t(b)$ and $s(a \circ b) = s(b)$. Also this composition must be an algebra morphism. This is true if the interchange law for addition and multiplication

$$(a \circ b) + (c \circ d) = (a + c) \circ (b \circ d)$$

and

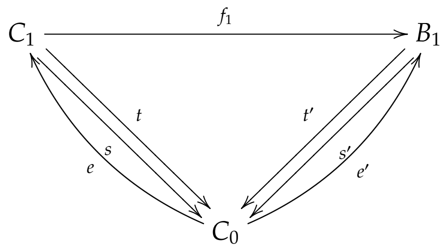
$$(a \circ b) \, . \, (c \circ d) = (a.c) \circ (b.d)$$

are satisfied whenever $a \circ b$ and $c \circ d$ are defined. We will denote such a categorical algebra as (C_1, C_0, s, t, e, o). A morphism $(f_1, f_0) : (C_1, C_0, s, t, e, o) \longrightarrow (B_1, B_0, s', t', e', o')$ of categorical algebras



is a pair of algebra morphisms f_1 and f_0 compatible with source, target and identity maps. That is $f_0 s = s' f_1, f_0 t = t' f_1$ and $e' f_0 = f_1 e$. These definitions give us the category **Cat** of categorical algebras.

Note that if $f = (f_1, f_0)$ is a categorical algebra morphism between categorical algebras with the same base C_0, i.e. f_0 is identity on C_0, then the following triangle



commutes. Since the composition of two morphisms of categorical algebras over C_0 is a morphism of categorical algebras we get a subcategory $\mathbf{Cat}/_{\mathbf{C_0}}$ of $\mathbf{Cat}$.

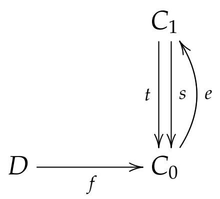
Definition 3.1. A subcategorical algebra, $(L_1, L_0, s', t', e', o)$ of a categorical algebra (C_1, C_0, s, t, e, o) is a categorical algebra such that L_1,L_0 are subalgebras of C_1 and C_0 respectively, and the morphisms s', t' and $e'$are the restrictions of s, t and e, respectively.

We know that the category of crossed modules and cat-groups are natural equivalent due to the Brown-Spencer theorem [14]. The algebra adaptation of this result has been proved by Porter in [23]. In this section, we will give a construction of a topology on the category of categorical algebras.

Proposition 3.2. *The category* $\mathbf{Cat}/_{\mathbf{C_0}}$ *has pullbacks.*

Proof. We will construct the pullback object along with a morphism in the category **Cat** of categorical algebras which is a more general case.

Let (C_1, C_0, s, t, e, o) be a categorical algebra and $f{:}D \longrightarrow C_0$ is a morphism of algebras as follows:



Define

$$f^*(C_1) = \{(d, c_1, d_1) \in D \times C_1 \times D : s(c_1) = d, t(c_1) = d_1\}$$

$t^*(d, c_1, d_1) = d_1$, $s^*(d, c_1, d_1) = d$, $e^*(d) = (d, ef(d), d)$. The composition $a \circ b$ can be given by

$$a \circ b = (d, c_1, d_1) \circ \left(d', c_1', d\right) = \left(d', c_1 \circ c_1', d_1\right)$$

Next, we show that the interchange law holds for $f^*(C_1)$. For $x_i = \left(d_i, c_i, d_i'\right)$ and $i = 1, 2, 3, 4$ we obtain

$$(x_1 \cdot x_2) \circ (x_3 \cdot x_4) = \left(d_3 d_{4,} (c_1 \circ c_2) \cdot (c_3 \circ c_4), d_1' d_2'\right)$$

$$(x_1 \circ x_3) \cdot (x_2 \circ x_4) = \left(d_3 d_{4,} (c_1 \cdot c_3) \circ (c_2 \cdot c_4), d_1' d_2'\right)$$

since (C_1, C_0, s, t, e, o) is a categorical algebra $(c_1 \circ c_2) \cdot (c_3 \circ c_4) = (c_1 \cdot c_3) \circ (c_2 \cdot c_4)$. Thus interchange law holds for $f^*(C_1)$. We also obtain:

$$t^*(x_1 \circ x_2) = t^*\left(d_2, c_1 \circ c_2, d_1'\right) = d_1' = t^*(x_1)$$

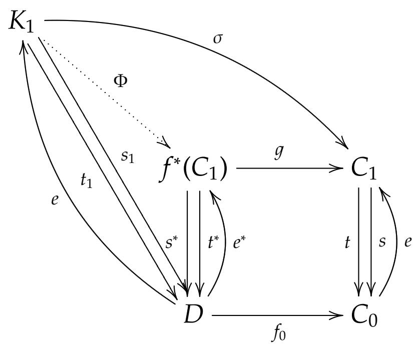
$$s^*(x_1 \circ x_2) = s^*\left(d_2, c_1 \circ c_2, d_1'\right) = d_2 = s^*(x_2)$$

$$s^*e^*(d) = d = t^*e^*(d) = id(d).$$

Then $(f^*(C_1), C_0, s^*, t^*, e^*, o)$ is a categorical algebra. Define $g : f^*(C_1) \longrightarrow C_1$ by $(d, c_1, d_1) \mapsto c_1$, then g is a homomorphism and we have

$$fs^* = sg\,,\; ft^* = tg.$$

Given any categorical algebra $(K_1, D, s_1, t_1, e_1, o)$ and a morphism $\sigma : K_1 \longrightarrow C_1$, such that $ft_1 = t\sigma$ and $fs_1 = s\sigma$, we get the universal morphism $\Phi : K_1 \longrightarrow f^*(C_1)$ by defining $k \mapsto (s_1(k), \sigma(k), t_1(k))$. We can illustrate this situation by the following diagram:



Since the identity morphism is an algebra homomorphism taking $D = C_0$ and $f = Id$ we get pullback object in $\mathbf{Cat}/_{\mathbf{C_0}}$. □

Proposition 3.3. *Let F be an idempotent filter on C_0, then there exists a topology τ on the category $\mathbf{Cat}/_{\mathbf{C_0}}$ such that $\tau-$dense subobjects of $(C_0, 0)$ are in F where $(C_0, 0) = C_0 \underset{t}{\overset{s}{\rightrightarrows}} 0$ with $s(a) = t(a) = 0$.*

Proof. Consider the subcategorical algebra $L_1 \underset{t'}{\overset{s'}{\rightrightarrows}} C_0$ of $C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$ for $x, y \in C_0$. Define

$$\widetilde{L_1}(x, y) = \{c_1 \in C_1(x, y) : (L, c_1) \in F\}$$

where

$$(L_1(x, y) : c_1) = \{c_0 \in C_0 : c_1 \circ c_0 \in L_1(xc_0, yc_0)\}.$$

1- $\widetilde{L_1}(x, y) \subseteq C_1(x, y)$.

2- To show that the composition is well defined, we will show that $c_1 \circ c_1' \in \widetilde{L_1}(x, z)$ for $c_1 \in \widetilde{L_1}(x, y)$ and $c_1' \in \widetilde{L_1}(y, z)$. If $c_1 \in \widetilde{L_1}(x, y)$, then we obtain

$$c_1 \in \widetilde{L_1}(x, y)(L_1(x, y) : C_1) = \{c_0 \in C_0 : C_1.C_0 \in L_1(xc_0, yc_0)\} \in F,$$

and if $c_1' \in \widetilde{L_1}(y, z)$ we get,

$$c_1' \in \widetilde{L_1}(y, z)(L_1(y, z) : C_1) = \{c_0 \in C_0 : C_1'.C_0 \in L_1(yc_0, zc_0)\} \in F,$$

that is

$$c_1 \circ c_1' \in \widetilde{L_1}(x, z) \Leftrightarrow (c_1 \circ c_1').c_0 \in L_1(xc_0, zc_0).$$

Since

$$(c_1.c_0) \circ (c_1'.c_0) = (c_1 \circ c_1').c_0$$

we have

$$c_1 \circ c_1' \in \widetilde{L_1}(x,z)$$

which shows that $(\widetilde{L_1}, C_0)$ is a subcategory of (C_1, C_0). Moreover, the sets $\widetilde{L_1}(x,y)$ are sub-R-modules of $C_1(x,y)$. For $c, c' \in \widetilde{L_1}(x,y)$ we have

$$\begin{aligned}(L_1(x,y) : c + c') &= \{c_0 \in C_0 : (c+c').c_0 \in L_1(xc_0, yc_0)\}\\ &= (L_1(x,y).c) \cap (L_1(x,y).c') \in F,\end{aligned}$$

for $c \in \widetilde{L_1}(x,y)$, $r \in C_0$,

$$c \in \widetilde{L_1}(x,y) \Rightarrow (L_1(x,y).c) = \{c_0 \in C_0 : c.c_0 \in L_1(xc_0, yc_0)\} \in F$$

$$\begin{aligned}((L_1(x,y) : c) : r) &= \{c_0' \in C_0 : r \circ c_0' \in (L_1(x,y) : c)\}\\ &= \{c_0' \in C_0 : c \circ (r \circ c_0') \in L_1(xrc_0', yrc_0')\}\\ &= \{c_0' \in C_0 : (c \circ r) \circ c_0' : c \circ r \in L_1(xr, yr)\}\\ &\Rightarrow (L_1(x,y) : c \circ r) \in F\\ &\Rightarrow c \circ r \in \widetilde{L_1}(x,y).\end{aligned}$$

Thus $\widetilde{L_1}(x,y)$ is a sub-R-module of $C_1(x,y)$. Next, we will show that $\widetilde{L_1}(x,x)$ is a subalgebra of $C_1(x,x)$. For $c, c' \in \widetilde{L_1}(x,x)$ since $(L_1(x).c) \subseteq (L_1(x) : cc')$ by **F3**, $(L_1(x) : cc')$ is in F. Consequently, $(\widetilde{L_1}, C_0)$ is a subcategorical algebra of (C_1, C_0). Now we will show that $\tau : Sub \to Sub$ given by

$$\tau(L_1) = \widetilde{L_1}$$

gives a topology on $\mathbf{Cat}/_{C_0}$. First, we will prove (ii) and (iii).

(ii) Let $y : a \to b \in L_1(a,b)$ since $L_1(a,b) \subseteq C_1(a,b)$, then we have

$$(L_1(a,b) : y) = \{x \in C_0 : y.x \in L_1(ax, bx)\} = C_0$$

If y is in $L_1(a,b)$ for each x in C_0, then $y.x$ is in $L_1(ax, bx)$. So

$$L_1(a,b) \subseteq \widetilde{L_1}(a,b) = \{c \in C_1(a,b) : (L_1(a,b) : c) \in F\}$$

that is we get $L \subseteq \tau(L)$.

(iii) Let L_1, L_1' be two subcategorical algebras of C_1 such that $L_1 \subseteq L_1'$. For $x \in \tau(L_1(a,b)) = \widetilde{L_1}(a,b)$, we have

$$(L_1(a,b) : x) = \{c \in C_1(a,b) : c \circ x \in L_1(ax, bx)\}$$

where $(L_1(a,b) : x) \subseteq (L_1'(a,b) : x)$ which implies

$$(L_1(a,b) : x) = \{c \in C_1(a,b) : c \circ x \in L_1'(ax, bx)\},$$

then we have

$$x \in \widetilde{L_1}(a,b) = \tau(L_1(a,b))$$

for all $a, b \in C_0$.

(i) To prove that τ is idempotent, we need only check that $\tau \circ \tau(L_1(a,b)) \subseteq \tau(L_1(a,b))$. Let $c \in \tau^2(L_1(a,b))$, then $(\widetilde{L_1}(a,b) : c) \in F$ where

$$(\widetilde{L_1}(a,b) : c) = \{c_0 \in C_0 : c \circ c_0 \in \widetilde{L_1}(ac_0, bc_0)\}.$$

For $x \in (\widetilde{L_1}(a,b) : c)$, $c \circ x \in \widetilde{L_1}(ax,bx) = \tau(L_1(ax,bx))$, i.e. $(L_1(a,b) : c \circ x) \in F$, then we get

$$(L_1(a,b) : c \circ x) = ((L_1(a,b) : c) : x) \in F.$$

Thus by axiom F3, we obtain $(\widetilde{L_1}(a,b) : c) \in F$. Therefore $c \in \tau(L_1) = \widetilde{L_1}$ and so $\tau^2(L_1 \subseteq \tau(L)$. We get that τ gives a topology on Cat/C_0. Finally, we will show that $I \in F$ if and only if $\tau(I) = \widetilde{I} = C$. Suppose $I \in F$, then

$$\tau(I) = \widetilde{I} = \{r \in C_0 : (I : r) \in F\} = C_0$$

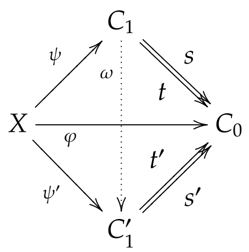
since for each $r \in C_0$, $(I : r) \in F$. Conversely, suppose $\tau(I) = \widetilde{I} = C_0$, then for all $r \in C_0$, we have

$$(I : r) = \{c_0 \in C_0 : r.c_0 \in I\},$$

since for each $c_0 \in C_0$ and each $r \in I$, $r.c_0$ is in I and $I \in F$. □

4. Free categorical algebra with same base

Using the universal morphism, we can give the free categorical algebra definition as follows: Let (C_1, C_0, s, t, e) be an object in $\mathbf{Cat}/_{\mathbf{C_0}}$ and $\varphi : X \longrightarrow C_0$ be a map from a set X to C_0. We say (C_1, C_0, s, t, e) is a free categorical algebra if there is a map $\psi : X \longrightarrow C_1$ such that $s\psi = \varphi$, $t\psi = \varphi$, $e\varphi = \psi$ for any categorical algebra $\left(C_1', C_0, s', t', e'\right)$ in $\mathbf{Cat}/_{\mathbf{C_0}}$ and a function $\psi' : X \longrightarrow C_1$ with $s\psi' = \varphi$, $t\psi' = \varphi$ and $e\varphi = \psi'$, there is a unique morphism $\omega : C_1 \longrightarrow C_1'$ such that $\omega\psi = \psi'$. We show this diagrammatically as :



Proposition 4.1. *A free categorical algebra on (X, f) exists in $\mathbf{Cat}/_{\mathbf{C_0}}$ and is uniquely determined up to isomorphism.*

Proof. Let C_0 be an algebra and $f : X \longrightarrow C_0$ be a function from a set X to C_0. Define $C_1 = C_0 < x >$ as the free monoid algebra on X. That is an element c in $C_0 < x >$ has the form

$$c = \sum c_i x_i c_i' c_j x_j c_j' ... c_m x_m c_m'$$

where x's $\in X$ and the c's and c''s $\in C_0$. The function f induces morphisms $s_1 : C_1 \longrightarrow C_0$ and $t_1 : C_1 \longrightarrow C_0$ of algebras defined on the generators by $s_1(\underline{x}) = t_1(\underline{x}) = f(x)$. Let P be the subcategorical algebra $Kers_1$. Since $s_1(P) = t_1(P) = 0$, taking $C(x) = C_1/P$ we obtain induced morphisms

$$\varepsilon_1, \varepsilon_2 : C(x) \longrightarrow C_0$$

then it is easily checked that $(C(x), C_0, \varepsilon_1, \varepsilon_2, e_x)$ is a categorical algebra.

Suppose that (C, C_0, s, t, e) be a categorical algebra with a morphism $\omega : X \longrightarrow C_0$ then $s\omega = f = t\omega$, then there exists a unique morphism $\bar{\omega} : C(x) \longrightarrow C_0$, therefore $(C(x), C_0, \varepsilon_1, \varepsilon_2, e_x)$ is the required free categorical algebra. □

Definition 4.2. Let C_0 be an algebra. For each $x \in C_0$, we take the symbol $\underline{x}$ and form the singly generated free categorical algebra $(C_1(\underline{x}), C_0, s_x, t_x, e_x)$ on $(\{\underline{x}\}, \omega_x)$, where ω_x takes
$\underline{x} \mapsto x \in C_0$ and

$$s_x(\underline{x}) = t_x(\underline{x}) = \omega_x(\underline{x}) = x$$

then an element $(\underline{x}) \in C_1(\underline{x})$ will have the form

$$\underline{x} : x \longrightarrow x$$

for some $x \in C_0$.

Next, we obtain the coproduct in the category of categorical algebras to get a topology from a localization.

Proposition 4.3. *Let (A, C_0, s_1, t_1, e_1) and (B, C_0, s_2, t_2, e_2) be any two objects in $\mathbf{Cat}/_{\mathbf{C_0}}$. Then $(A \ltimes B, C_0, s_*, t_*, e_*)$ is an object in $\mathbf{Cat}/_{\mathbf{C_0}}$. Where $A \ltimes B$ is the semidirect product of categorical algebras A and B.*

Proof. For injections $i_1 : A \longrightarrow A \ltimes B$ and $i_2 : B \longrightarrow A \ltimes B$ take an element $(\alpha, \beta) \in A \ltimes B$ and $c \in C_0$. If we define the morphisms

$$s_*(\alpha, \beta) = s_*(\alpha, 0).s_*(0, \beta) = s_1(f).s_2(g)$$

$$t_*(\alpha, \beta) = t_*(\alpha, 0).t_*(0, \beta) = t_1(f).t_2(g)$$

$$e_*(c) = (i_1 \circ e_1(c), i_2 \circ e_2(c))$$

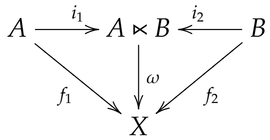
then the proof is clear. □

Proposition 4.4. *For $(A, C_0, s_1, t_1, e_1), (B, C_0, s_2, t_2, e_2) \in \mathbf{Cat}/_{\mathbf{C_0}}$, the categorical algebra $(A \ltimes B, C_0, s_*, t_*, e_*)$ together with the morphisms i_1 and i_2, is the coproduct of the categorical algebras (A, C_0, s_1, t_1, e_1) and (B, C_0, s_2, t_2, e_2).*

Proof. Let (X, C_0, s_x, t_x, e_x) be any categorical algebra in $\mathbf{Cat}/_{\mathbf{C_0}}$ and $f_1 : A \longrightarrow X$,$f_2 : B \longrightarrow X$ be two morphisms of categorical algebras. Then there is a map $\omega : A \ltimes B \longrightarrow X$ given by

$$\omega(a, b) = f_1(a) + f_2(b)$$

is the necessary unique morphism of categorical algebras for the diagram



to commute. □

Proposition 4.5. *The category $\mathbf{Cat}/_{\mathbf{C_0}}$ has a set of generators.*

Proof. Let Γ be the full subcategory of $\mathbf{Cat}/_{\mathbf{C_0}}$ such that

$$\Gamma = \{(C(x), C_0, s_x, t_x, e_x, \circ) \in \mathbf{Cat}/_{\mathbf{C_0}} : C(x) \text{ free on } \omega_x : X \longrightarrow C_0 \text{ and } X \text{ finite set}\}$$

Let (A, C_0, s_a, t_a, e_a) and (B, C_0, s_b, t_b, e_b) be any two objects in $\mathbf{Cat}/_{\mathbf{C_0}}$ and $h : (A, C_0, s_a, t_a, e_a) \longrightarrow (B, C_0, s_b, t_b, e_b)$ be any morphism of categorical algebras which is not an isomorphism. For $a,b \in B$ such that $b \notin Imh$. Let $y_1 = s_b(b)$ and $y_2 = t_b(b)$. Form the singly generated categorical algebras $\left(C\{\underline{y}_1\}, C_0, s_{y_1}, t_{y_1}, e_{y_1}\right)$ and $\left(C\{\underline{y}_2\}, C_0, s_{y_2}, t_{y_2}, e_{y_2}\right)$ where

$$s_{y_1}\left(\underline{y}_1\right) = t_{y_1}\left(\underline{y}_1\right) = \omega_{y_1}\left(\underline{y}_1\right) = y \in C_0,$$

$s_{y_2}\left(\underline{y}_2\right) = t_{y_2}\left(\underline{y}_2\right) = \omega_{y2}\left(\underline{y}_2\right) = y \in C_0.$

Let us define the morphisms

$$g_i : C\{\underline{y}_i\} \longrightarrow B$$

by $g_i\left(\underline{y}_i\right) = b$ for $i = 1, 2$. If g_i factors through a morphism n_i:$C\{\underline{y}_i\} \longrightarrow A$, then $hn_i = g_i$ for $i = 1, 2$. This shows that $b \in Imh$. Hence morphisms g_i does not factor through h. □

Proposition 4.6. *For* $(C\{\underline{x}\}, C_0, s_x, t_x, e_x)$ *and* $\left(C\{y\}, C_0, s_y, t_y, e_y\right)$ *in* Γ , $f : C\{\underline{x}\} \longrightarrow C\{y\}$ *exists if and only if* $x = c.y$ *for some* $c \in C_0$.

Proof. If f exists then $fs_y = s_x$ and $ft_y = t_x$ for some $c \in C_0$. Thus the images of s_x and t_x are the subsets of image of s_y and t_y, respectively. For example, for $\underline{x} \in \{\underline{x}\}$, we have $x = c.y$ for some $c \in C_0$.

On the other hand, suppose that $x = c.y$ for some $c \in C_0$, define $f : C\{\underline{x}\} \longrightarrow C\{y\}$. It is obvious by defining

$$f(t\underline{x}) = f\left(\underline{c}\underline{y}\right) = tc\underline{y},$$

then f is a morphism of categorical algebras. □

Definition 4.7. ([15]) A subcategory A of B is called reflective in B if the inclusion functor $a : A \to B$ has a left adjoint functor $i : B \to A$ called reflector.

Definition 4.8. A localization (L, i) on $\mathbf{Cat}/_{\mathbf{C_0}}$ is a full reflective subcategory L such that the reflector i : $\mathbf{Cat}/_{\mathbf{C_0}} \to L$ is left exact.

Lemma 4.9. *If*

$$\begin{array}{ccc} B_0 & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & A \end{array}$$

is a pullback diagram then the morphism $\tau_B(B_0) \longrightarrow \tau_A(A_0)$ *making the diagram*

$$\begin{array}{ccc} \tau_B(B_0) & \dashrightarrow & \tau_A(A_0) \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & A \end{array}$$

commutative is unique [7].

Proposition 4.10. *A localization* (L, i) *on the category* $\mathbf{Cat}/_{\mathbf{C_0}}$ *will give us a topology* τ *on* $\mathbf{Cat}/_{\mathbf{C_0}}$.

Proof. Define for each $(C, C_0, s, t, e, \circ)$ in $\mathbf{Cat}/_{\mathbf{C_0}}$ a function $\tau : Sub(C) \longrightarrow Sub(C)$ in the following way:

For a subcategorical algebra $(C', C_0, s', t', e', \circ)$ of $(C, C_0, s, t, e, \circ)$, $\tau_C(C')$ is the inverse image of $i(C')$ along $\eta_C : C \longrightarrow i(C')$ and the following diagram is a pullback diagram since i is left exact the upper and lower morphisms are

$$\begin{array}{ccc} \tau_C(C') & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \eta_C} \\ i(C') & \longrightarrow & i(C) \end{array}$$

monomorphism. Next, we will show that τ_C is a topology on $\mathbf{Cat}_{/\mathbf{C_0}}$.

i. Consider the following diagram

$$\begin{array}{ccccc} \tau_C(\tau_C(C'')) & \longrightarrow & \tau_C(C') & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \eta_C \\ i(C'') & \longrightarrow & i(C') & \longrightarrow & i(C) \end{array}$$

where $(C'', C_0, s'', t'', e'', \circ)$ is a subcategorical algebra of $(C', C_0, s', t', e', \circ)$. Since the pullback of a pullback is a pullback we get $\tau_C^2(C') = \tau_C(C')$.

ii. The universal property of η_C' and the inclusion of C' in C implies $C' \subset \tau_C(C')$ for a subcategorical algebra $(C', C_0, s', t', e', \circ)$ of $(C, C_0, s, t, e, \circ)$,.

iii. Let A_0 and A_1 be two subobjects of C such that $A_0 \subset A_1$. Using Lemma 4.9 we get a commutative diagram

$$\begin{array}{ccc} \tau_C(A_0) & \longrightarrow & \tau_C(A_1) \\ \downarrow & & \downarrow \eta_C \\ C & = & C \end{array}$$

Since $\tau_C(A_0)$ and $\tau_C(A_1)$ are in $Sub(C)$ the vertical arrows are monomorphism. That is top arrow is also a monomorphism. Then we get $\tau_C(A_0) \subset \tau_C(A_1)$.

Now we will define a localizing system F for the category of categorical algebras using the subobjects of members Γ_1, the set of singly generated free categorical algebras where $\Gamma_1 = \{C_1(\underline{x}) : x \in C_0\}$ and $Sub(\Gamma_1)$ be the set of all subobjects of $(C_1(\underline{x}), C_0, s_x, t_x, e_x, \circ)$. Given a morphism $f : C_1(\underline{x}) \longrightarrow C_1(\underline{y})$ of categorical algebras we have $Ims(x) = Imt(x) \subset Ims(y) = Imt(y)$,i.e $\{x\} \subset \{y\}$ and hence $x = c.y$ for some $c \in C_0$. Therefore f will looks like multiplication by c, i.e, $t\underline{c}\underline{y} \mapsto tc\underline{y} \in C_1(\underline{y})$. We will write $\alpha_c^x : f : C_1(\underline{cx}) \longrightarrow C_1(\underline{x})$. □

Definition 4.11. A categorical idempotent filter for $\mathbf{Cat}/_{\mathbf{C_0}}$ is a family

$$F = \{F_x : x \in C_0\}$$

where F_x is a nonempty family of subobjects of $(C_1(\underline{x}), C_0, s_x, t_x, e_x, \circ)$ satisfying

Cif$_1$. If $S \in F_x$ and S' is any subobject of $(C_1(\underline{x}), C_0, s_x, t_x, e_x, \circ)$ with $S \subset S'$ then $S' \in F_x$,

Cif$_2$. If $S, S' \in F_x$ then $S \cap S' \in F_x$,

Cif$_3$. If $S \in F_x$, then for $c \in C_0$, $(\alpha_c^x)^{-1}(S) \in F_{cx}$,

Cif$_4$. For each $S \in F_x$ and $T \in F_y$ define $f : C_1(\underline{x}+\underline{y}) \longrightarrow C_1(\underline{x},\underline{y})$ by $f(\underline{x}+\underline{y}) = \underline{x}+\underline{y}$ then the inverse image of $S \circ T$ along f is $F_{\underline{x}+\underline{y}}$,

Cif$_5$. Let S be a subcategorical algebra of $(C_1(\underline{x}), C_0, s_x, t_x, e_x, \circ)$ if there exists $T \in F_x$ such that for each $\underline{t} = t\underline{x} \in T$, $\left(\alpha_t^x\right)^{-1}(S) \in F_{tx}$ then $S \in F_x$.

Proposition 4.12. *A topology τ on $\mathbf{Cat}/_{\mathbf{C_0}}$ gives a categorical idempotent filter F on $\mathbf{Cat}/_{\mathbf{C_0}}$.*

Proof. Let $F_x = \{\underline{S} \in Sub(C(\underline{x})) : \tau_{C_1(\underline{x})}(\underline{S}) = C(\underline{x})\}$, i.e, F_x consists of all the $\tau-$dense subobjects of $(C(\underline{x}), C_0, s_x, t_x, e_x, \circ)$. Take $F = \{F_x : x \in C_0\}$. We will show the categorical idempotent filter axioms.

Cif$_1$. Let $\underline{S}' \in Sub(C_1(\underline{x}))$ and $\underline{S} \subset \underline{S}'$ for a $\underline{S} \in F_x$ then $\tau_{C_1(\underline{x})}(\underline{S}) = C_1(\underline{x}) \subset \tau_{C_1(\underline{x})}(\underline{S}')$. Therefore we get $\tau_{C_1(\underline{x})}(\underline{S}') = C_1(\underline{x})$. This shows that $\underline{S}' \in F_x$

Cif$_2$. Let $\underline{S}, \underline{T} \in F_x$. Since $\tau_{C_1(\underline{x})}(\underline{S} \cap \underline{T}) = \tau_{C_1(\underline{x})}(\underline{S}) \cap \tau_{C_1(\underline{x})}(\underline{T}) = C_1(\underline{x}) \cap C_1(\underline{x}) = C_1(\underline{x})$, $\underline{S} \cap \underline{T} \in F_x$.

Cif$_3$. Let $\underline{T} \in F_x$ and $f : C_1(\underline{x}) \longrightarrow C_1(\underline{y})$ be a morphism of categorical algebras. Since $\tau_{C_1(\underline{x})} \circ f^{-1}(\underline{T}) = f^{-1} \circ \tau_{C_1(\underline{y})} = C_1(\underline{x})$, $f^{-1}(\underline{T}) \in F_x$.

Cif$_4$. If we apply τ to

$$\begin{array}{ccccc} \underline{S} & \longleftarrow & \underline{S}\circ\underline{T} & \longleftarrow & \underline{T} \\ \uparrow & & \downarrow & & \uparrow \\ C_1(\underline{x}) & \longleftarrow & C_1(\underline{x},\underline{y}) & \longleftarrow & C_1(\underline{y}) \end{array} \qquad (1)$$

where $\underline{S}\in F_x$ and $\underline{T}\in F_y$, we get a commutative triangle

$$\begin{array}{ccccc} C_1(\underline{x}) & \longrightarrow & \tau(\underline{S}\circ\underline{T}) & \longleftarrow & C_1(\underline{y}) \\ & \searrow & \downarrow \mu & \swarrow & \\ & & C_1(\underline{x},\underline{y}) & & \end{array}$$

where $C_1(\underline{x},\underline{y}) = C_1(\underline{x}) \circ C_1(\underline{y})$. Therefore there exists a unique map $\eta : C_1(\underline{x},\underline{y}) \longrightarrow \tau(\underline{S}\circ\underline{T})$ such that $\mu\eta = i_{C_1(\underline{x},\underline{y})}$. Since

$$\mu \circ i = i \circ \mu = (\mu \circ \eta) \circ \mu = \mu \circ (\mu \circ \eta)$$

and μ is a monomorphism, we get $\eta \circ \mu = i$. Therefore μ is an isomorphism. Thus $\tau(\underline{S}\circ\underline{T}) = C_1(\underline{x},\underline{y})$. Thus

$$\left(\tau_{C_1(\underline{x}+\underline{y})} \circ \beta^{-1}\right)(\underline{S}\circ\underline{T}) = \left(\beta^{-1} \circ \tau_{C_1(\underline{x},\underline{y})}\right)(\underline{S}\circ\underline{T}) = C_1(\underline{x},\underline{y})$$

so $\beta^{-1}(\underline{S}\circ\underline{T}) \in F_{\underline{x}+\underline{y}}$.

Cif$_5$. Let $\underline{S}\in Sub(C_1(\underline{x}))$ and suppose there is an element $\underline{T}\in F_x$ such that for each $\underline{t}\in\underline{T}$, $<\underline{t}>^{-1}(S) \in F_y$ where $y = s_x\underline{t} = t_x\underline{t}$, $<\underline{t}>: C_1(\underline{y}) \longrightarrow C_1(\underline{x})$ and $<\underline{t}>^{-1}$is the ring of fractions. If $<\underline{t}>^{-1}(S) \in F_y$ then for all $\underline{t}\in\underline{T}$ $\tau_{C_1(\underline{y})}\left(<\underline{t}>^{-1}(S)\right) = C_1(\underline{y})$. So by Lemma 3.1 we get a pullback. Therefore for all $c \in C_1(\underline{y})$, $<\underline{t}>(c) \in \tau_{C_1(\underline{x})}$. Hence by **Cif**$_1$ $\tau_{C_1(\underline{x})}(\underline{S}) \in F_x$ implies $\underline{S}\in F_x$. Therefore F is a categorial idempotent filter. As a result we show that given a localization (L, i) on $\mathbf{Cat}/_{\mathbf{C_0}}$ gives a topology τ on $\mathbf{Cat}/_{\mathbf{C_0}}$ and τ gives a categorical idempotent filter F on $\mathbf{Cat}/_{\mathbf{C_0}}$ Thus we obtain a categorical idempotent filter from a topology τ on $\mathbf{Cat}_{/\mathbf{C_0}}$ □

5. A topology on 2-crossed modules

Conduche defined the notion of a 2-crossed module as an algebraic model of homotopy 3-types and showed how to obtain a 2-crossed module from a simplicial group [8]. The notion of a 2-crossed module for commutative algebras was defined by Grandjean and Vale [12]. For more results about 2-crossed modules of commutative algebras and simplicial commutative algebras see Arvasi and Porter [2, 3, 22].

We recall from [12], the definition of a 2-crossed module: A 2-crossed module of **k**-algebras consists of a complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with ∂_2, ∂_1 morphisms of C_0-algebras, where the algebra C_0 acts on itself by multiplication, such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module in which C_1 acts on C_2, (thus we require that for all $x \in C_2$, $y \in C_1$ and $r \in R$ that $(xy)r = x(yr)$), further, there is a C_0-bilinear function

$$\{\_ \otimes \_\} : C_1 \otimes_{C_0} C_1 \longrightarrow C_2$$

called a Peiffer lifting, which satisfies the following axioms:
PL1 : $\partial_2\{y_0 \otimes y_1\} = y_0 y_1 - y_0 \cdot \partial_1(y_1)$,
PL2 : $\{\partial_2(x_1) \otimes \partial_2(x_2)\} = x_1 x_2$,
PL3 : $\{y_0 \otimes y_1 y_2\} = \{y_0 y_1 \otimes y_2\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\}$,
PL4 : $\{y \otimes \partial_2(x)\} + \{\partial_2(x) \otimes y\} = \partial_1(y) \cdot x$,
PL5 : $\{y_0 \otimes y_1\} \cdot r = \{y_0 \cdot r \otimes y_1\} = \{y_0 \otimes y_1 \cdot r\}$,
for all $x, x_1, x_2 \in C_1$, $y, y_0, y_1, y_2 \in C_2$ and $r \in C_0$.

We simply denote such a 2-crossed module as $(C_2, C, R, \partial_2, \partial)$. A morphism of 2-crossed modules of algebras can be defined in an obvious way. We thus define the category of 2-crossed module denoting it by $\mathbf{X_2Mod}$. If we fix the pre-crossed module $C \longrightarrow R$ we get a subcategory $\mathbf{X_2Mod}/_{\mathbf{C \to R}}$.

Definition 5.1. Let $\{L, M, P, \mu, \partial\}$ be a 2-crossed module of algebras. If
(i) L' is a subalgebra of L and M' is a subalgebra of M,
(ii) $\mu' = \mu_{|L'}$ and $\partial' = \partial_{|M'}$ are the restrictions of μ and ∂, respectively,
(iii) the action of P' on L' and M' is the restriction of the action of P on L and M,
(iv) $\{\_, \_\} : M' \times M' \longrightarrow L'$ is the restriction of $\{\_, \_\} : M \times M \longrightarrow L$ to $M' \times M'$,
then we call $\{L', M', P', \mu_{|L'}, \partial_{|M'}\}$ is sub-2-crossed module of $\{L, M, P, \mu, \partial\}$.

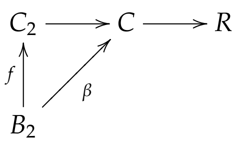
Using an idempotent filter on the algebra R, we will construct a topology on $\mathbf{X_2Mod}_{\ /\mathbf{C \longrightarrow R}}$. The set

$$\{L \longrightarrow R : L \longrightarrow R \text{ is the subcrossed module of } C \longrightarrow R\}$$

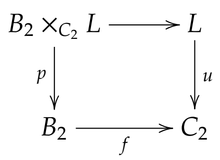
becomes a lattice by defining

$$\begin{aligned} L \wedge L' &= L \cap L' \\ L \vee L' &= < L \cup L' > \end{aligned}$$

Let $C_2 \longrightarrow C \longrightarrow R$ be a 2-crossed module and $B_2 \longrightarrow C \longrightarrow R$ be a sub-2-crossed module of $(C_2, C, R, \alpha, \partial)$ as shown diagrammatically:



Let $f : B_2 \longrightarrow C_2$ be a morphism of 2−crossed modules. For any sub-2-crossed module (L, C, R) of $(C_2, C, R, \alpha, \partial)$, the pullback of f with the inclusion $u : L \longrightarrow C$ given by the following diagram



becomes the sub-2-crossed module of (B, C, R), where $B_2 \times_{C_2} L$ is defined by

$$B_2 \times_{C_2} L = \{(b, x) \in B_2 \times L : f(b) = x\}$$

and

$$B_2 \times_{C_2} L \xrightarrow{\gamma} C \xrightarrow{\partial} R$$

is a 2-crossed module where $\gamma(b,x) = \beta(x)$ is called the inverse image of f. Thus we get the contravariant functor

$$Sub : \mathbf{X_2Mod}/_{\mathbf{C}\to\mathbf{R}} \longrightarrow \mathbf{Lat}$$

from the category of 2-crossed modules to that of lattices. Using the following definition, we can give the main result of this section.

Definition 5.2. Given a crossed module of algebras (C,R,∂) the set

$$F' = \{X \longrightarrow R : \text{X is an ideal of C}\}$$

is called the pre-crossed idempotent filter.

Theorem 5.3. *Let (C,R,∂) be a crossed module of algebras and F' be the pre-crossed idempotent filter of (C,R,∂). Then there exists a topology τ on the category $\mathbf{X_2Mod}/_{\mathbf{C}\to\mathbf{R}}$ and the $\tau-$ dense subobjects of the trivial 2-crossed module are in F'.*

Proof. We denote the sub-2-crossed module $(L, C_1, R, \alpha_{/L}, \partial_{/L})$ of $(C_2, C_1, R, \alpha, \partial)$ by L. Define

$$(L : y) = \{x \in R : y.x \in L\}$$

and

$$\tilde{L} = \{c_2 \in C_2 : (L : c) \longrightarrow R \in F'\}.$$

First, we will show that L is an ideal of C.

1. For $c_1, c_2 \in C_2$, since $(L : c_1)$ and $(L : c_2)$ are in F', the intersection $(L : c_1) \cap (L : c_2) \in F'$. By condition F_1 of an idempotent filter, we obtain $(L : c_1) \cap (L : c_2) \subset (L : c_1 + c_2)$ and $(L : c_1 + c_2) \in F'$.

2. For $l \in L$ and $r \in R$, since R acts on L, we have

$$\begin{aligned} x \in ((L : l) : r) &\Leftrightarrow r \cdot x \in (L : l) \\ &\Leftrightarrow l \cdot (r \cdot x) \in (L : l) \\ &\Leftrightarrow l \cdot (rx) \in (L : l) \\ &\Leftrightarrow l\,(rx) \in (L : l) \\ &\Leftrightarrow (lr)\,x \in (L : l) \\ &\Leftrightarrow (lr) \cdot x \in (L : l) \\ &\Leftrightarrow x \in (L : l \cdot r) \end{aligned}$$

Thus, we have

$$((L : l) : r) = (L : l \cdot r)$$

3. For $c_1, c_2 \in C_{2,}$ if $x \in ((L : c_1) : \alpha(c_2))$ then $\alpha[(c_2)] \cdot x\,(L : c_1)$, similarly we can obtain $c_1 \cdot [\alpha(c_2)]\,x \in L$ and $(c_1 c_2) \cdot x \in L$. Finally we obtain $x \in (L : c_1 c_2)$. Thus, $\left(\tilde{L}, C_1, R\right)$ becomes a sub-2-crossed module. If we define

$$\tau(L) = \tilde{L}$$

the operator

$$\tau : Sub \longrightarrow Sub$$

gives a topology on $\mathbf{X_2Mod}/_{\mathbf{C}\to\mathbf{R}}$.

ii. For $l \in L$,

$$(L : l) = \{x \in R : l \cdot x \in L\} = (C_1 \longrightarrow R)$$

since $(C_1 \longrightarrow R)$ is in F', we can write $l \in \tau(L)$. Thus

$$L \subset \tau(L).$$

iii. Let L and L' be two sub-2-crossed modules of (C_2, C_1, R). Assume that

$$L \subset L'$$

for $x \in \tau(L) = \tilde{L}$, $(L, x) \in F'$. We have

$$(L : x) \subset (L' : x)$$

from the first condition F_1 of an idempotent filter $(L' : x) \in F'$. Thus $x \in \tilde{L}' = \tau(L')$. Consequently, we can write

$$L \subset L' \Rightarrow \tau(L) \subset \tau(L').$$

i. To show that τ is idempotent we need to show

$$\tau(L) \circ \tau(L) \subset \tau(L)$$

From (ii) we already know the fact that $\tau(L)$ is a subset of $\tau(L) \circ \tau(L) = \tau^2(L)$.

Conversely, for $c \in \tau^2(L)$,

$$c \in \tau^2(L) \Rightarrow J = (\tau(L) : c) \in F'.$$

Thus for $x \in J$, $c \cdot x \in \tau(L)$, we get

$$c \cdot x \in \tau(L) \Rightarrow J = (L : cx) \in F',$$

since

$$(L : cx) = ((L : c) : x).$$

By condition F_3, it is clear that $(L : c)$ is in F'. Thus we have $c \in \tau(L)$. Finally, we have

$$\tau(L) \circ \tau(L) \subset \tau(L).$$

Finally, we will show that

$$I \in F' \Leftrightarrow \tau(I) = (C_1 \longrightarrow R).$$

Let $I \in F'$. Then we have

$$\tau(I) = \tilde{I} = \{r \in R : (l : r) \in F'\} = \partial(c_1).$$

Conversely, if

$$\tau(I) = \tilde{I} = (C_1 \longrightarrow R),$$

then for all $r \in R$, we can write $(l : r) \in F'$. Since this result holds for all $r \in R$, we obtain that F' is an idempotent filter and I is in F. □

Acknowledgment

The authors gratefully thank the referee for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

References

[1] Z. Arvasi, Crossed squares and 2-crossed modules of commutative algebras, Theory and Applications of Categories 3 (1997) 160–181.
[2] Z. Arvasi, T. Porter, Higher dimensional peiffer elements in simplicial commutative algebras, Theory and Applications of Categories 3 (1997) 1–23.
[3] Z. Arvasi, T. Porter, Freeness conditions for 2-crossed module of commutative algebras, Applied Categorical Structures 6 (1998) 455–477.
[4] Z. Arvasi, E. Ulualan, Quadratic and 2-crossed modules of algebras, Algebra Colloquium 4 (2007) 669–686.
[5] J.C. Baez, A.S. Crans, Higher dimensional algebra V:Lie 2-algebras, Bulletin of the American Mathematical Society 12 (2004) 492–528.
[6] J.C. Baez, A.D. Lauda, Higher dimensional algebra V: 2-groups, Theory and Applications of Categories 12 (2004) 423–491.
[7] M. Barr, C. Wells, Toposes, Triples, and Theories, Springer Verlag, Berlin, 2000.
[8] D. Conduché, Modules croisés généralisés de longueur 2, Journal of Pure and Applied Algebra 34 (1984) 155–178.
[9] G.J. Ellis, Crossed modules and higher dimensional analogues, PhD, University of Wales, Bangor, 1984.
[10] L. Euler, Elementa doctrinae solidorum, Novi commentarii academiae scientiarum Petropolitanae (1758) 109-140.
[11] P. Gabriel, Des categories abeliennes, Bulletin de la Société Mathématique de France 90 (1962) 323–448.
[12] A.R. Grandjean, M.J. Vale, 2-modulos cruzados an la cohomologia de André-Quillen, Memorias de la Reai Academia de Ciencias 22 (1986) 1–28.
[13] A. Grothendieck, Technique de descente et théoremes d'existence en géométrie algébrique. V. Les schémas de Picard: théoremes d'existence, Séminaire Bourbaki, 7 (1961), 143-161.
[14] J.L. Loday, Spaces with finitely many non-trivial homotopy groups, Journal of Pure and Applied Algebra 24 (1982) 179–202.
[15] S. Mac Lane. Categories for the working mathematician, (2nd edition), Springer Verlag, New York 1978.
[16] H. Poincaré, Analysis situs, Journal de l'École Polytechnique 2 (1895) 1—123.
[17] H. Poincaré, Complément à l'Analysis Situs, Rendiconti del Circolo Matematico di Palermo 13 (1899) 285-–343.
[18] H. Poincaré, Second complément à l'Analysis Situs, Proceedings of the London Mathematical Society 32 (1900) 277—308.
[19] H. Poincaré, Sur certaines surfaces algébriques: troisième complément à l'Analysis Situs, Bulletin de la Société Mathématique de France 30 (1902) 49-–70.
[20] H. Poincaré, Sur les cycles des surfaces algébriques: quatrième complément à l'Analysis Situs, Journal de Mathématiques Pures et Appliquées 5 (1902) 169-–214.
[21] H. Poincaré, Cinquième complément à l'analysis situs, Rendiconti del Circolo Matematico di Palermo 18 (1904) 45-–110.
[22] T. Porter, Some categorical results in the theory of crossed modules in commutative algebras, Journal of Algebra 109 (1987) 415–429.
[23] T. Porter, Extensions, crossed modules and internal categories of groups with operations, Proceedingsn of the Edinburgh Mathematical Society 30 (1987) 373–381.
[24] N.M. Shammu, Algebraic and categorical structure of categories of crossed modules of algebras, PhD thesis, University of Wales, Bangor, 1992.
[25] J.H.C. Whitehead, Combinatorial homotopy II, Bullettin of the American Mathematical Society 55 (1949) 453–496.