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Generalized m-Quasi-Einstein Metric on Certain Almost Contact Manifolds

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Abstract. In this paper, we study the generalized m-quasi-Einstein metric in the context of contact geometry. First, we prove if an H-contact manifold admits a generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ, then M is K-contact and η-Einstein. Moreover, it is also true when H-contactness is replaced by completeness under certain conditions. Next, we prove that if a complete K-contact manifold admits a closed generalized m-quasi-Einstein metric whose potential vector field is contact then M is compact, Einstein and Sasakian. Finally, we obtain some results on a 3-dimensional normal almost contact manifold admitting generalized m-quasi-Einstein metric.

1. Introduction

The study of Einstein manifolds and their several generalizations have received a lot of attention in recent decades. One such generalization is the so-called Ricci solitons, which play a crucial role in Ricci flow. A Riemannian manifold (M^n, g) together with vector field V is called Ricci soliton if it satisfies:

$$\mathcal{L}_V g + 2S = 2\lambda g, \tag{1}$$

where $\mathcal{L}_V$ denotes the Lie-derivative operator along a vector field V, S is the Ricci tensor of g and λ a constant. Clearly, for Killing vector V, the soliton equation becomes Einstein i.e., $S = \lambda g$. When $V = Df$ i.e., a gradient of smooth function f on M, it is called a gradient Ricci soliton. For a detailed survey, we recommend Cao [6].

A generalized notion of Einstein metric and gradient Ricci soliton, called m-quasi-Einstein metric has become an attractive topic in modern Riemannian geometry. It is mainly since an n-dimensional m-quasi-Einstein manifold is exactly those manifolds that are the base of an $(n + m)$-dimensional Einstein warped product (see [17]). A Riemannian manifold (M^n, g) together with constant λ is said to be m-quasi-Einstein if it satisfies:

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{2}$$

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where $0 < m \leq \infty$ and $\nabla^2 f$ denotes the Hessian form of the smooth function f on M. Here, $S+\nabla^2 f-\frac{1}{m}df\otimes df = S^m_f$ is known as m-Bakery-Emery Ricci tensor which arises from the warped product $(M\times N, \bar{g})$ of two Riemannian manifolds (M^n, g) and (N^m, h) with the Riemannian metric $\bar{g} = g + e^{-\frac{2f}{m}}h$. Moreover, when $m = \infty$, (2) reduces to the gradient Ricci soliton, and when f is constant, it is Einstein. The m-quasi-Einstein metric has been analyzed deeply by Case [7] and Case et al. [8].

Independently, by considering 1-form V^b instead of df, Barros-Ribeiro Jr [1] and Limoncu [18] generalized m-quasi-Einstein metric as follows:

$$S + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m}V^b \otimes V^b = \lambda g. \tag{3}$$

Here, V^b is the 1-form associated with the potential vector field V. Notice that if a 1-form V^b is closed i.e., $dV^b = 0$ then (3) reduces to (2) with $V^b = \nabla f$, where f is a smooth function on M. m-quasi-Einstein manifold with closed 1-form V^b is called closed m-quasi-Einstein manifold. Ghosh [12] studied an m-quasi-Einstein structure on K-contact, contact metric manifolds, and H-contact manifolds and gave several examples. Later, Chen [11] studied it in almost cosympletic manifolds.

Extending the notion of m-quasi-Einstein, Catino [10] introduced and studied the concept of a generalized m-quasi-Einstein manifold. A particular case of this was proposed by Barros-Ribeiro Jr [2] which is defined as follows:

A Riemannian manifold (M^n, g) is said to be generalized m-quasi-Einstein if there exists a function $\lambda : M^n \to \mathbb{R}$ such that

$$S + \nabla^2 f - \frac{1}{m}df \otimes df = \lambda g. \tag{4}$$

If $m = \infty$ then (4) reduces to Ricci almost soliton. Using the terminology of Ricci soliton we say that generalized m-quasi-Einstein metric is said to be expanding, shrinking or steady accordingly as $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$ respectively. Hu et al. [14, 15] studied generalized m-quasi-Einstein manifolds with constant Ricci curvatures and constant scalar curvature. Recently, Ghosh [13] studied generalized m-quasi-Einstein metric in Sasakian and K-contact manifolds and proved, "*Let $(M^{2n+1}, g, m, \lambda)$ be a generalized quasi-Einstein manifold. If g represents a K-contact metric and $m \neq 1$, then it is compact, Einstein, Sasakian, and isometric to the unit sphere S^{2n+1}.*" In continuation, we studied the generalized m-quasi-Einstein metric with 1-form V^b. Ghosh [12] on H-contact manifold proved, "*Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an H-contact manifold. If g represents an m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ, then M is K-contact and η-Einstein.*" Generalizing this we prove the following result.

Theorem 1.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an H-contact manifold. If g represents a generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ, then M is K-contact and η-Einstein. Moreover, λ is constant.*

In [3], Boyer and Galicki studied Einstein K-contact and η-Einstein K-contact manifolds. In particular, they proved that a *compact Einstein K-contact is Sasakian*. This is also true for compact η-Einstein ($S = \alpha g + \beta\eta\otimes\eta$ for constant α, β) K-contact with $\alpha > -2$. These results are also valid if one relaxes compactness by completeness (see [24]). Because of the above theorem and the Boyer-Galicki result, we can state the following:

Corollary 1.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a complete H-contact manifold. If g admits shrinking generalized m-quasi-Einstein metric with non-zero potential vector field V collinear with ξ then M is compact Sasakian and η-Einstein.*

Consider a special case when $\lambda = \lambda' + \rho r$ $(\lambda', \rho \in \mathbb{R})$ in (4), then it is said to be (m, ρ)-quasi-Einstein manifold. In particular, if $m = \infty$, then it is exactly the ρ-Einstein soliton [9]. For details on (m, ρ)-quasi-Einstein see [16, 23, 25] and references therein.

Replacing H-contactness by a compact contact metric manifold and generalizing ([23], Theorem 3) we prove the following result.

Theorem 1.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a complete contact metric manifold. If g admits a generalized m-quasi-Einstein metric with non-zero potential vector field collinear with ξ and $\|\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda\|_g \in L^1(M, g)$ then M is K-contact and η-Einstein.*

Using the similar argument as in Corollary 1.2, we can state the following:

Corollary 1.4. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a complete contact metric manifold. If g admits shrinking generalized m-quasi-Einstein metric with non-zero potential vector field collinear with ξ and $\|\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda\|_g \in L^1(M, g)$ then M is compact Sasakian and η-Einstein.*

Firstly we review an important definition: A vector field V on a contact metric manifold M is said to be contact if there exists a smooth function $\varrho : M \to \mathbb{R}$ satisfying

$$(\mathcal{L}_V \eta)(Y) = \varrho\eta(Y), \tag{5}$$

for all $Y \in \chi(M)$ and if $\varrho = 0$, then the vector field V is called strict. In [21], the author proved that *if a K-contact metric g represents a Ricci almost soliton with the potential vector field V is contact and the Ricci operator Q commutes with the constant structure ϕ, then V is killing and g is Einstein with constant scalar curvature* $2n(2n+1)$. Based on the above result, a natural question can be posed:

Does a generalized m-quasi-Einstein metric with contact potential vector field on K-contact manifold is Einstein?

We answer the above question by proving the following result.

Theorem 1.5. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a complete K-contact manifold. If g admits a closed generalized m-quasi-Einstein metric whose potential vector field is contact then M is compact, Einstein and Sasakian. Moreover, V is strict and λ is constant.*

Finally, we studied the generalized m-quasi-Einstein metric in the framework of 3-dimensional normal almost contact metric manifold and prove the following result.

Theorem 1.6. *If a 3-dimensional normal almost contact metric manifold with β =constant admits a generalized m-quasi-Einstein metric whose non-zero potential vector field is collinear with ξ then M^3 is either η-Einstein, β-Kenmotsu or locally the product of a Kähler manifold and an interval or unit circle S^1.*

2. Preliminaries

In this section, we review some of the results and the definitions and properties of certain contact structures (see [4]).

A $(2n+1)$-dimensional smooth manifold M^{2n+1} is said to be a contact metric manifold if there exists a global 1-form η, known as the contact form, such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, a unit vector field ξ, called the Reeb vector field, corresponding to 1-form η such that $d\eta(\xi, \cdot) = 0$, a $(1,1)$ tensor field ϕ and Riemannian metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \ \ \eta(X) = g(X, \xi), \ \ d\eta(X, Y) = g(X, \phi Y), \tag{6}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie-algebra of all vector fields on M. The metric g is called the associate metric and the structure (ϕ, ξ, η, g) is called the contact metric structure. A Riemannian manifold M^{2n+1} together with contact structure (ϕ, ξ, η, g) is called a contact metric manifold. It follows from (6) that

$$\begin{aligned} &\phi(\xi) = 0, \quad \eta \cdot \phi = 0, \ \ g(X, \phi Y) = -g(\phi X, Y), \\ &\qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{7}$$

for any $X, Y \in \chi(M)$. Further, we define two self-adjoint operators h and l by $h = \frac{1}{2}(\mathcal{L}_\xi \phi)$ and $l = R(\cdot, \xi)\xi$ respectively, where R is the Riemannian curvature of M. These operators satisfy

$$h\xi = l\xi = 0, \ \ h\phi + \phi h = 0, \ \ Tr.h = Tr.h\phi = 0. \tag{8}$$

Here, "Tr." denotes trace. The following formulas hold on a contact metric manifold [4]

$$\nabla_X \xi = -\phi X - \phi h X, \tag{9}$$

$$Tr.l = S(\xi, \xi) = 2n - \|h\|^2. \tag{10}$$

When unit vector ξ is Killing (i.e. $h = 0$ or $Tr.l = 2n$) then contact metric manifold is called K-contact. On K-contact manifold the following formulas hold [4]

$$\nabla_X \xi = -\phi X, \tag{11}$$

$$R(X, \xi)\xi = X - \eta(X)\xi, \tag{12}$$

$$Q\xi = 2n\xi, \tag{13}$$

where Q is the Ricci operator associated with the Ricci tensor S and ∇ is the operator of covariant differentiation of g. A contact structure is said to be normal if the almost complex structure J on $M \times \mathbb{R}$ is defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where t is the coordinate of $\mathbb{R}$ and f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called Sasakian. A Sasakian manifold is K-contact but the converse is true only in dimension 3. Olszak [20] showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla\xi \cdot \phi = \phi \cdot \nabla\xi$, or, equivalently,

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{14}$$

where $2\alpha = div\xi$ and $2\beta = Tr.(\phi\nabla\xi)$, $div\xi$ is the divergence of ξ defined by $div\xi = Tr.\{X \to \nabla_X \xi\}$ and $Tr.(\phi\nabla\xi) = Tr.\{X \to \phi\nabla_X \xi\}$. On 3-dimensional normal almost contact metric manifold the following relations hold [20]

$$S(Y, \xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2\}\eta(Y), \tag{15}$$

$$\xi\alpha + 2\alpha\beta = 0. \tag{16}$$

A vector field V is said to be harmonic vector field if it is a critical point of the energy functional E defined by

$$E(V) = \frac{1}{2}\int \|dV\|^2 dM = \frac{n}{2} vol(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dM$$

on the space χ^1 of all unit vector fields on M. A contact metric manifold whose Reeb vector field is harmonic is called an H-contact manifold. In [22], Perrone proved that *a contact metric manifold is an H-contact manifold, that is ξ is a harmonic vector field, if and only if ξ is an eigenvector of the Ricci operator*. This implies $Q\xi = (Tr.l)\xi$. This is valid for K-contact manifolds, (k, μ)-contact manifolds and unit sphere S^{2n+1} with standard contact metric structure.

In the next section, we give some examples of generalized m-quasi-Einstein metrics.

3. Examples

Example 3.1. On a standard unit sphere $(\mathbb{S}^n, g_0), n \geq 2$, considering the function $f = -m\ In(\tau - \frac{h_v}{n})$, where τ is a real parameter lying in $(1/n, +\infty)$ and h_v is some height function. Then considering $\lambda = (n-1) - m\frac{\tau - u}{u}$, we find that $(\mathbb{S}^n, g_0)$ admits generalized m-quasi-Einstein metric. For details, see [2].

Example 3.2. On the Euclidean space $(\mathbb{R}^n, g_0), n \geq 2$ together with function $f = -m\ In(\tau + |x|^2)$, where τ is a positive real perimeter and $|x|$ is the Euclidean norm of x, we see that $u = e^{-\frac{f}{m}} = \tau + |x|^2$ and considering $\lambda = -2\frac{m}{u}$, it admits generalized m-quasi-Einstein structure (see [2]).

Next, we will construct an example in a warped product manifold. Let us consider $M = \mathbb{R} \times_\sigma N^{n-1}$ with the product metric $g = dt^2 + \sigma^2(t)g_0$, where g_0 is a fixed metric in N^{n-1} and σ is a positive function on $\mathbb{R}$.

Example 3.3. For a positive $m \in \mathbb{R}$, let us assume,

$$f(x,t) = f(t) = m(t - e^t), \qquad \sigma(t) = e^{-t}$$

Inserting the value of σ in Eq. 2-3, 2.4 (see [26]) together with the assumption that N^{n-1} is a Ricci flat manifold we get

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g,$$

where $\lambda = e^t(e^t + 2 - m) - n$. Hence M admits generalized m-quasi-Einstein metric.

Example 3.4. Consider a Hyperbolic space $\mathbb{H}^n(-1) \subset \mathbb{R}^{n+1} : \langle x, x\rangle_0 = -1,$. Now, consider a height function $h_v : \mathbb{H}^n(-1) \to \mathbb{R}$ given by $h_v(X) = \langle x, v\rangle_0$ for a fixed point $v \in \mathbb{H}^n(-1)$. Let us assume $u = e^{-\frac{f}{m}} = \tau + h_v,\ \ \tau > -1$, then $\mathbb{H}^n(-1)$ admits generalized m-quasi-Einstein metric for $\lambda = -(n-1) - m\frac{\tau - u}{u}$. For details, see [2].

4. Proof of main results

Proof of Theorem 1.1: A potential vector field V collinear with Reeb vector field ξ implies $V = \sigma\xi$, for some smooth function σ on M. Differentiating this along any $X \in \chi(M)$ we get

$$\nabla_X V = X(\sigma)\xi - \sigma(\phi X + \phi h X). \tag{17}$$

In consequence of (17), Eq. (4) reduces to the following

$$\begin{aligned} &X(\sigma)\eta(Y) + Y(\sigma)\eta(X) - 2\sigma g(\phi h X, Y) \\ &+2S(X,Y) - \frac{2}{m}\sigma^2\eta(X)\eta(Y) = 2\lambda g(X,Y), \end{aligned} \tag{18}$$

for any $X, Y \in \chi(M)$. Replacing X and Y by ξ in (18) and using (10) yields

$$\xi\sigma + Tr.l - \frac{\sigma^2}{m} = \lambda. \tag{19}$$

Putting $Y = \xi$ in (18) and using (19) we obtain

$$Q\xi - (Tr.l)\xi = -\frac{1}{2}\{D\sigma - (\xi\sigma)\xi\}. \tag{20}$$

Moreover, contracting (18) we obtain the following result

$$\xi\sigma + r - \frac{\sigma^2}{m} = (2n+1)\lambda. \tag{21}$$

By hypothesis, H-contactness implies ξ is an eigenvector of the Ricci operator at each point of M i.e. $Q\xi = (Tr.l)\xi$. Making use of this in (20), we get $D\sigma = (\xi\sigma)\xi$. By Lemma 1 in [21], σ is constant on M. Then (18) reduces to

$$QX = -\sigma h\phi X + \frac{\sigma^2}{m}\eta(X)\xi + \lambda X, \tag{22}$$

for any $X \in \chi(M)$. Differentiating (22) along arbitrary $Y \in \chi(M)$ and using (9) we obtain

$$\begin{aligned} (\nabla_Y Q)X &= -\sigma(\nabla_Y h\phi)X - \frac{\sigma^2}{m}[g(X, \phi Y + \phi h Y)\xi \\ &+ \eta(X)(\phi X + \phi h Y)] + (Y\lambda)X. \end{aligned} \tag{23}$$

Contracting (23) over Y and making use of (8) gives

$$\frac{1}{2}Xr = -\sigma(divh\phi)X + (X\lambda). \tag{24}$$

Recalling that for any contact metric manifold $div(\phi h)X = 2n\eta(X) - g(Q\xi, X)$. By hypothesis, since $Q\xi = Tr.l\xi$, we get $div(\phi h)X = (2n - Tr.l)\eta(X)$. Applying this in the forgoing eq. (24) infers

$$\frac{1}{2}Xr = \sigma(2n - Tr.l)\eta(X) + (X\lambda). \tag{25}$$

Also differentiating (21) along $X \in \chi(M)$ gives $Xr = (2n+1)(X\lambda)$. Using this in (25) and replacing X by ϕX gives $g(\phi X, D\lambda) = 0$, which implies $D\lambda = (\xi\lambda)\xi$. Then by Lemma 1 in [21], we have λ is constant and hence $Xr = 0$ i.e. r is constant on M. In consequence of this (25) reduces to $\sigma(2n - Tr.l) = 0$. Thus either $\sigma = 0$ or $Tr.l = 2n$. Since V is non-zero implies $\sigma \neq 0$. Hence, $Tr.l = 2n$ which implies the manifold is K-contact. From (22) we see that m is η-Einstein i.e. $QX = \lambda X + \frac{\sigma^2}{m}\eta(X)\xi$, where $\frac{\sigma^2}{m} = \lambda - 2n$. This completes the proof. □

***Proof of Theorem 1.3*:** By our assumption $V = \sigma\xi$ and hence Eq. (17)-(21) are valid. Making use of (17) generalized m-quasi-Einstein equation becomes

$$QX + \frac{1}{2}[g(X, D\sigma)\xi + \eta(X)D\sigma] + \sigma h\phi X = \lambda X + \frac{\sigma^2}{m}\eta(X)\xi. \tag{26}$$

Differentiate (26) along arbitrary $Y \in \chi(M)$ then contracting the obtain result along Y and taking $X = \xi$ together with $div(\phi h)\xi = ||h||^2$ we get

$$\frac{1}{2}\{\xi r + \xi(\xi\sigma) + divD\sigma\} - \sigma||h||^2 = \frac{2}{m}\sigma(\xi\sigma) + \xi\lambda. \tag{27}$$

Differentiating (21) along ξ yields

$$\xi r = (2n+1)(\xi\lambda) + \frac{2\sigma}{m}(\xi\sigma) - \xi(\xi\sigma). \tag{28}$$

Using convention $divD\sigma = -\Delta\sigma$ and combining (27) and (28) we obtain

$$\frac{1}{2}\Delta\sigma + \sigma||h||^2 + \frac{\sigma}{m}(\xi\sigma) = \frac{1}{2}(2n-1)(\xi\lambda). \tag{29}$$

In contact metric manifold $div\xi = 0$ and hence $g(D\sigma, \xi) = \xi\sigma = divV$. Now contracting the well-known formula $\nabla_X(\sigma^2 V) = X(\sigma^2)V + \sigma^2(\nabla_X V)$ over X gives

$$div(\sigma^2 V) = g(\nabla\sigma^2, V) + \sigma^2 divV = 3\sigma^2\xi(\sigma). \tag{30}$$

Multiplying (29) by σ and using (30) and $(\Delta\sigma)\sigma = \frac{1}{2}\Delta(\sigma^2) + ||\Delta\sigma||^2$ we obtain the following relation

$$div(\nabla(\sigma^2) - \frac{4}{3m}\sigma^2 V + 2(2n-1)\sigma\xi\lambda) = 4\sigma^2||h||^2 + 2||\nabla\sigma||^2, \tag{31}$$

Here we have used the fact that $div(\xi\lambda) = \lambda div\xi + \xi(\lambda)$. Applying Proposition 1 in [5], the foregoing equation (31) infers

$$2\sigma^2||h||^2 + ||\nabla\sigma||^2 = 0. \tag{32}$$

This implies $\nabla\sigma = 0$ and $h = 0$, hence M is K-contact and σ is constant. Moreover, from (26) it is η-Einstein. This completes the proof. □

Proof of Theorem 1.5: Taking the exterior derivative of (5) and by properties of Lie-derivative we obtain

$$\begin{aligned}(\mathcal{L}_V d\eta)(X,Y) &= d(\mathcal{L}_V \eta)(X,Y)\\ &= \frac{1}{2}[X(\varrho)\eta(Y) - Y(\varrho)\eta(X)] + \varrho d\eta(X,Y),\end{aligned} \tag{33}$$

for any $X, Y \in \chi(M)$. Taking the Lie-derivative of $d\eta(X,Y) = g(X,\phi Y)$ along V and using (33) gives

$$\begin{aligned}(\mathcal{L}_V \phi)Y = \frac{1}{2}[D\varrho\eta(Y) - Y(\varrho)\xi] + \varrho\phi Y\\ -\frac{2}{m}V^b(\phi Y)V + 2Q\phi Y - 2\lambda\phi Y.\end{aligned} \tag{34}$$

Replacing Y by ξ in generalized m-quasi-Einstein equation becomes

$$(\mathcal{L}_V g)(X,\xi) = \frac{2}{m}V^b(X)\eta(V) - 4n\eta(X) + 2\lambda\eta(X), \tag{35}$$

for any $X \in \chi(M)$. Combining the forgoing equation and (5) on the Lie-derivative of $\eta(X) = g(X,\xi)$ yields

$$g(X,\mathcal{L}_V\xi) = (\varrho + 4n - 2\lambda)\eta(X) - \frac{2}{m}V^b(X)\eta(V), \tag{36}$$

for all $X \in \chi(M)$. Replacing Y by ξ in (34) and making use of the fact that $\phi\xi = 0$ implies $(\mathcal{L}_V\phi)\xi = 0$ we obtain $D\varrho = \xi(\varrho)\xi$. By Lemma 1 in [21], we see that ϱ is constant. In consequence of this (34) becomes

$$(\mathcal{L}_V\phi)Y = \varrho\phi Y - \frac{2}{m}V^b(\phi Y)V + 2Q\phi Y - 2\lambda\phi Y. \tag{37}$$

On the other hand, taking Lie-derivative of $g(\xi,\xi) = 1$ and using (35) we get

$$\lambda = 2n + \varrho - \frac{1}{m}\eta(V)\eta(V). \tag{38}$$

Now taking Lie-derivative of (6) along V we obtain

$$(\mathcal{L}_V\phi)\phi X + \phi(\mathcal{L}_V\phi)X = (\mathcal{L}_V\eta)(X)\xi + \eta(X)\mathcal{L}_V\xi, \tag{39}$$

for all $X \in \chi(M)$. Making use of (5), (36) and (37) in (39) infers

$$\begin{aligned}(2\lambda - \varrho)X + \frac{1}{m}[V^b(X)V - V^b(\phi X)\phi V]\\ -QX + \phi Q\phi X - \lambda\eta(X)\xi = 0.\end{aligned} \tag{40}$$

Replacing X by ξ in (40) and inserting (38) we get $\eta(V)[V - \eta(V)\xi] = 0$ which implies $V = \eta(V)\xi$ or $\eta(V) = 0$ i.e. $V = 0$. Assume $V \neq 0$, then taking derivative of $V = \eta(V)\xi$ along arbitrary $X \in \chi(M)$ and using (11) gives $\nabla_X V = g(\nabla_X V,\xi) - \eta(V)\phi X$, which implies

$$dV^b(X,Y) = 2\eta(V)g(X,\phi Y) + g(\nabla_X V,\xi)\eta(Y) - g(\nabla_Y V,\xi)\eta(X).$$

Replacing X by ϕX and Y by ϕY in the forgoing equation and using the fact that V^b is closed we get $\eta(V)d\eta(X,Y) = 0$. Since $d\eta$ is non-vanishing everywhere on M implies $\eta(V) = 0$, a contradiction. Hence $V = 0$, consequently M is Einstein i.e. $QX = \lambda X$. Making use of this in (40) shows $\varrho = 0$. Then (38) implies M is Einstein with Einstein constant $2n$. Suppose M is complete. Since M is complete Einstein by Myer's theorem [19] it is compact. Finally, applying the Boyer-Gallicki theorem [3] we can conclude that M is Sasakian. This completes the proof. □

Proof of Theorem 1.6: In a 3-dimensional Riemannian manifold the curvature tensor is given by [4]

$$\begin{aligned}R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X\\ -g(QX,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned} \tag{41}$$

By our hypothesis, $V = \sigma\xi$, for some smooth σ. Differentiating this and using (14), the generalized m-quasi-Einstein equation becomes

$$QX = (\sigma\beta + \frac{\sigma^2}{m})\eta(X)\xi + (\lambda - \sigma\beta)X - \frac{1}{2}[\eta(X)D\sigma + (X\sigma)\xi]. \tag{42}$$

Inserting (42) in (41) and replacing Z by ξ gives

$$R(X,Y)\xi = \frac{1}{2}[(Y\sigma)\eta(X)\xi - (X\sigma)\eta(Y)\xi] + \frac{1}{2}[(X\sigma)Y - (Y\sigma)X] + (\frac{\sigma^2}{m} - \frac{\xi\sigma}{2} + 2\lambda - \sigma\beta - \frac{r}{2})[\eta(Y)X - \eta(X)Y]. \tag{43}$$

Replacing X by ϕX and Y by ϕY in (43) we obtain

$$\phi X(\sigma)\phi Y = \phi Y(\sigma)\phi X. \tag{44}$$

Taking $X = D\sigma$ in (44) gives $\phi Y(\sigma)\phi D\sigma = 0$ which implies $D\sigma = \xi(\sigma)\xi$. Differentiating forgoing equation along any $X \in \chi(M)$ infers

$$(\nabla_X D\sigma) = X(\xi\sigma)\xi - \alpha(\xi\sigma)\phi X + \beta[X - \eta(X)\xi](\xi\sigma). \tag{45}$$

Making use of the fact that $g(\nabla_X D\sigma, Y) = g(\nabla_Y D\sigma, X)$ from (45) we get

$$X(\xi\sigma)\eta(Y) - Y(\xi\sigma)\eta(X) - 2\alpha(\xi\sigma)g(\phi X, Y) = 0. \tag{46}$$

Choosing $X, Y \perp \xi$ above equation reduces to $\alpha(\xi\sigma) = 0$. Therefore, either $\alpha = 0$ or $\xi\sigma = 0$. I f $\alpha = 0$ then M is either β-Kenmotsu (for $\beta \neq 0$) or cosympletic manifold (for $\beta = 0$). Assuming the next case when $\xi\sigma = 0$, implies $D\sigma = 0$ and hence σ is constant. In consequence, from (42) we see that M is η-Einstein. This completes the proof. □

Replacing X by ξ in (42) and differentiating it along any $Y \in \chi(M)$ results in

$$(\nabla_Y Q)\xi = (\lambda + \frac{\sigma^2}{m})\nabla_Y\xi + Y(\lambda + \frac{\sigma^2}{m})\xi - \frac{1}{2}[(\nabla_Y D\sigma) + Y(\xi\sigma)\xi + (\xi\sigma)(\nabla_Y\xi)]. \tag{47}$$

Contracting the foregoing (47) yields

$$\frac{1}{2}\xi r = 2\beta(\lambda + \frac{\sigma^2}{m}) + \xi\lambda + \frac{2\sigma}{m}(\xi\sigma) - \frac{1}{2}[\Delta\sigma + \xi(\xi\sigma) + 2\beta(\xi\sigma)]. \tag{48}$$

Contracting (42) and then differentiating the obtained result by ξ and finally inserting it in (48) we obtain

$$\frac{1}{2}\Delta\sigma = (\xi\lambda) + (\frac{\sigma}{m} + \beta)(\xi\sigma) + 2\beta(\lambda + \frac{\sigma^2}{m} + \alpha\sigma). \tag{49}$$

For the case when $\alpha = 0$ and β a non-zero constant, M is β-Kenmotsu manifold. In a β-Kenmotsu manifold we have $Q\xi = -2\beta^2\xi$. Replacing X by ξ in (42) and using the forgoing equation along with $D\sigma = (\xi\sigma)\xi$ infers

$$\xi\sigma = \lambda + \frac{\sigma^2}{m} + 2\beta^2. \tag{50}$$

Making use of the fact $\Delta\sigma = div(D\sigma) = \xi(\xi\sigma) + 2\beta(\xi\sigma)$ and inserting (50) we get

$$\Delta\sigma = \xi\lambda + 2(\beta + \frac{\sigma}{m})(\xi\sigma). \tag{51}$$

Combining (51) and (49) infers

$$\xi\lambda = -4\beta(\lambda + \frac{\sigma^2}{m}). \tag{52}$$

Now, for the second case when σ is constant, Eq. (49) gives

$$\xi\lambda = -2\beta(\lambda + \frac{\sigma^2}{m} + \alpha\sigma). \tag{53}$$

Choosing λ as constant, Eq. (52) implies either $\beta = 0$ or $\lambda = -\frac{\sigma^2}{m}$. Assume $\beta \neq 0$ then σ is constant. Therefore, inserting value of λ in (50) shows $\beta = 0$, a contradiction. Hence, $\beta = 0$ and M is cosympletic. From second case, (13) implies either $\beta = 0$ or $\lambda + \frac{\sigma^2}{m} + \alpha\sigma = 0$. Fix $\beta \neq 0$ then it is obvious that α is a non-zero constant. Therefore M is α-Sasakian manifold and hence has constant scalar curvature. Hence we can state the following:

Corollary 4.1. *If a 3-dimensional normal almost contact metric manifold with β =constant admits m-quasi-Einstein metric whose potential vector field is collinear with ξ then M^3 is locally the product of a Kähler manifold and an interval or unit circle S^1 or has constant scalar curvature. Moreover, σ is constant.*

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