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Some Topological and Cardinal Properties of the Space of Permutation Degree

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Abstract. In this paper, we prove a few facts and some cardinal properties of the space of permutation degree introduced in [6]. More precisely, we prove that if the product X^n is a Lindelöf (resp. locally Lindelöf) space, then the space $\mathsf{SP}^n X$ is also Lindelöf (resp. locally Lindelöf). We also prove that if the product X^n is a weakly Lindelöf (resp. weakly locally Lindelöf) space, then the space $\mathsf{SP}^n X$ is also weakly Lindelöf (resp. weakly locally Lindelöf). Moreover, we investigate the preservation of the network weight, π-character and local density of topological spaces by the functor of G-permutation degree. It is proved that this functor preserves the network weight, π-character and local density of infinite topological T_1-spaces.

1. Introduction

In recent research the interest in the theory of cardinal invariants and their behavior under various covariant functors is increasing rapidly. In [3–5, 9–11, 17–19] the authors investigated behavior of several cardinal properties under some seminormal and normal functors and of the space of permutation degree.

The current paper is devoted to the investigation of cardinal invariants the network weight, π-character, local density and some other topological properties of the space of permutation degree. In addition, some geometric properties of the space of permutation degree are studied.

The concept of symmetric product of a topological space was introduced by K. Borsuk in [6]. As it turned out, functors such as the functor of G-permutation degree SP^n_G and the exponential functor of finite degree $\exp_n$ are similar to each other in many ways, and for $n = 2$ they coincide. However, there is an example of a topological space X showing that for $n > 2$ the spaces $\mathsf{SP}^n_G X$ and $\exp_n X$ are not homeomorphic (see [8, p. 19-20]).

In [17], the density and the weak density of spaces of permutation degree were investigated. Also, in [17] it was proved that the density (the weak density) of topological spaces coincides with the density (the weak density) of spaces of the permutation degree.

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In this paper, we study the behavior of the network weight, π-character and local density of topological spaces under the functor of G-permutation degree. It is proved that this functor preserves the network weight, π-character and local density of infinite topological T_1-spaces. We also prove that if the product X^n is a Lindelöf (resp. locally Lindelöf) space, then the space $\mathsf{SP}^n X$ is also a Lindelöf (resp. locally Lindelöf) space. Besides, we prove that if the product X^n is weakly Lindelöf (resp. weakly locally Lindelöf), then the space $\mathsf{SP}^n X$ is also weakly Lindelöf (resp. weakly locally Lindelöf).

Throughout the paper all spaces are assumed to be infinite T_1-spaces, and τ denotes an infinite cardinal number.

2. Some topological properties of the space of permutation degree

First, we give some notations, concepts, and statements that are widely used in this article. The *permutation group* of a set X is the group of all permutations of X. The permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, 3, ..., n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the n-th power of a compact space X. The permutation group S_n acts on X^n as the permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by $\mathsf{SP}^n X$. Thus, points of the space $\mathsf{SP}^n X$ are finite subsets (the equivalence classes) of the product X^n; two points $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in X^n$ are considered to be equivalent, if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space $\mathsf{SP}^n X$ is called the *n-permutation degree* of the space X. The equivalence relation by which we obtain the space $\mathsf{SP}^n X$ is called the *symmetric equivalence relation*. The n-th permutation degree is always a quotient of X^n. Thus, the quotient mapping is denoted as

$$\pi_n^s : X^n \to \mathsf{SP}^n X,$$

where

$$\pi_n^s\ ((x_1, x_2, ..., x_n)) = [x = (x_1, x_2, ..., x_n)]$$

is an orbit of the point $x = (x_1, x_2, ..., x_n) \in X^n$ [8].

The concept of permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as a group of permutations of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. This quotient space of the product of X^n under the G-symmetric equivalence relation is called *G-permutation degree* of the space X and it is denoted by $\mathsf{SP}^n_G X$. SP^n_G is also the covariant functor in the category of compact spaces and it is said to be the *functor of G-permutation degree*. (If $G = S_n$, then $\mathsf{SP}^n_G = \mathsf{SP}^n$. If the group G consists of one element, then $\mathsf{SP}^n_G X = X^n$.) The quotient mapping is this case is denoted as

$$\pi_{n,G}^s : X^n \to \mathsf{SP}^n_G X.$$

In [15], it was proved that the mapping $\pi_{n,G}^s$ is an open, closed and continuous onto mapping.

Definition 2.1. ([7]) A topological space X is a Lindelöf space if X is regular and every open cover of X has a countable subcover.

Theorem 2.2. *If the product X^n is a Lindelöf space, then $\mathsf{SP}^n X$ is also a Lindelöf space.*

Proof. Suppose that X^n is a Lindelöf space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of the space $\mathsf{SP}^n X$. For each $\alpha \in \Lambda$ set $V_\alpha = (\pi_n^s)^{\leftarrow}(U_\alpha)$. Since the mapping $\pi_n^s : X^n \to \mathsf{SP}^n X$ is continuous and $\{U_\alpha : \alpha \in \Lambda\}$ is an open cover of $\mathsf{SP}^n X$, we have that the family $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ is an open cover of X^n. As X^n is a Lindelöf space, there is a countable collection $V_{\alpha_i}{}_{i \in \mathbb{N}}$ with $\bigcup_{i \in \mathbb{N}} V_{\alpha_i} = X^n$. It follows

$$\pi_n^s(X^n) = \pi_n^s\left(\bigcup_{i \in \mathbb{N}} V_{\alpha_i}\right) = \bigcup_{i \in \mathbb{N}} \pi_n^s(V_{\alpha_i}) = \bigcup_{i \in \mathbb{N}} U_{\alpha_i} = \mathsf{SP}^n X,$$

i.e. $\{U_{\alpha_i}\}_{i \in \mathbb{N}}$ is an open countable subcover of $\mathcal{U}$. The theorem is proved. $\square$

Remark 2.3. If X is a Lindelöf space, it does not always mean that $\mathsf{SP}^{\mathsf{n}}X$ is a Lindelöf space. The Sorgenfery line is such an example.

Definition 2.4. ([16]) A topological space X is *almost Lindelöf* if for every open cover μ of X, there exists a countable subfamily $\eta \subset \mu$ with $\bigcup_{V \in \eta} \overline{V} = X$.

It follows immediately from the definition that every Lindelöf space is almost Lindelöf. But the converse is not true.

Example 2.5. Let $A = \{(a_\alpha; -1) : \alpha < \omega_1\}$ be an ω_1-long sequence in the set $\{(x; -1) : x \geq 0\} \subset R^2$, where ω_1 is first uncountable cardinal number. Let $Y = \{(a_\alpha; n) : \alpha < \omega_1, n \in N\}$, $a = (-1; 1)$ and $X = Y \bigcup A \bigcup \{a\}$. Define a topology in X as follows:

- all points in Y are isolated;

- for $\alpha < \omega_1$ the basic neighbourhoods of $(a_\alpha; -1)$ will be of the form

$$U_n(a_\alpha; -1) = \{(a_\alpha; -1)\} \bigcup \{(a_\alpha; m) : m \geq n\}$$

for all $n \in N$.

- basic neighbourhoods of $a = (-1; 1)$ are of the form

$$U_\alpha(a) = \{a\} \bigcup \{(a_\beta; n) : \beta > \alpha, n \in N\}$$

for $\alpha < \omega_1$.

It is obvious that this topological space is almost Lindelöf, but not Lindelöf.

It is known that a continuous image of an almost Lindelöf space is almost Lindelöf. In particular, a quotient space of an almost Lindelöf space is also almost Lindelöf. Therefore, we have the following corollary.

Corollary 2.6. *If the space X^n is almost Lindelöf, then the space $\mathsf{SP}^{\mathsf{n}}X$ is also almost Lindelöf.*

Remark 2.7. If X is an almost Lindelöf space, it is not true in general that $\mathsf{SP}^{\mathsf{n}}X$ is almost Lindelöf.

Almost Lindelöfness is not preserved under the products of the spaces. For instance, the Sorgenfrey line K is almost Lindelöf but the Sorgenfrey plane $K \times K$ is not almost Lindelöf.

The following notion is well known.

Definition 2.8. A topological space X is *weakly Lindelöf* if for every open cover μ of X, there exists a countable subfamily $\eta \subset \mu$ with $\overline{\bigcup_{V \in \eta} V} = X$.

It is clear from the definition of weakly Lindelöf space that any Lindelöf space is weakly Lindelöf. In addition, every almost Lindelöf space is weakly Lindelöf, but the converse is not true. For instance, the Sorgenfrey plane is regular weakly Lindelöf, but not almost Lindelöf.

It is known that a continuous image of a weakly Lindelöf space is weakly Lindelöf. So, a quotient space of a weakly Lindelöf space is also weakly Lindelöf. Hence we have the following corollary.

Corollary 2.9. *If the product X^n is a weakly Lindelöf space, then the space $\mathsf{SP}^{\mathsf{n}}X$ is also weakly Lindelöf.*

A closed subspace of a weakly Lindelöf space needs not be weakly Lindelöf.

Definition 2.10. ([1]) A topological space is called *quasi-Lindelöf* if every closed subspace of it is weakly Lindelöf.

It is easy to check that if a topological space is normal and weakly Lindelöf, then the space is quasi-Lindelöf. See [14] for a discussion about weaker forms of the Lindelöf property.

It is known that a continuous image of a quasi-Lindelöf space is quasi-Lindelöf, therefore a quotient space of a quasi-Lindelöf space is also quasi-Lindelöf. Hence we have the following corollary.

Corollary 2.11. *If the product X^n is quasi-Lindelöf space, then the space $\mathsf{SP}^{\mathsf{n}}X$ is also quasi-Lindelöf.*

Definition 2.12. ([13]) A topological space X is said to be *locally Lindelöf* if every $x \in X$ has a closed neighbourhood F_x which is Lindelöf.

Evidently, every Lindelöf space is locally Lindelöf, but the converse is not true in general.

Example 2.13. Every uncountable discrete space is locally Lindelöf, but not Lindelöf.

Proposition 2.14. *If the product X^n is a locally Lindelöf space, then $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is also a locally Lindelöf space.*

Proof. Since $\pi^s_{n,G}$ is an onto mapping for every point $y \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ we have that there exist some $x \in X^n$ such that $\pi^s_{n,G}(x) = y$. Since X^n is locally Lindelöf, x has a Lindelöf closed neigbourhood F_x. Then, by continuity and closedness of $\pi^s_{n,G}$, the set $\pi^s_{n,G}(F_x)$ is a Lindelöf closed neigbourhood of y. It means that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is locally Lindelöf. □

Local Lindelöfness is not a finitely productive property. For instance, the Sorgenfrey line K is Lindelöf, hence it is locally Lindelöf. But the Sorgenfrey plane $K \times K$ is not locally Lindelöf.

In general, the continuous image of a locally Lindelöf space need not be locally Lindelöf. For example, let X be an uncountable discrete space and Y be the Sorgenfrey plane $K \times K$. Then any mapping from X onto Y is continuous, but Y is not locally Lindelöf.

Remark 2.15. If X is a locally Lindelöf space, then the space $\mathsf{SP}^{\mathsf{n}}X$ need not be locally Lindelöf.

Definition 2.16. ([13]) A topological space X is said to be *weakly locally Lindelöf* if every $x \in X$ has a neighbourhood U_x which is Lindelöf.

Example 2.17. Let (Y, τ_Y) be a non-Lindelöf but locally Lindelöf space. Put $X = Y \bigcup \{x_0\}$, where x_0 is a point not belonging to Y. We define a topology τ_X on X as follows:

$$\tau_X = \{U \cup \{x_0\} : U \in \tau_Y\} \cup \{\emptyset\}.$$

In other words, τ_X consists of all open sets in Y augmented by x_0 and the empty set. Then the closure of any neighbourhood of x_0 is the whole space X which is not Lindelöf.

This example shows that the closure of a Lindelöf set is not always Lindelöf.

Every locally Lindelöf space is weakly locally Lindelöf, but weak local Lindelöfness need not imply local Lindelöfness. For instance, (R, τ) where $\tau = \{U \subseteq R : 0 \in U\} \cup \{\emptyset\}$. The space is weakly locally Lindelöf, but it is not locally Lindelöf.

Proposition 2.18. *If the product X^n is a weakly locally Lindelöf regular space, then $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a weakly locally Lindelöf space.*

Proof. Since $\pi^s_{n,G}$ is an onto mapping for every point $y \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ we have that there exist some $x \in X^n$ such that $\pi^s_{n,G}(x) = y$. Since X^n is weakly locally Lindelöf, every $x \in X^n$ has a Lindelöf neigbourhood U_x. Then, by openness of $\pi^s_{n,G}$, $\pi^s_{n,G}(U_x)$ is a neigbourhood of y. In addition, the Lindelöf property is preserved under an open mapping for every regular space. Therefore, $\pi^s_{n,G}(U_x)$ is a Lindelöf neigbourhood of y. It means that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is weakly locally Lindelöf. □

Remark 2.19. If X is a weakly locally Lindelöf space, it does not always mean that $\mathsf{SP}^{\mathsf{n}}X$ is a weakly locally Lindelöf space.

Recall that a subset A of a space X is said to be a G_τ-set if it is the intersection of $\leq \tau$ many open sets in X.

Theorem 2.20. *If the set $\mathsf{SP}^{\mathsf{n}}E$ is a G_τ-set, $\tau \geq \aleph_0$, in the space $\mathsf{SP}^{\mathsf{n}}X$, then the set $(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}E)$ is also a G_τ-set in the space X^n.*

Proof. Suppose that the set $\mathsf{SP}^{\mathsf{n}}E$ is a G_τ-set in $\mathsf{SP}^{\mathsf{n}}X$. Then there exists a family of open sets $\mathsf{SP}^{\mathsf{n}}\gamma = \{\mathsf{SP}^{\mathsf{n}}U_\alpha : \alpha < \tau\}$ in $\mathsf{SP}^{\mathsf{n}}X$, such that $\mathsf{SP}^{\mathsf{n}}E = \bigcap_{\alpha<\tau}\mathsf{SP}^{\mathsf{n}}U_\alpha$. Therefore, we have

$$(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}E) = (\pi_n^s)^{-1}(\bigcap_{\alpha<\tau}\mathsf{SP}^{\mathsf{n}}U_\alpha) = \bigcap_{\alpha<\tau}((\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}U_\alpha)).$$

Since the quotient mapping $\pi_n^s : X^n \to \mathsf{SP}^{\mathsf{n}}X$ is continuous and for every $\alpha < \tau$ the set $\mathsf{SP}^{\mathsf{n}}U_\alpha$ is open, we have that the subset $((\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}U_\alpha))$ of X^n is open. This shows that a family $\gamma = \{((\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}U_\alpha)) : \alpha < \tau\}$ is a family of open sets in X^n and

$$|\gamma| = |\{((\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}U_\alpha)) : \alpha < \tau\}| \leq n! \cdot |\{\mathsf{SP}^{\mathsf{n}}U_\alpha : \alpha < \tau\}| = n! \cdot \tau = \tau.$$

This means that the set $(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}E)$ is a G_τ-set in X^n. □

Definition 2.21. ([2]) A subset $A \subset X$ is said to be τ-placed in X, if for each $x \in X \setminus A$ there exists a G_τ-set $E \subset X$ such that $x \in E \subset X \setminus A$.

Theorem 2.22. *If a set $\mathsf{SP}^{\mathsf{n}}A$ is τ-placed in $\mathsf{SP}^{\mathsf{n}}X$, then the set $(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}A)$ is τ-placed in X^n.*

Proof. Assume that the set $\mathsf{SP}^{\mathsf{n}}A$ is τ-placed in $\mathsf{SP}^{\mathsf{n}}X$. Then we have that for every point $y \in \mathsf{SP}^{\mathsf{n}}X \setminus \mathsf{SP}^{\mathsf{n}}A$ there exists a set $\mathsf{SP}^{\mathsf{n}}E$ which is of the type G_τ in $\mathsf{SP}^{\mathsf{n}}X$ and such that $y \in \mathsf{SP}^{\mathsf{n}}E \subset \mathsf{SP}^{\mathsf{n}}X \setminus \mathsf{SP}^{\mathsf{n}}A$. The mapping $\pi_n^s : X^n \to \mathsf{SP}^{\mathsf{n}}X$ is an onto mapping and so for every point $x \in X^n$ there exists a point $y \in \mathsf{SP}^{\mathsf{n}}X$ such that

$$x \in (\pi_n^s)^{\leftarrow}(y) \subset (\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}E) \subset (\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}X \setminus \mathsf{SP}^{\mathsf{n}}A) = X^n \setminus ((\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}A)).$$

On the other hand, by the above theorem the set $(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}E)$ is a G_τ-set in X^n. This shows that the set $(\pi_n^s)^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}X \setminus \mathsf{SP}^{\mathsf{n}}A)$ is τ-placed in X^n. □

Proposition 2.23. *Let X be a topological space, n a natural number and G an arbitrary subgroup of the group S_n. The space X is regular (resp. completely regular) space if and only if $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a regular (resp. completely regular) space.*

Proof. Let X be a regular (resp. completely regular) space. Then the nth power X^n of X is also a regular (resp. completely regular) space. It is known that regularity is preserved by closed mappings, while complete regularity is preserved by open-closed mappings. Hence, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is regular (resp. completely regular) space whenever X is regular (resp. completely regular). The converse is clear since X is a subspace of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, and regularity and complete regularity are hereditary properties. □

The Sorgenfrey line K is a normal space, but the Sorgenfrey plane $K \times K$ is not normal. In addition, exp_2K is not also normal. It is clear that the spaces SP^2X and exp_2X are homeomorphic and so SP^2K is not normal.

Remark 2.24. The functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ does not preserve the normality of topological spaces.

Definition 2.25. ([7]) We say that a topological space X is *locally connected* if for every $x \in X$ and any neighbourhood U of the point x there exists a connected set $C \subset U$ such that $x \in \mathrm{int}(C)$.

Theorem 2.26. ([7]) *A space X is locally connected if and only if the components of all open subspaces of X are open.*

The local connectedness is not an invariant of continuous mappings, but it is an invariant of quotient mappings.

Theorem 2.27. ([7]) *The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for all $s \in S$, is locally connected if and only if all spaces X_s are locally connected and there exists a finite set $S_0 \subset S$ such that X_s is connected for $s \in S \setminus S_0$.*

Corollary 2.28. *Let X be a locally connected space and G be an arbitrary subgroup of the permutation group S_n. Then the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is locally connected.*

In [12] it is proved that the exponential functor $\exp$ and the functor of superextension λ preserve the conditions (i) and (ii) below with respect to the topology of any T_1-space, and the functor of complete linked systems $\mathcal{N}$ preserves the conditions (i) and (ii) with respect to the topology of any compact space, where

(i) $\tau_1 \subseteq \tau_2$;

(ii) τ_1 is a π-base for τ_2, i.e. for each non-empty element $O \in \tau_2$ there exists an element $V \in \tau_1$ such that $V \subset O$.

Problem 2.29. *Suppose that a topological space X satisfies conditions (i) and (ii). Do spaces $\mathsf{SP}^{\mathsf{n}}X$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ satisfy conditions (i) and (ii) too?*

3. Some cardinal properties of the space of permutation degree

A family $\mathcal{N} = \{M_s\}_{s\in S}$ of subsets of a topological space X is called a *network* of X if for each point $x \in X$ and each neighborhood U of the point x there exists $s \in S$ such that $x \in M_s \subset U$. The *network weight* of the space X is defined as the smallest cardinal of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network of X. This cardinal is denoted by $nw(X)$.

Proposition 3.1. *Let X be an infinite topological T_1-space and G be an arbitrary subgroup of the permutation group S_n. Then $nw(X) = nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X)$.*

Proof. First, show that $nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X) \leq nw(X^n)$. Suppose that $nw(X^n) = \tau \geq \aleph_0$ and let the family $\mathcal{N} = \{M_\alpha : \alpha < \tau\}$ be a network in X^n. Consider the family $\mathcal{N}' = \{\pi^s_{n,G}(M_\alpha) : \alpha < \tau\}$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. It is clear that $|\mathcal{N}'| \leq \tau$. We will show that $\mathcal{N}'$ is a network in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. The image of every open element of the family $\mathcal{N}$ is an open set, since $\pi^s_{n,G}$ is an open mapping. Let $y \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ be an arbitrary point and U_y an arbitrary neighborhood of the point y in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Then there exists a point $x \in X^n$ such that $y = \pi^s_{n,G}(x)$. The set $(\pi^s_{n,G})^{\leftarrow}(U_y)$ is an open set in X^n, since the mapping $\pi^s_{n,G}$ is continuous. Then there exists $M_\alpha \in \mathcal{N}$ such that $x \in M_\alpha \subset (\pi^s_{n,G})^{\leftarrow}(U_y)$. Hence, we have that $y = \pi^s_{n,G}(x) \in \pi^s_{n,G}(M_\alpha) \subset U_y$. Therefore, the family $\mathcal{N}'$ is a network in the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. It means that $nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X) \leq nw(X^n)$. In addition, it is known that $nw(X^n) \leq nw(X)$.

Now we will show the inverse inequality $nw(X) \leq nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X)$. Indeed, $nw(X) \leq nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X)$ since X is a subspace of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and the network weight is inherited by any subspace. Therefore, we have that $nw(X^n) \leq nw(X) \leq nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X) \leq nw(X^n)$. Hence, $nw(X) = nw(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X)$. □

Recall that a family ν of non-empty open sets of X is called a *local π-base* of x in X if for every neighbourhood V of x, there is $U \in \nu$ with $U \subset V$.

The *π-character* $\pi\chi(x, X)$ of x in X, denoted by $\pi\chi(x, X)$, is $\min\{|\nu| : \nu$ is a local π-base of x in $X\} + \omega$. The π-character $\pi\chi(X)$ of X is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.

Proposition 3.2. *Let X be an infinite topological T_1-space and G be an arbitrary subgroup of the permutation group S_n. Then $\pi\chi(X) = \pi\chi(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X)$.*

Proof. First, we will show that $\pi\chi(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X) \leq \pi\chi(X^n)$. Suppose that $\pi\chi(X^n) = \tau \geq \aleph_0$. Take an arbitrary point $y \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Then there exists a point $x \in X^n$ such that $y = \pi^s_{n,G}(x)$. Let the family $\Im = \{U_\alpha : \alpha \in A\}$ be the local π-base of X^n at the point x. Consider the family $\Im' = \{\pi^s_{n,G}(U_\alpha) : \alpha \in A\}$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. It is clear that $|\Im'| \leq \tau$. We will show that $\Im'$ is a local π-base at the point $y \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Each element of the family $\Im'$ is an open set and contains the point y, since the mapping $\pi^s_{n,G}$ is open. Let V be a non-empty open subset of $\mathsf{SP}^n_G X$ which contains the point y. Then the set $(\pi^s_{n,G})^{\leftarrow}(V)$ is an open set and contains the point x. It follows that there exists $U_\alpha \in \Im$ such that $U_\alpha \in (\pi^s_{n,G})^{\leftarrow}(V)$. Therefore, we have $(\pi^s_{n,G})(U_\alpha) \subset V$. It means that the system $\Im'$ is a local π-base in $\mathsf{SP}^n_G X$ at the point y. So, $\pi\chi(y, \mathsf{SP}^n_G X) \le \tau$. It means that $\pi\chi(\mathsf{SP}^n_G X) \le \pi\chi(X^n)$. It is clear that $\pi\chi(X^n) \le \pi\chi(X)$ so that $\pi\chi(\mathsf{SP}^n_G X) \le \pi\chi(X)$

Now we will show the inverse inequality that $\pi\chi(X) \le \pi\chi(SP^n_G X)$. Since X is a subspace of the space $\mathsf{SP}^n_G X$ and the π-character is inherited by every subspace. Therefore, we have that $\pi\chi(X) \le \pi\chi(\mathsf{SP}^n_G X)$.

Hence, $\pi\chi(X) = \pi\chi(\mathsf{SP}^n_G X)$. □

We say that the *local density* of a topological space X at a point $x \in X$ is τ if τ is the smallest cardinal number such that x has a neighbourhood of density τ. The local density at a point x is denoted by $ld(X, x)$. The local density of a topological space X is defined as the supremum of all numbers $ld(X, x)$: $ld(X) = \sup\{ld(X, x) : x \in X\}$ [17]. It is known that, for any topological space we have $ld(X) \le d(X)$.

Proposition 3.3. *Let X be a space of local density τ and $f : X \to Y$ be an open onto mapping. Then Y is a space of local density τ.*

Proof. Since the mapping f is onto, for every point $y \in Y$ the pre-image $f^{\leftarrow}(y)$ is nonempty in X. For each point $x \in f^{\leftarrow 1}(y)$ there exists a neighborhood O_x such that the density of O_x is τ. Since f is open, $f(O_x)$ is an open set in Y and contains the point y. It is known that the density is preserved under a continuous mapping, therefore the density of $f(O_x)$ is τ. □

Theorem 3.4. *Let X be an infinite topological T_1-space and G be an arbitrary subgroup of the permutation group S_n. Then $ld(X) = ld(\mathsf{SP}^n_G X)$.*

Proof. First, we will show that $ld(\mathsf{SP}^n_G X) \le ld(X^n)$. Consider the mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^n_G X$, where $n \in \mathbb{N}$. Since the mapping $\pi^s_{n,G}$ is open, by Proposition 3.3 we have $ld(\mathsf{SP}^n_G X) \le ld(X^n)$.

Now we will show the inverse inequality $ld(X^n) \le ld(\mathsf{SP}^n_G X)$. The mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^n_G X$ is finite-to-one because for every $y \in \mathsf{SP}^n_G X$ we have $|(\pi^s_{n,G})^{\leftarrow}(y)| \le n!$. Then it is shown that $ld(X^n) \le ld(\mathsf{SP}^n_G X)$.

Now we will show that $ld(X^n) \le ld(X)$. Let $ld(X) = \tau \ge \aleph_0$ and take an arbitrary point $x = (x_1, x_2, ..., x_n) \in X^n$. There exist neighborhoods $O_{x_1}, O_{x_2}, ..., O_{x_n}$ of the points $x_1, x_2, ..., x_n \in X$ such that $d(O_{x_j}) \le \tau$ for all $j = 1, 2, ..., n$. By virtue of the well know Hewitt-Marczewski-Pondiczery theorem [7, Theorem 2.3.15] we have $d(\prod_{j=1}^n O_{x_j}) \le \tau$. The set $\prod_{j=1}^n O_{x_j}$ is a neighbourhood of the point $x \in X^n$, hence $ld(X^n) \le \tau$.

The inverse inequality $ld(X) \le ld(X^n)$ is clear by Proposition 3.3 because the projection $pr : X^n \to X$ is an open continuous mapping. Hence, $ld(X) = ld(X^n)$.

It means that $ld(X) = ld(X^n) = ld(\mathsf{SP}^n_G X)$ and the theorem is proved. □

References

[1] A.V. Arhangel'skii, A theorem on cardinality, Russian Math. Surveys 34 (1979) 153–154.
[2] A.V. Arhangel'skii, Functional tightness, Q-spaces and τ-embeddings, Comment. Math. Univ. Carolin. 24 (1983) 105–120.
[3] R.B. Beshimov, Some properties of the functor O_β, J. Math. Sci. 133 (2006) 1599–1601.
[4] R.B. Beshimov, Nonincrease of density and weak density under weakly normal functors, Math. Notes 84 (2008) 493–497.
[5] R.B. Beshimov, F.G. Mukhamadiev, Cardinal properties of Hattori spaces and their hyperspaces, Questions Answers Gen. Topology 33 (2015) 33–38.
[6] K. Borsuk, S. Ulam, On the symmetric products of topological spaces, Bull. Amer. Math. Soc. 37 (1931) 875–882.
[7] R. Engelking, General Topology (Revised and completed edition), Heldermann, Berlin, 1989.
[8] V.V. Fedorchuk, V.V. Filippov, Topology of hyperspaces and its applications, Znanie, Moscow, Mathematica, Cybernetica 4 (1989), 48 p. [in Russian].
[9] Lj.D.R. Kočinac, F.G. Mukhamadiev, A.K. Sadullaev, Some cardinal and geometric properties of the space of permutation degree, Axioms 11 (2022) 290, 9 pages.
[10] F.G. Mukhamadiev, On certain cardinal properties of the N^φ_τ-nucleus of a space X, J. Math. Sci. 245 (2020) 411–415.
[11] F.G. Mukhamadiev, On local weak τ-density of topological spaces, Vestnik Tomskogo Gos. Univ. Mat. Mekh. 70 (2021) 16–23.
[12] F.G. Mukhamadiev, Some topological properties of the spaces $expX$, λX and NX, Math. Moravica 21 (2017) 17–25.
[13] D. Ali Reyadh, On locally Lindelöf spaces, CO-Lindelöf topologies and locally LC-Spaces, J. AL-Qadisiyah Comp. Sci. Math. 7 (2015) 53–71.

[14] P. Staynova, A comparison of Lindelöf-type covering properties of topological spaces, Rose Hulman Undergrad. Math. J. 12 (2011) 163–204.
[15] H. Wagner, Symmetric, cyclic, and permutation products of manifolds, Dissertationes Mathematicae, Warszawa, 1980. 47 p.
[16] S. Willard, U.N.B. Dissanayake, The almost Lindelöf degree, Canad. Math. Bull. 27 (1984) 452–455.
[17] T.K. Yuldashev, F.G. Mukhamadiev, The local density and the local weak density in the space of permutation degree and in Hattori space, Ural Math. J. 6 (2020) 108–116.
[18] T.K. Yuldashev, F.G. Mukhamadiev, Local τ-density and the local weak τ-density in topological spaces, Siberian Electronic Math. Reports 18 (2021) 474–478.
[19] T.K. Yuldashev, F.G. Mukhamadiev, Caliber of space of subtle complete coupled systems, Lobachevskii J. Math. 42 (2021) 3043–3047.