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Geometric Intersection Numbers Between Multiple and Elementary Curves on Punctured Torus

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Abstract. We obtain formulas that give the geometric intersection numbers between multiple curves and elementary curves, which are particular types of multiple curves, on the punctured torus with boundary by using Dynnikov coordinates. To do this, we first determine what the elementary curves on this surface are, and then find separately the formulas that give the number of geometric intersections of multiple curves with each elementary curve.

1. Introduction

Let S_n be an orientable genus-1 surface with n ($n \geq 2$) punctures and one boundary component as shown in Figure 1. Moreover, let $\mathcal{L}_n$ denote the set of multiple curves (a multiple curve is a finite union of mutually disjoint essential simple closed curves up to isotopy) on S_n and $\mathcal{L} \in \mathcal{L}_n$. Throughout the paper, we always work with the minimal representative (a multiple curve in the same isotopy class intersecting minimally with coordinate curves) of $\mathcal{L}$ and denote it by L (you can see an example of a multiple curve in Figure 2). Such curves are usually described combinatorially by train track or Dehn-Thurston coordinates [5]. An alternative and effective way to coordinate a multiple curve on S_n is achieved by the Dynnikov coordinate system which provides an explicit bijection between the isotopy classes of multiple curves and $\mathbb{Z}^{2n+2} \setminus \mathcal{V}_n$ where $\mathcal{V}_n = \{(a; b; T; c) : c \leq 0 \text{ and } T \neq 0\} \cup \{0\}$ [3]. For the multiple curves on a finitely punctured disk, this coordinate system was created by Dynnikov in 2002 [1]. Given $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}_n$, the geometric intersection number $i(\mathcal{L}_1, \mathcal{L}_2)$ is the minimum number of intersections between representatives in the isotopy classes of $\mathcal{L}_1$ and $\mathcal{L}_2$.

The aim of this paper is to generalize the formula of the geometric intersection number between a multiple curve and an elementary curve, which is a particular type of multiple curve, on the punctured disk [2, 6–8] to the surface S_n. To do this, first, we introduce elementary curves on S_n. Secondly, we give formulas to calculate the geometric intersection number of a multiple curve with each elementary curve on S_n. Since a multiple curve is given by its Dynnikov coordinates, we use Theorem 2.11 for using formulas and solving all examples. To calculate the geometric intersection number of two arbitrary multiple curves on S_n, one could calculate the action of the mapping class group of S_n in terms of Dynnikov coordinates given in Definition 2.10 and combine it with the formulas stated in this paper.

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The paper is organized as follows. In Section 2, we give a brief summary of the Dynnikov coordinate system on S_n, and explain the properties of multiple curves on this surface [3]. The results of this paper are given in Section 3, where we introduce the elementary curves on S_n [4], and obtain formulas for the geometric intersection numbers between multiple curves and these elementary curves by using Dynnikov coordinates on S_n.

2. Dynnikov coordinates on S_n

We consider the arcs α_i $(1 \leq i \leq 2n)$, β_i $(1 \leq i \leq n+1)$, γ, and the curve c as depicted in Figure 1. That

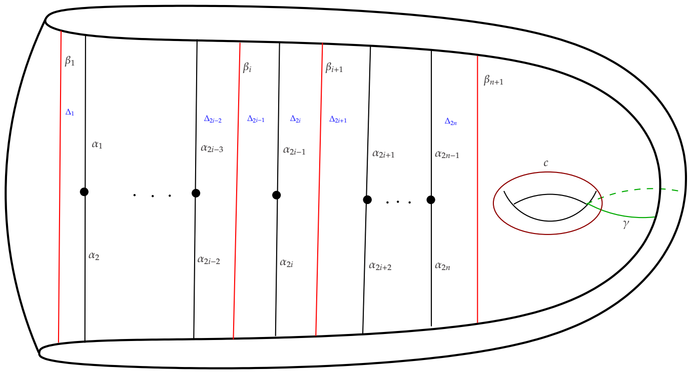

Figure 1: Coordinate curves on S_n.

is, the endpoints of the arcs α_i and β_i are either on the boundary of S_n or on the puncture. While c is the longitude of the torus, γ is the arc whose both endpoints are on the boundary. Note that γ intersects c once transversally. Moreover, let the vector $(\alpha_1, \cdots, \alpha_{2n}; \beta_1, \cdots, \beta_{n+1}; \gamma; c) \in \{\mathbb{Z}^{3n+3}_{\geq 0}\} \setminus \{0\}$ show the intersection numbers of L with the corresponding arcs and the simple closed curve c. It will always be clear from the context whether we mean the coordinate curve or the geometric intersection number assigned on the coordinate curve. For example, $(6, 6, 8, 4, 4, 2; 4, 12, 4, 6; 6; 2)$ are the intersection numbers of the multiple curve L shown in Figure 2.

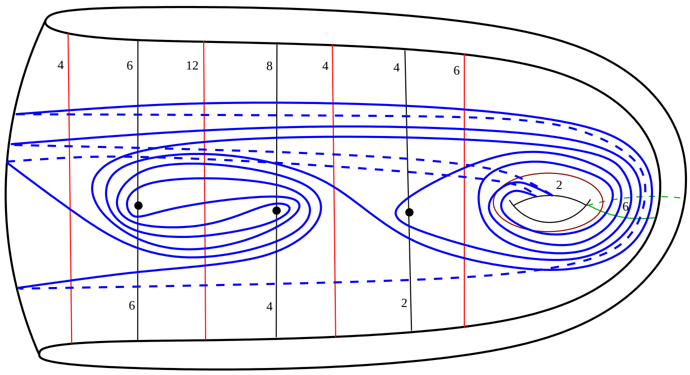

Figure 2: Intersection numbers of the multiple curve L with the coordinate curves.

Let U_i $(1 \leq i \leq n)$ be the region that is bounded by β_i and β_{i+1} (see Figure 3) and G be the region bounded by β_1, β_{n+1} and the boundary of S_n (∂S_n) (see Figures 4 and 5). Each component of $L \cap U_i$ and $L \cap G$ is called the *path component* of L in U_i and G, respectively. Since L is minimal, there are 4 types of path components in U_i $(1 \leq i \leq n)$ as on the disk [7] (see Figure 3). An *above component* has endpoints on β_i and β_{i+1} and intersects α_{2i-1}. A *below component* has endpoints on β_i and β_{i+1} and intersects α_{2i}. A *left loop component* has both endpoints on β_{i+1} and intersects α_{2i-1} and α_{2i} (see Figure 3a). A *right loop component* has both endpoints on β_i and intersects α_{2i-1} and α_{2i} (see Figure 3b). There are 6 types of path components in G. The first three of these are curve c, front genus component and back genus component. The *curve c* is the longitude of the torus (see Figure 4a). The *front genus component* has both endpoints on β_{n+1} and does not intersect the curve c (see Figure 4b). The *back genus component* has both endpoints on β_1 and does not intersect the curve c (see Figure 4c). The other three components are called *twisting*, which have endpoints on β_1 and β_{n+1} and intersect the curve c (see Figure 5). These components are nontwist, negative twist, and positive twist components. The *nontwist component* does not make any twist (see Figure 5a). The *negative twist component* makes clockwise twist (see Figure 5b). The *positive twist component* makes counterclockwise twist (see Figure 5c).

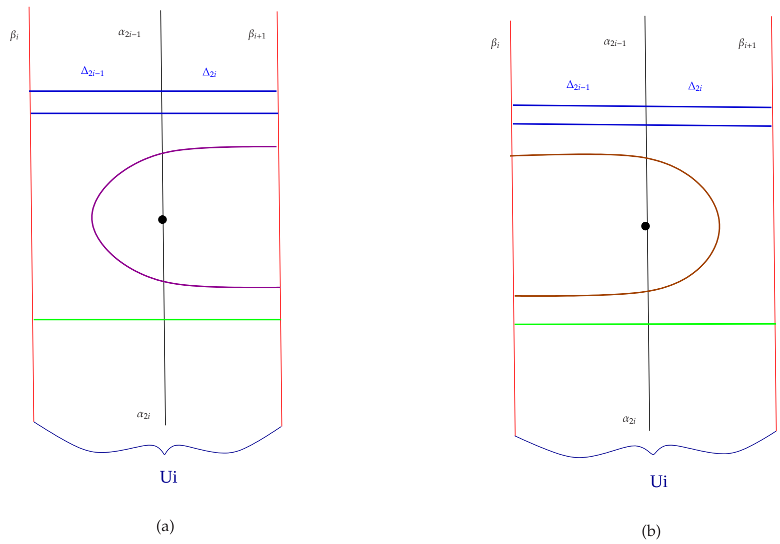

Figure 3: Above and below components, left and right loop components in the region U_i.

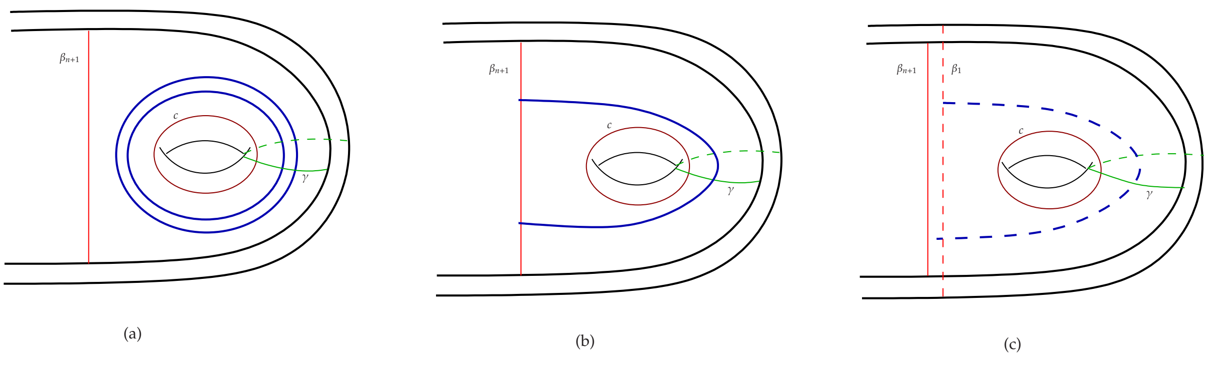

Figure 4: (a) c curves, (b) front genus component, (c) back genus component in the region G.

The *twist number* of a twisting component is the signed number of intersections between this component and the curve γ.

Remark 2.1. If L contains the $p(c)$ copies of the curve c, then let

$$c = -p(c), \tag{1}$$

where $p(c) > 0$. Moreover, let c^+ give the number of the twisting components in the region G. c^+ is defined

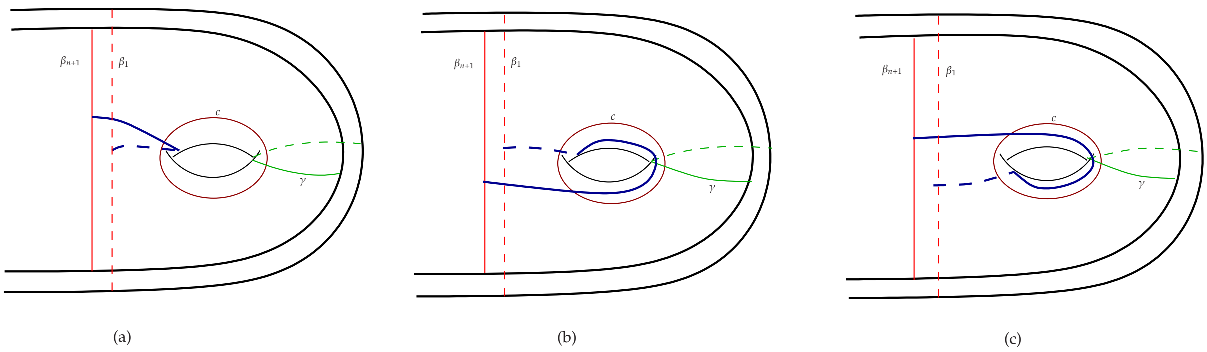

Figure 5: (a) Nontwist component, (b) negative twist component, (c) positive twist component.

by $\max(c, 0)$ throughout the paper. That is, if the integral lamination contains the twisting components, then $c > 0$ and the number of the twisting components is c^+; if the integral lamination contains the curve c, then $c < 0$ and the number of the copies of curve c is given by $p(c)$; and if the integral lamination contains neither twisting components nor curve c, then $c = 0$.

Let us denote the *smaller twist number* by t and the *bigger twist number* by $t + 1$. The *total twist number* T in G is the sum of the twist numbers of such components. If the difference between the twist numbers of any two twisting components is 0, then $T = tc^+$. On the other hand, if the difference between the twist numbers of any two twisting components is 1, then $T = m(t + 1) + (c^+ - m)t$, where $m \in \mathbb{Z}_{\geq 0}$ is the number of the twisting components with the twist number $t + 1$, and $c^+ - m$ is the number of the twisting components with the twist number t.

Remark 2.2. Although T gives the total twist number in G, it cannot show the directions of the twists by itself. Therefore, we first calculate the number T, and then we add a sign in front of T, denoting the negative direction by $-T$ and the positive direction by T. However, since only the total number of the twists is required in the formulas throughout the paper, $|T|$ shall be used as the total number of the twists in order not to cause any confusion.

Lemma 2.3. ([3]) *Let L be given with the intersection numbers $(\alpha; \beta; \gamma; c)$, and the number of the front genus components and the number of the back genus components be ρ and ρ', respectively. Then,*

$$\rho = \frac{\beta_{n+1} - c^+}{2} \quad \textit{and} \quad \rho' = \frac{\beta_1 - c^+}{2}.$$

The following lemma gives the total twist number of the twisting components:

Lemma 2.4. ([3]) *Let L be given with the intersection numbers $(\alpha; \beta; \gamma; c)$, denoting the signed total twist number of the twisting components by T. We have*

$$|T| = \begin{cases} 0 & \textit{if } c^+ = 0, \\ \gamma - \frac{\beta_{n+1} - c^+}{2} - \frac{\beta_1 - c^+}{2} & \textit{if } c^+ \neq 0. \end{cases} \tag{2}$$

The sign of the negative twist component is -1 and the sign of the positive twist component is 1.

By using the following lemma, we calculate the number of the curves c (see Figure 4a).

Lemma 2.5. ([3]) *Let L be given with the intersection numbers $(\alpha; \beta; \gamma; c)$. We find the number of the curves c in L with*

$$p(c) = \begin{cases} \gamma - \frac{\beta_{n+1}}{2} - \frac{\beta_1}{2} & \textit{if } c^+ = 0, \\ 0 & \textit{if } c^+ \neq 0. \end{cases} \tag{3}$$

We find the twist numbers of each twisting component of a multiple curve with the following lemma.

Lemma 2.6. ([3]) *Let L be given with the intersection numbers $(\alpha;\beta;\gamma;c)$. Let $|T|$ and m be the total twist number and the number of twisting components, each with $t+1$ twists, respectively. In this case,*

$$m \equiv |T| \ (mod\ c^+) \quad and \quad t = \frac{|T| - m}{c^+}, \tag{4}$$

where $c^+ \neq 0$.

Corollary 2.7. *The number of nontwist components of a multiple curve L, n, is given by*

$$n = \max(c^+ - |T|, 0).$$

Proof. This follows from Lemma 2.6 when $t = 0$ is taken. □

Lemma 2.8. ([1]) *In each region U_i $(1 \leq i \leq n)$, let the number of the loop components be denoted by $|b_i|$, where*

$$b_i = \frac{\beta_i - \beta_{i+1}}{2}. \tag{5}$$

If $b_i < 0$, the loop component is called left*; if $b_i > 0$, the loop component is called* right.

Lemma 2.9. ([1]) *Let $L \in \mathcal{L}_n$ be given with the intersection numbers $(\alpha;\beta;\gamma;c)$. For each $1 \leq i \leq n$, the number of above, A_i, and below, B_i, components in U_i can be found by*

$$A_i = \alpha_{2i-1} - |b_i| \quad and \quad B_i = \alpha_{2i} - |b_i|.$$

Definition 2.10. ([3]) Let $\mathcal{V}_n = \{(a;b;T;c) : c \leq 0 \text{ and } T \neq 0\} \cup \{0\}$. The Dynnikov coordinate function $\Theta : \mathcal{L}_n \to \mathbb{Z}^{2n+2} \setminus \mathcal{V}_n$ on S_n is defined by

$$\Theta(L) = (a;b;T;c) = (a_1,\cdots,a_n;b_1,\cdots,b_n;T;c),$$

where for each $1 \leq i \leq n$,

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1} - c^+}{2}, \quad b_i = \frac{\beta_i - \beta_{i+1}}{2}, \tag{6}$$

and

$$|T| = \begin{cases} 0 & \text{if } c^+ = 0, \\ \gamma - \frac{\beta_{n+1} - c^+}{2} - \frac{\beta_1 - c^+}{2} & \text{if } c^+ \neq 0. \end{cases} \tag{7}$$

The following theorem gives the inversion of the Dynnikov coordinate function on S_n.

Theorem 2.11. ([3]) *Let $(a;b;T;c) \in \mathbb{Z}^{2n+2} \setminus \mathcal{V}_n$. Then, the vector $(a;b;T;c)$ corresponds to one and only one multiple curve $L \in \mathcal{L}_n$ whose intersection numbers are given by*

$$\beta_i = 2\sum_{j=i}^{n} b_j + \max(c^+, c^+ - 2\sum_{i=1}^{n} b_i, \kappa), \quad \beta_{n+1} = \max(c^+, c^+ - 2\sum_{i=1}^{n} b_i, \kappa), \tag{8}$$

$$\alpha_i = \begin{cases} \frac{2(-1)^i a_{\lceil i/2 \rceil} + (-1)^i c^+ + \beta_{\lceil i/2 \rceil}}{2} & \textit{if } b_{\lceil i/2 \rceil} \geq 0, \\ \frac{2(-1)^i a_{\lceil i/2 \rceil} + (-1)^i c^+ + \beta_{(1+\lceil i/2 \rceil)}}{2} & \textit{if } b_{\lceil i/2 \rceil} \leq 0, \end{cases} \tag{9}$$

and

$$\gamma = \begin{cases} |T| + \sum_{j=1}^{n} b_j + \max(c^+, c^+ - 2\sum_{i=1}^{n} b_i, \kappa) - c^+ & \text{if } c > 0, \\ |c| + \sum_{j=1}^{n} b_j + \max(c^+, c^+ - 2\sum_{i=1}^{n} b_i, \kappa) - c^+ & \text{if } c \leq 0, \end{cases} \tag{10}$$

where

$$\kappa = \max_{1 \leq k \leq n} \left[2\max(b_k, 0) + |2a_k + c^+| - 2\sum_{j=k}^{n} b_j \right].$$

3. Geometric intersection of multiple curves with elementary curves on S_n

In this section, the elementary curves defined on S_n shall be explained, and obtained formulas for the geometric intersection numbers between multiple curves and these elementary curves.

Definition 3.1. Let us take the path x, which is the interval between x_1 and x_2 on S_n as seen in Figure 6. Given an essential and untwisted simple closed curve C, $||C||$ denotes the minimum number of intersections

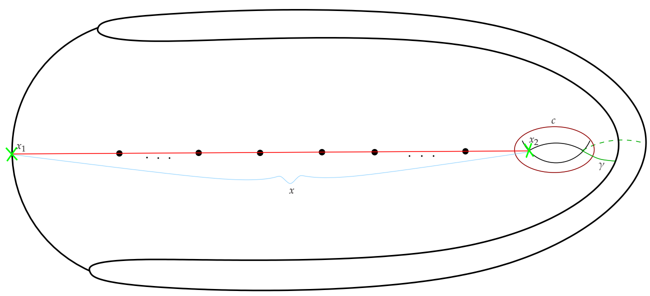

Figure 6: The path x

of C with path x. If C intersects the path x at most twice (i.e., $||C|| \leq 2$), we say C is *elementary*.

There are 6 types of elementary curves on S_n:
1. elementary curve: The curve that intersects the path x exactly twice (Figure 7);

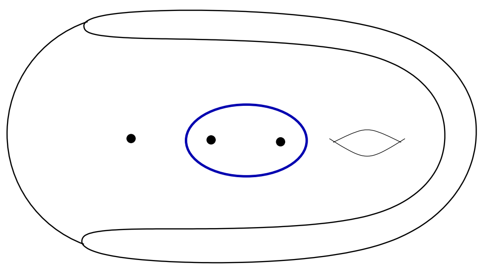

Figure 7: 1. elementary curve

2. elementary curve: The curve that never intersects the path x, intersects the curve c once, and is above all the punctures (Figure 8);
3. elementary curve: The curve that never intersects the path x, intersects the curve c once, and is below all the punctures (Figure 9);
4. elementary curve: The curve that intersects the path x once, and includes the genus (Figure 10);

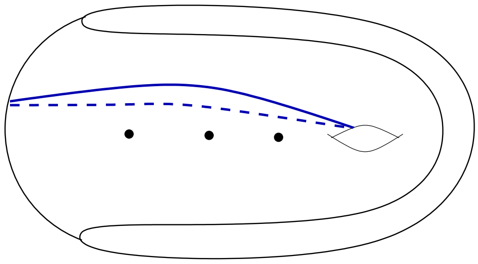

Figure 8: 2. elementary curve

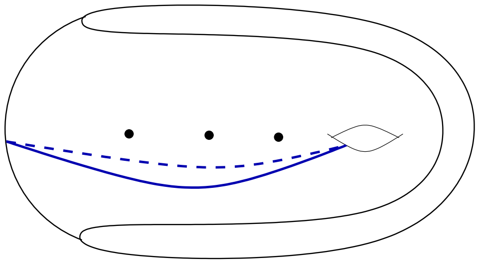

Figure 9: 3. elementary curve

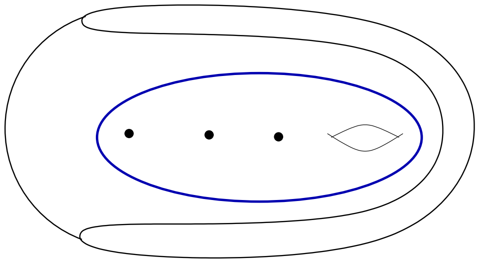

Figure 10: 4. elementary curve

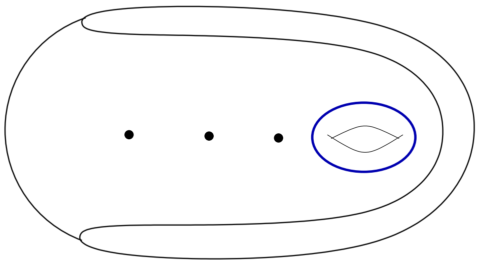

Figure 11: 5. elementary curve

5. elementary curve: The curve c itself (Figure 11);

6. elementary curve: The curve that intersects the path x once, and includes the genus on the invisible part of the surface S_n (Figure 12).

In the following lemma, we find the geometric intersection number between a given multiple curve L and the 1. elementary curve.

Lemma 3.2. *The geometric intersection number between a multiple curve L, whose generalized Dynnikov coordinates* $(a; b; T; c)$ *are given by Definition 2.10 and whose intersection numbers with the coordinate curves* $(\alpha; \beta; \gamma; c)$ *are as*

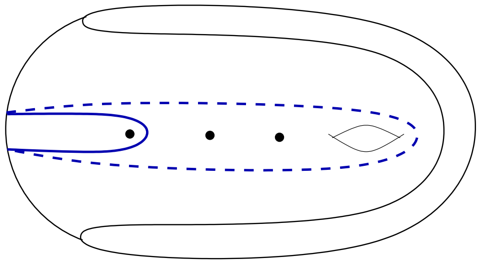

Figure 12: 6. elementary curve

in the Theorem 2.11, and the 1. elementary curve $C_{1,i,j}$ $(1 \le i, j \le n)$ *is found as follows:*

$$i(L, C_{1,i,j}) = \beta_i + \beta_{j+1} - 2(r + l + A_{i,j} + B_{i,j}), \tag{11}$$

where

$$A_i = \alpha_{2i-1} - |b_i|, \quad B_i = \alpha_{2i} - |b_i|,$$
$$A_{l,m} = \min_{l \le k \le m} A_k, \quad B_{l,m} = \min_{l \le k \le m} B_k;$$
$$r = \min(A_{i,j-1} - A_{i,j}, B_{i,j-1} - B_{i,j}, \max(b_j, 0)),$$
$$l = \min(A_{i+1,j} - A_{i,j}, B_{i+1,j} - B_{i,j}, \max(-b_i, 0)).$$

Proof. Lemma 3.2 is proven in a similar way to the works in [1, 8], and also using Theorem 2.11. □

Corollary 3.3. *Let* κ^a *(respectively* κ^b*) be the number of the parts of the front genus components between the arcs* β_1 *and* β_{n+1} *and above (respectively below) the* 2. *(respectively* 3.*) elementary curve (for example, the green curve in Figure 13 (a) (respectively Figure 13 (b))). In this case,*

$$\kappa^a = \rho - \max(\rho - A_{1,n}, 0) \quad \textit{and} \quad \kappa^b = \rho - \max(\rho - B_{1,n}, 0).$$

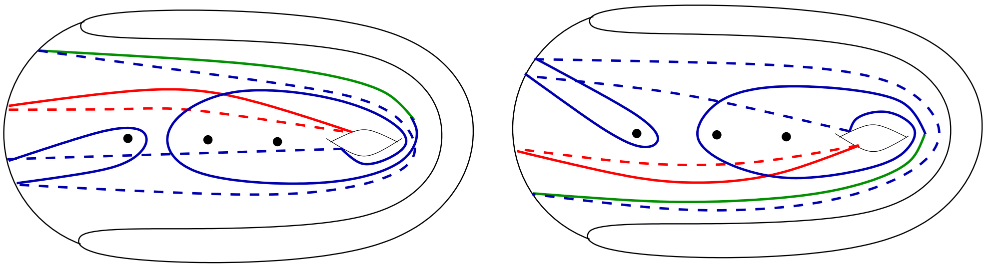

Figure 13: (a) $\kappa^a = 1$, and (b) $\kappa^b = 1$

Proof. The number of the intersections of front genus components with the 2. (respectively 3.) elementary curve is $\max(\rho - A_{1,n}, 0)$ (respectively $\max(\rho - B_{1,n}, 0)$). Thus, κ^a (respectively κ^b) is the number of the parts of front genus components that do not intersect the 2. (respectively 3.) elementary curve. □

Assumption 3.4. Throughout the work, it shall be assumed that the parts of the front genus components between β_1 and β_{n+1} and above (respectively below) the 2. (respectively 3.) elementary curve, such as the green curve (a) (respectively (b)) in Corollary 3.3, never intersect the 2. (respectively 3.) elementary curve.

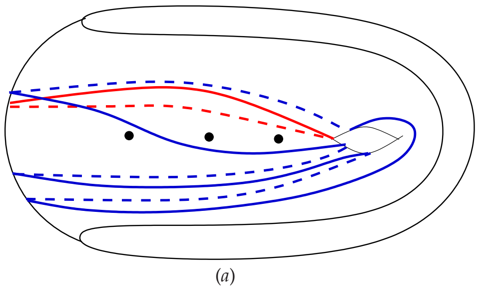

Figure 14: (a) The invisible parts of all nontwist components are below the 2. elementary curve. (b) The invisible parts of all nontwist components are above the 3. elementary curve.

Assumption 3.5. If there are twists in the multiple curve L (i.e., $|T| \neq 0$) and there are also nontwist components (Figure 5 (a)), we assume that the invisible parts of all nontwist components are below (respectively above) the 2. (respectively 3.) elementary curve (see Figure 14).

Assumption 3.6. If the twisting components in L consist of only nontwist components ($|T| = 0$ and $c^+ \neq 0$), we assume that all nontwist components are either above (Figure 15 (a)) or below (Figure 15 (b)) the 2. elementary curve; similarly, we assume that all nontwist components are either above (Figure 15 (c)) or below (Figure 15 (d)) the 3. elementary curve.

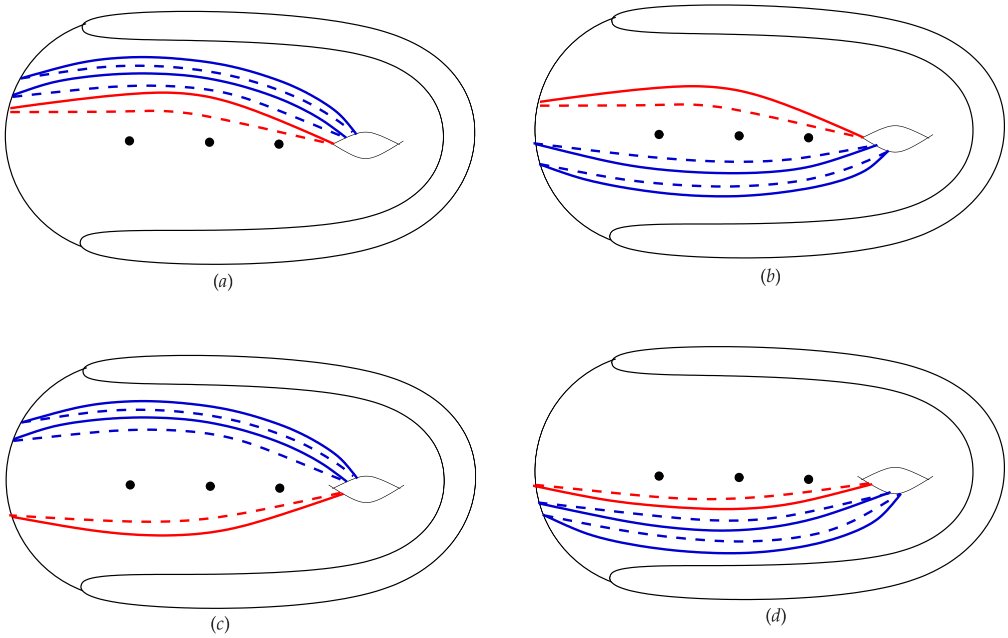

Figure 15: (a) All nontwist components are above the 2. elementary curve. (b) All nontwist components are below the 2. elementary curve. (c) All nontwist components are above the 3. elementary curve. (d) All nontwist components are below the 3. elementary curve

Thanks to the following theorem, we can find the number of geometric intersections between a multiple curve L on the surface S_n, given the Dynnikov coordinates, and the 2. elementary curve on the same surface.

Theorem 3.7. *Let L on S_n be given with Dynnikov coordinates $(a; b; T; c)$, and let $(\alpha; \beta; \gamma; c)$ be the geometric intersection numbers, corresponding to these Dynnikov coordinates, between L and the coordinate curves on S_n. The geometric intersection number between L and the* 2*. elementary curve is as follows:*

If $T > 0$,

$$\begin{aligned} i(L, C_2) &= \max(|T| - c^+, 0) + \left| \frac{\beta_1 - c^+}{2} - A_{1,n} \right| \\ &+ \max\left(\frac{\beta_{n+1} + c^+}{2} - \max(c^+ - |T|, 0) - A_{1,n}, 0 \right); \end{aligned} \tag{12}$$

If $T < 0$,

$$\begin{aligned} i(L, C_2) &= \max(|T| - c^+, 0) + \left| \frac{\beta_{n+1} - c^+}{2} - A_{1,n} \right| \\ &+ \max\left(\frac{\beta_1 + c^+}{2} - \max(c^+ - |T|, 0) - A_{1,n}, 0 \right); \end{aligned} \tag{13}$$

If $T = 0$,

i. *Whenever* $c^+ \neq 0$,

$$i(L, C_2) = \begin{cases} \frac{\beta_1 + c^+}{2} - A_{1,n} + \left| A_{1,n} - \frac{\beta_{n+1} + c^+}{2} \right| & \text{if } \alpha_{2n-1} > \alpha_{2n}, \\ \Big[\max\left(\frac{\beta_1 - \beta_{n+1}}{2} + \max\left(\frac{\beta_{n+1} - c^+}{2} - A_{1,n}, 0 \right), 0 \right) & \\ + \max\left(\frac{\beta_{n+1} - c^+}{2} - A_{1,n}, 0 \right) \Big] & \text{if } \alpha_{2n} > \alpha_{2n-1}; \end{cases} \tag{14}$$

ii. *Whenever* $c^+ = 0$,

$$i(L, C_2) = \frac{\beta_1 - c^+}{2} - A_{1,n} + \left| \frac{\beta_{n+1} - c^+}{2} - A_{1,n} \right| + \max(-c, 0). \tag{15}$$

Proof. If $T \neq 0$, since the number of intersections is minimal, the number of geometric intersections in the region G (i.e., $\max(|T| - c^+, 0)$) is obtained by subtracting the number of twisting components from the total twist number of twisting components.

Suppose we divide the surface S_n in two in any way. Let us call the left side *first region* and the right side *second region* (except for the region G).

When $T > 0$, if the curves coming from the side β_1 pass between any two punctures, the number of geometric intersections is $\max(\frac{\beta_1 - c^+}{2} - A_{1,n}, 0)$ using the formula of the number of back genus components (Lemma 2.3); if the curves from the side β_1 pass through the left side of the first puncture, the number of geometric intersections is $\max(A_{1,n} - \frac{\beta_1 - c^+}{2}, 0)$. This gives the number of geometric intersections in the first region as $\left| \frac{\beta_1 - c^+}{2} - A_{1,n} \right|$. Now, let us find the number of geometric intersections in the second region. The front genus components intersect β_{n+1} twice, and the twisting components intersect β_{n+1} once. Since the nontwist components due to Assumption 3.5, and also the minimum of the above components between the first puncture and the $(n+1)$th puncture cannot intersect 2. elementary curve on the side of β_{n+1}, these two terms are subtracted from the sum of the front genus components and twisting components, and found the number of geometric intersections on the side β_{n+1}. Thus, the number of geometric intersections in the second region is $\max\left(\frac{\beta_{n+1} + c^+}{2} - \max(c^+ - |T|, 0) - A_{1,n}, 0 \right)$.

Let us calculate the geometric intersection number in the first region when $T < 0$. In this case, the back genus components intersect β_1 twice, and the twisting components intersect β_1 once. The nontwist components and the minimum of the above components between the first puncture and the $(n+1)$th puncture cannot intersect 2. elementary curve on the side of β_1 because of the similar reasons above. Therefore, these two terms are subtracted from the sum of the back genus components and twisting components, and found the number of geometric intersections on the side β_1 (in the first region) as $\max\left(\frac{\beta_1 + c^+}{2} - \max(c^+ - |T|, 0) - A_{1,n}, 0 \right)$. The number of geometric intersections in the second region (i.e., $\left| \frac{\beta_{n+1} - c^+}{2} - A_{1,n} \right|$) is proved similarly to the geometric intersection number when $T > 0$ in the first region. Here, the passing of the curves from the right side of the n. puncture is considered.

Now, we find the number of the geometric intersections between L and the 2. elementary curve when T = 0. Let all nontwist components be above the 2. elementary curve (see Figure 15 (a)), and $c^+ \neq 0$ (i.e., when $\alpha_{2n-1} > \alpha_{2n}$). In this case, the back genus and twisting components are located above the 2. elementary curve. The back genus components intersect α_1 once, and the twisting components intersect α_1 once. Since the minimum of the above components between the first puncture and the $(n+1)$th puncture never intersects the 2. elementary curve, the number of geometric intersections in the first region is $\frac{\beta_1 - c^+}{2} + c^+ - A_{1,n}$. Let us calculate the geometric intersection number in the second region. If *both* branches of *some* front genus components intersect α_{2i-1}, the geometric intersection number is $\max(A_{1,n} - (\frac{\beta_{n+1}-c^+}{2} + c^+), 0)$; if *only one* branch of *all* front genus components intersects α_{2i-1}, the geometric intersection number is $\max((\frac{\beta_{n+1}-c^+}{2} + c^+) - A_{1,n}, 0)$. Hence, the geometric intersection number in the second region is $\left|A_{1,n} - (\frac{\beta_{n+1}-c^+}{2} + c^+)\right|$. Now, let all nontwist components be below the 2. elementary curve (see Figure 15 (b)), and $c^+ \neq 0$ (i.e., when $\alpha_{2n} > \alpha_{2n-1}$). We find the geometric intersection number in the first region. Only the back genus components intersect α_1. From Corollary 3.3 and Assumption 3.4, the parts of the front genus components between the arcs β_1 and β_{n+1} and above the 2. elementary curve do not intersect the 2. elementary curve. Accordingly, the number of geometric intersections with the 2. elementary curve in the first region is calculated as $\max\left(\frac{\beta_1-\beta_{n+1}}{2} + \max\left(\frac{\beta_{n+1}-c^+}{2} - A_{1,n}, 0\right), 0\right)$. Let us examine the number of geometric intersections in the second region. When all nontwist components are below the 2. elementary curve (Figure 15 (b)), the second region of the 2. elementary curve intersects only the front genus components. Solely, the parts of the front genus components between the arcs β_1 and β_{n+1} and above the 2. elementary curve cannot intersect the 2. elementary curve. Thus, the geometric intersection number in the second region is $\max\left(\frac{\beta_{n+1}-c^+}{2} - A_{1,n}, 0\right)$.

Let us examine the case that there is not any twisting component for the 2. elementary curve (i.e., when $c^+ = 0$). In this case, in the first region, α_1 intersects only the back genus component. The minimum of the above components between the first puncture and the $(n+1)$th puncture does not intersect the 2. elementary curve. Therefore, the number of geometric intersection with the 2. elementary curve in the first region is $\frac{\beta_1-c^+}{2} - A_{1,n}$. Let us calculate the geometric intersection number in the second region. If *only one* branch of *all* front genus components intersects α_{2i-1}, the geometric intersection number in the second region becomes $\max\left(\frac{\beta_{n+1}-c^+}{2} - A_{1,n}, 0\right)$. If *both* branches of *some* front genus components intersect α_{2i-1}, then the geometric intersection number in the second region becomes $\max\left(A_{1,n} - \frac{\beta_{n+1}-c^+}{2}, 0\right)$. Hence, the geometric intersection number between L and 2. elementary curve in the second region is $\left|\frac{\beta_{n+1}-c^+}{2} - A_{1,n}\right|$. Since there is no twisting component in the region G, curves c intersect the 2. elementary curve by the number of $p(c)$. Thus, the total number of geometric intersections is equal to the sum of the number of geometric intersections in the first region, the number of geometric intersections in the second region, and the number of curves c. That is, when $T = 0$ and $c^+ = 0$, $i(L, C_2) = \frac{\beta_1-c^+}{2} - A_{1,n} + \left|\frac{\beta_{n+1}-c^+}{2} - A_{1,n}\right| + \max(-c, 0)$. □

The number of geometric intersections between the multiple curve L, given Dynnikov coordinates in Definition 2.10, and the 3. elementary curve is obtained by the following theorem.

Theorem 3.8. *Let L on the surface S_n be given with Dynnikov coordinates $(a; b; T; c)$, and let $(\alpha; \beta; \gamma; c)$ be the geometric intersection numbers, corresponding to these Dynnikov coordinates, between L and the coordinate curves on S_n. The geometric intersection number between L and* 3. *elementary curve is as follows:*

If $T > 0$,

$$\begin{aligned} i(L, C_3) &= \max(|T| - c^+, 0) + \left|\frac{\beta_{n+1} - c^+}{2} - B_{1,n}\right| \\ &+ \max\left(\frac{\beta_1 + c^+}{2} - \max(c^+ - |T|, 0) - B_{1,n}, 0\right); \end{aligned} \tag{16}$$

If $T < 0$,

$$\begin{aligned} i(L, C_3) &= \max(|T| - c^+, 0) + \left|\frac{\beta_1 - c^+}{2} - B_{1,n}\right| \\ &+ \max\left(\frac{\beta_{n+1} + c^+}{2} - \max(c^+ - |T|, 0) - B_{1,n}, 0\right); \end{aligned} \tag{17}$$

If $T = 0$,

i. *Whenever* $c^+ \neq 0$,

$$i(L, C_3) = \begin{cases} \left[\max\left(\frac{\beta_1 - \beta_{n+1}}{2} + \max\left(\frac{\beta_{n+1} - c^+}{2} - B_{1,n}, 0\right), 0\right) \right. & \\ \left. + \max\left(\frac{\beta_{n+1} - c^+}{2} - B_{1,n}, 0\right)\right] & \text{if } \alpha_{2n-1} > \alpha_{2n}, \\ \frac{\beta_{n+1} + c^+}{2} - B_{1,n} + \left|B_{1,n} - \frac{\beta_1 + c^+}{2}\right| & \text{if } \alpha_{2n} > \alpha_{2n-1}; \end{cases} \tag{18}$$

ii. *Whenever* $c^+ = 0$,

$$i(L, C_3) = \frac{\beta_1 - c^+}{2} - B_{1,n} + \left|\frac{\beta_{n+1} - c^+}{2} - B_{1,n}\right| + \max(-c, 0). \tag{19}$$

Proof. The geometric intersection number in only region G is found as in the case of the 2. elementary curve.

Let $T > 0$. The first region of the 3. elementary curve intersects the back genus component once, the twisting components (except the nontwist components) once. However, the nontwist components and the minimum of the below components cannot intersect this region. Therefore, the geometric intersection number in the first region is obtained by subtracting the sum of the nontwist and the minimum of the below components from the sum of the twisting and the back genus components. Thus, we have $\max\left(\frac{\beta_1 + c^+}{2} - \max(c^+ - |T|, 0) - B_{1,n}, 0\right)$ in the first region. Now, let us find the geometric intersection number in the second region. If $\rho < B_{1,n}$, then it is $\max(B_{1,n} - \rho, 0)$. If $B_{1,n} < \rho$, then $\max(\rho - B_{1,n}, 0)$. Hence, in the second region, the number of geometric intersection with the 3. elementary curve is $\left|\frac{\beta_{n+1} - c^+}{2} - B_{1,n}\right|$.

Let $T < 0$. The geometric intersection number in the first region is proved in a similar manner to the second region of the case ($T > 0$). The geometric intersection number in the second region is proved in a similar manner to the first region of the case ($T > 0$).

Let $T = 0$ and $c^+ \neq 0$. If all nontwist components are above the 3. elementary curve, the geometric intersection number in the first region of the 3. elementary curve is proved similarly to the second region of the 2. elementary curve (when all nontwist components are below the 2. elementary curve). The geometric intersection number in the second region of the 3. elementary curve is proved similarly to the first region of the 2. elementary curve (when all nontwist components are below the 2. elementary curve). If all nontwist components are below the 3. elementary curve, the geometric intersection number in the first region of the 3. elementary curve is proved similarly to the second region of the 2. elementary curve (when all nontwist components are above the 2. elementary curve). The geometric intersection number in the second region of the 3. elementary curve is proved similarly to the first region of the 2. elementary curve (when all nontwist components are above the 2. elementary curve).

When there is no twisting component (i.e., when $c^+ = 0$), the total number of geometric intersections of the multiple curve L with the 3. elementary curve is proved similarly to the same case of the 2. elementary curve. □

In the following corollary, we calculate the geometric intersection numbers between the multiple curve L whose Dynnikov coordinates are given in Definition 2.10 and the 4. elementary curve.

Corollary 3.9. *The geometric intersection numbers between L and the* 4. *elementary curve are given by*

$$i(L, C_{4,i}) = \beta_i - 2r, \tag{20}$$

where $r = \min(A_{i,n}, B_{i,n}, \rho)$. ρ *is the number of the front genus components.*

Proof. In order to find the number of geometric intersections between L and the 4. elementary curve, Lemma 3.2 is used. Here, genus is considered the final puncture (i.e., as $(n+1)$th puncture). □

Corollary 3.10. *The number of geometric intersections between L and the* 5. *elementary curve is* c^+. *That is,* $i(L, C_5) = c^+$. *To tell explicitly, this number is the total number of twisting components in L.*

Proof. The proof of Corollary 3.10 is obvious. All twisting components have to intersect the 5. elementary curve. But, other path components do not intersect it. □

Assumption 3.11. Let the red dashed curve in Figure 16 show the invisible part of the 6. elementary curve. We shall assume that there are no back genus components within the red dashed curve (that is, we shall not accept the behavior of L (blue closed curve) as seen in Figure 16) and that all back genus components are outside the red dashed curve.

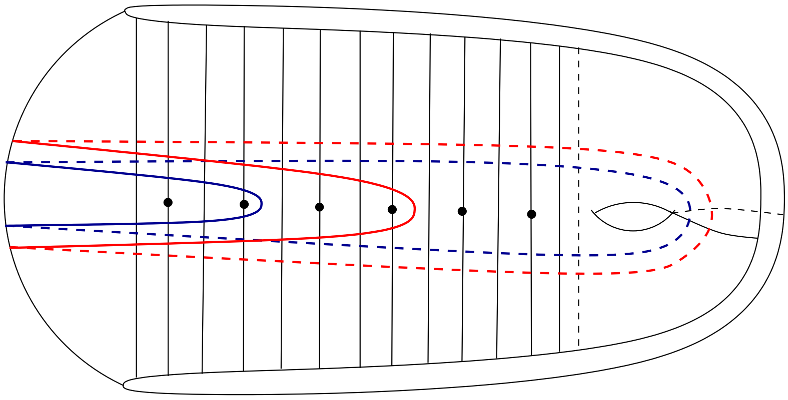

Figure 16: There should not be the back genus components within the invisible part of the 6. elementary curve (i.e., the red dashed curve).

The following theorem gives the number of geometric intersections between L and the 6. elementary curve.

Theorem 3.12. *Let L be a multiple curve on* S_n, *given by Dynnikov coordinates in Definition 2.10, and let* $1 \leq j, k \leq n$. *The geometric intersection number between L and the* 6. *elementary curve is found as follows:*

$$\begin{aligned} i(L, C_{6,j}) &= \max(c^+ - 2h, 0) + \max(\frac{\beta_1 - c^+}{2} - A_{1,j}, 0) + \max(\frac{\beta_1 - c^+}{2} - B_{1,j}, 0) \\ &+ \max(\alpha_{2j-1} - \max(-b_j, 0) - A_{1,j} - \max(b_j, 0), 0) \\ &+ \max(\alpha_{2j} - \max(-b_j, 0) - B_{1,j} - \max(b_j, 0), 0) + 2\max(-b_j, 0) \\ &- \max(c^+ - \max(A_{1,j} - \frac{\beta_1 - c^+}{2}, 0) - \max(B_{1,j} - \frac{\beta_1 - c^+}{2}, 0) - 2h, 0), \end{aligned} \tag{21}$$

where

$$h = \sum_{k=1}^{j} \left\lfloor \max(\max(\frac{c^+ - \beta'_{k+1}}{2}, 0) - \max(\frac{c^+ - \beta'_k}{2}, 0), 0) \right\rfloor.$$

Here, $\lfloor x \rfloor$ *is the greatest integer smaller than or equal to x. Moreover,* $\beta'_1 = \beta_1$ *and*

$$\beta'_{k+1} = \max(\alpha_{2k-1} - \max(b_k, 0) - \frac{\beta_1 - c^+}{2}, 0) + \max(\alpha_{2k} - \max(b_k, 0) - \frac{\beta_1 - c^+}{2}, 0).$$

Proof. Let us find the geometric intersection number between L and the 6. elementary curve in the first region. Since the twisting component intersects the arc β_1 once, and the back genus component intersects β_1 twice; when the parts that do not intersect the visible part of the 6. elementary curve (the closed solid red curve in Figure 16) are removed from these path components, the parts that intersect this curve remain. Hence, the number of geometric intersections in the first region is derived as $\max(c^+ - 2h, 0) + \max(\frac{\beta_1 - c^+}{2} - A_{1,j}, 0) + \max(\frac{\beta_1 - c^+}{2} - B_{1,j}, 0)$. Here, h is an auxiliary component used for the formula to work correctly.

Now, we shall calculate the number of geometric intersections in the second region. The expression $\max(\alpha_{2j-1} - \max(-b_j, 0) - A_{1,j} - \max(b_j, 0), 0) + \max(\alpha_{2j} - \max(-b_j, 0) - B_{1,j} - \max(b_j, 0), 0) + 2\max(-b_j, 0)$ has been created in such a way that the path components passing on the arcs α_{2j-1} and α_{2j} have the number of geometric intersections on the visible part of the 6. elementary curve (the closed solid red curve in Figure 16) by coming from the right side of the closed solid red curve. The expression $\max(c^+ - \max(A_{1,j} - \frac{\beta_1 - c^+}{2}, 0) - \max(B_{1,j} - \frac{\beta_1 - c^+}{2}, 0) - 2h, 0)$ has been subtracted from this expression to prevent the extra intersections that twisting components may cause in the second region. The reason why the latest $A_{1,j}$ and $B_{1,j}$ have been subtracted is that they had already been subtracted beforehand. That's why we have subtracted them at the end to ignore them. □

Remark 3.13. Notice that the formulas given in this paper can be written by using Dynnikov coordinates on S_n since one can write each $\alpha_i, \beta_i, \gamma$ and c in terms of a_i, b_i, T and c by Theorem 2.11.

Example 3.14. Let us take a multiple curve $L \in \mathcal{L}_3$ on S_3 whose Dynnikov coordinates are given by

$$\Theta(L) = (a_1, a_2, a_3; b_1, b_2, b_3; T; c) = (1, 0, -1; -1, 0, 1; 2; 3).$$

We calculate the number of geometric intersections between L and the 2. elementary curve C_2.

Using Theorem 2.11 and Equation (12), we find the geometric intersection number between L and C_2 as $i(L, C_2) = 4$, and Figure 17 depicts this case.

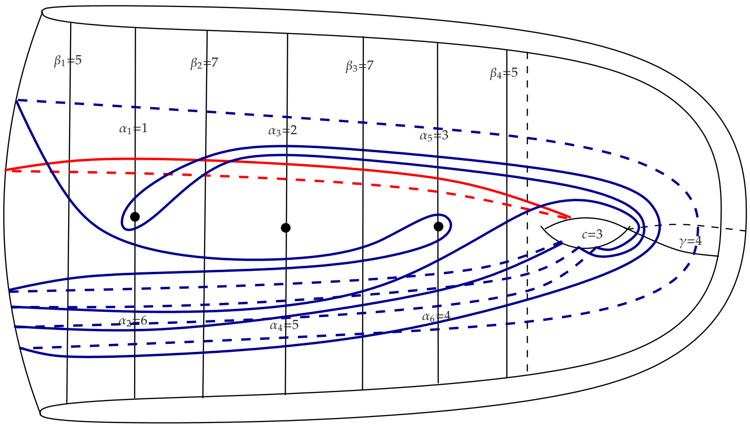

Figure 17: $i(L, C_2) = 4$

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