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Topological Properties of Multi-Valued Functions

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Abstract. In this paper, we generalize properties for single-valued functions in homotopy theory to multi-valued functions. We present new definitions for multi-valued functions such as locally m-pathwise connected, m-simply connected, m-covering space, and regular m-covering space. We apply some topological properties from algebraic topology to the multi-valued functions.

1. Introduction

In some studies, single-valued functions were insufficient to obtain the desired results. For this reason, the notion of the multi-valued function has emerged. Functions in which the image of the points in the domain is a set are multi-valued functions. Therefore, we can say that multi-valued functions are a generalization of single-valued functions. Multi-valued functions have a lot of applications in many fields such as optimal control theory, calculus of variation, probability, statistics, and economy.

The concept of continuity is one of the most fundamental issues in topology. At the same time, continuity is an indispensable concept in the field of homotopy theory. For this reason, the continuity of multivalued functions is a topic that has been dealt with by many mathematicians in years. The continuity for multi-valued functions has been put forward for special cases by [30], [11], and [10]. Many continuous definitions that are equivalent to each other have been tried to be generalized to multivalued functions, separately. For this reason, Kurotowski [16] has described notions of semi-continuous (lower semi-continuous, upper semi-continuous). The concept of continuity has later defined in many different ways for multi-valued functions and Strother [29] has studied the relationship that exists between different definitions of continuity. Over the years, many articles have been published dealing with multi-valued functions. Kakutani's work [12] is one of the most important results because he has given a generalization of Brouwer's Fixed Point Theorem in this paper. Choquet [4] has studied the lower semi-continuous and upper semi-continuous of multi-valued functions by the concepts of limit on families of sets. Michael [19] has described a topology on the collection of non-empty closed subsets of a topological space. Then he has applied this topology to the subject of multi-valued functions and the well-known definitions and properties for single-valued functions in topology have tried to be shown for multi-valued functions. Strother [27] has defined a fixed point for multi-valued functions and given a definition of a trace of a multi-valued function and studied the fixed point theory. The definition of homotopy for functions is a very important notion in algebraic topology.

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Strother's work [28] is one of the main papers in this field. In this paper, he has given a multi-valued version of this notion and some related properties. Also, Kruse [15] and Hahn [9] have extensively studied multi-valued functions. Further, Ponomarev [22] has investigated the basic properties of multi-valued functions and made applications on a space built on closed sets and he [23] investigated which properties of the spaces are preserved under multi-valued functions. Also, Rhee [24] has investigated homotopy theorems which can be applied to the study of multi-valued functions. Karaca and Ozkan [14] give the definition of multi-category and generalize homeomorphism to multi-valued functions using tools from multi-category. Also, Ozkan and Karaca [21] deal with multi-homology and multi-cohomology and define a cup product on multi-cohomology.

In this work, we give some background about multi-valued functions that we need to make proofs. We apply some properties from topology to the multi-valued functions. Then we generalize some properties from homotopy theory to a multi-valued function. We shall give multi-valued versions of some important definitions such as locally pathwise connectedness, simply connectedness, covering space, and regular covering space for some special classes of spaces and sometimes under some special conditions. In short, we aim to prove properties for multi-valued functions which is in the homotopy theory for single-valued functions.

2. Preliminaries

All spaces in this article will be considered Hausdorff spaces. To state the definition of a multi-valued function, the following notation is introduced. Multi-valued functions will be denoted by uppercase letters such as F, G, H. Let A be a non-empty set and B be any topological spaces. If for every element a of A, $F(a)$ is a subset of B, then $F : A \rightrightarrows B$ is said to be a multi-valued function.

Remark 2.1. 1) If single-valued functions are considered as multi-valued functions, it can be seen that single-valued functions are a special case of multi-valued functions.

2) A multi-valued function $F : A \rightrightarrows B$ can be considered as a single-valued function because it maps $a \in A$ to the set in $\mathcal{P}(B)$.

Let $F : A \rightrightarrows B$ be a multi-valued function. Then the *range* of F is $R(F) = \bigcup_{a\in A} F(a)$ by [2]. Also, for each $A_0 \subset A$ we get

$$F(A_0) = \bigcup_{a\in A_0} F(a)$$

from [3]. Given that F is a multi-valued function, F is called *one-to-one* from [2] if for any $a, a' \in A$, $a \neq a'$, we have $F(a) \cap F(a') = \emptyset$. F is *surjective (onto)* if $R(F) = B$. F is called *closed valued* (*open valued*) if for each $a \in A$, $F(a)$ is closed (open) in B by [7]. The *composition* of $F : A \rightrightarrows B$, $G : B \rightrightarrows C$ is a function denoted by $G \circ F : A \rightrightarrows C$ given by $(G \circ F)(a) = G(F(a)) = \bigcup_{b\in F(a)} G(b)$. The *graph* of a multi-valued function $F : A \rightrightarrows B$ is the set

$$\mathrm{gr}F = \{(a, b) \in A \times B | b \in F(a) \text{ and } a \in A\}.$$

Now, we have the following lemma for the continuity of multi-valued functions due to Strother [28]:

Lemma 2.2. *Let $F : A \rightrightarrows B$ be a closed valued function. F is continuous if and only if the following conditions are held:*

(1) *If $a_0 \in A$, V is open in B and if $F(a_0) \cap V \neq \emptyset$, then there exists an open set U of A with $a_0 \in U$ such that $F(a) \cap V \neq \emptyset$ for all $a \in U$.*

(2) *If $a_0 \in A$ and $F(a_0) \subset V$, where V is open in B, then there exists an open set U containing a_0 such that $F(U) \subset V$.*

In Lemma 2.2, the first statement is called lower semi-continuous (lsc) of a multi-valued function F and the second is called upper semi-continuous (usc) of a multi-valued function F. Geletu has given useful properties [7, Proposition 5.3.5] and [7, Proposition 5.3.14] regarding semi-continuities of multi-valued functions. We can conclude this property based on these propositions.

Proposition 2.3. *Let us assume that $F : A \rightrightarrows B$ is a multi-valued function. Then the following conditions hold:*

(*i*) *F is usc if and only if $F^{-}(W) \subset A$ is closed for every closed set $W \subset B$;*

(*ii*) *F is lsc if and only if $F^{-}(V) \subset A$ is open for every open set $V \subset B$.*

Lemma 2.4. ([28]) *Let B be a topological space, A_0 and A_1 both open (or both closed) subsets of a topological space A such that $A = A_0 \cup A_1$. Assume that $F : A_0 \rightrightarrows B$ and $G : A_1 \rightrightarrows B$ be continuous multi-valued functions such that for all $a \in A_0 \cap A_1$, $F(a) = G(a)$. Then the multi-valued function defined by*

$$\begin{aligned} H : A &\rightrightarrows B \\ a &\mapsto \begin{cases} F(a), & a \in A_0 \\ G(a), & a \in A_1 \end{cases} \end{aligned}$$

is continuous.

We define the following new continuous multi-valued function with the help of two continuous multi-valued functions.

Lemma 2.5. ([13]) *Let $F : A \rightrightarrows C$ and $G : B \rightrightarrows D$ be two continuous multi-valued functions. Then the function defined by*

$$\begin{aligned} H : A \times B &\rightrightarrows C \times D \\ (a, b) &\mapsto F(a) \times G(b) \end{aligned}$$

is continuous.

Let A and B be compact and let $F : A \rightrightarrows B$ be continuous. Then F is closed [26]. The inverse of the multi-valued function $F : A \rightrightarrows B$ is the function denoted by $F^{-} : B \rightrightarrows A$ and given by $F^{-}(B_0) = \{a \in A \mid F(a) \cap B_0 \neq \emptyset\}$ for each $B_0 \subset B$ [3].

Proposition 2.6. ([8]) *Let F be a closed valued multi-valued function from A to B and G be a closed valued multi-valued function from B to C. In this case, for any $C_0 \subset C$ we obtain*

$$(G \circ F)^{-}(C_0) = F^{-} \circ G^{-}(C_0).$$

Lemma 2.7. ([26]) *If $F_1 : A \rightrightarrows B$ and $F_2 : B \rightrightarrows C$ are continuous and if A, B and C are compact, then $F = F_2 \circ F_1$ is continuous.*

The restriction of a multi-valued function is defined in [26]. Let $F : A \rightrightarrows B$ be a multi-valued function. If A_0 is a subspace of A, then $F|_{A_0}$ is defined by $F|_{A_0}(a) = F(a)$ for all $a \in A_0$ and called the restriction of F to A_0. Then $F|_{A_0}$ is continuous [26]. The multi-valued function $\mathrm{id}_A : A \rightrightarrows A$, $\mathrm{id}_A(a) = \{a\}$ is called the *identical function* of the set A [2]. A function $F : A \rightrightarrows B$ is called *constant* if $F(a) = B_0$, for all $a \in A$, where B_0 is a fixed subset of B [2].

Lemma 2.8. *If $F : A \rightrightarrows B$ is one-to-one, $F^{-} \circ F = id_A$.*

Proof. Let $a \neq a'$ for $a, a' \in A$. Assume that $a' \in F^{-} \circ F(a)$. Since $F^{-} \circ F(a) = \{a_i \in A \mid F(a_i) \cap F(a) \neq \emptyset\}$, this is a contradiction with F is one-to-one. So $F^{-} \circ F(a) = \{a\} = id_A(a)$ for all $a \in A$. $\square$

Definition 2.9. ([13]) Let A be a topological space and $A_0 \subset A$. Then the m-inclusion function is defined by

$$\begin{aligned} \mathcal{I} : A_0 &\rightrightarrows A \\ a &\mapsto \{a\}. \end{aligned}$$

Moreover, $\mathcal{I}(a) \cap \mathcal{I}(a^{'}) = \emptyset$ for all $a \neq a^{'} \in A_0$. Then the m-inclusion function is one to one. Also, the continuity of the m-inclusion function can be shown easily by Lemma 2.2.

Lemma 2.10. *Assume that A and B are two topological spaces, $A = U_1 \cup U_2$ and $F : A \rightrightarrows B$ is multi-valued maps such that $F|_{U_1} : U_1 \rightrightarrows B$ and $F|_{U_2} : U_2 \rightrightarrows B$ are continuous. If U_1 and U_2 is open in A, then F is continuous.*

Proof. Suppose that $V \subset B$ is open.

$$\begin{aligned} F^{-}(V) &= \{a \in A \mid F(a) \cap V \neq \emptyset\} \\ &= \{a \in U_1 \cup U_2 \mid F(a) \cap V \neq \emptyset\} \\ &= \{a \in U_1 \mid F(a) \cap V \neq \emptyset\} \cup \{a \in U_2 \mid F(a) \cap V \neq \emptyset\} \\ &= \{a \in U_1 \mid F|_{U_1}(a) \cap V \neq \emptyset\} \cup \{a \in U_2 \mid F|_{U_2}(a) \cap V \neq \emptyset\} \\ &= (F|_{U_1})^{-}(V) \cup (F|_{U_2})^{-}(V). \end{aligned}$$

Since V is open, $F|_{U_1}$ and $F|_{U_2}$ are continuous; $(F|_{U_1})^{-}(V) \cup (F|_{U_2})^{-}(V) \subset A$ is open. Then F is lower semi-continuous. Similarly, we can show that F is upper semi-continuous. Consequently, F is continuous. □

Lemma 2.11. *If B has the indiscrete topology, then $F : A \rightrightarrows B$ is a continuous multi-valued function for any topological space A.*

Proof. For all $a \in A$, there is only one open subset such that $F(a) \cap V \neq \emptyset$. In this case, V must be B. So there exists neighborhood U of a such that $F(a^{'}) \cap B \neq \emptyset$ for all $a^{'} \in U$. Thus F is lower semi-continuous. Similarly, we can show that F is upper semi-continuous. Therefore, F is a continuous multi-valued function. □

Lemma 2.12. *Let $F : A \rightrightarrows C$ and $G : B \rightrightarrows D$ be multi-valued functions. Then $(F \times G)^{-} = F^{-} \times G^{-}$.*

Proof. The functions $(F \times G)^{-}$ and $F^{-} \times G^{-}$ are defined as follows.

$$\begin{aligned} (F \times G)^{-} : C \times D &\rightrightarrows A \times B \\ (c, d) &\mapsto \{(a, b) \in A \times B \mid (F \times G)(a, b) \cap \{(c, d)\} \neq \emptyset\} \end{aligned}$$

and

$$\begin{aligned} F^{-} \times G^{-} : C \times D &\rightrightarrows A \times B \\ (c, d) &\mapsto F^{-}(c) \times G^{-}(d). \end{aligned}$$

Therefore, we have

$$\begin{aligned} F^{-}(c) \times G^{-}(d) &= \{(a, b) \in A \times B \mid F(a) \cap \{c\} \neq \emptyset \text{ and } G(b) \cap \{d\} \neq \emptyset\} \\ &= \{(a, b) \in A \times B \mid (F(a) \times G(b)) \cap \{(c, d)\} \neq \emptyset\} \\ &= \{(a, b) \in A \times B \mid F \times G(a, b) \cap \{(c, d)\} \neq \emptyset\} \\ &= (F \times G)^{-}(c, d). \end{aligned}$$

□

Lemma 2.13. *Let $F : A \rightrightarrows C$ and $G : B \rightrightarrows D$ be multi-valued functions. Then $F|_{A_0} \times G|_{B_0} = F \times G|_{A_0 \times B_0}$ for $A_0 \subset A$ and $B_0 \subset B$.*

Proof. By the definition of restriction, we know that $F(a) = F|_{A_0}(a)$ for $a \in A_0$ and $G(b) = G|_{B_0}(b)$ for $b \in B_0$. So

$$(F \times G)|_{A_0 \times B_0}(a, b) = F(a) \times G(b) = F|_{A_0}(a) \times G|_{B_0}(b) = (F|_{A_0} \times G|_{B_0})(a, b).$$

□

3. *m*-homeomorphisms

Definition 3.1. ([13]) Let the multi-valued function $H : X \rightrightarrows Y$ be one-to-one and surjective such that X and Y are two topological spaces. If H and its inverse H^- are continuous, then H is called an m-homeomorphism.

Lemma 3.2. ([13]) *Let us assume that X, Y, and Z are compact topological spaces. If $H : X \rightrightarrows Y$ and $K : Y \rightrightarrows Z$ are m-homeomorphisms, then $K \circ H : X \rightrightarrows Z$ is an m-homeomorphism.*

Example 3.3. Let X be a topological space. The identical multi-valued function $id_X : X \rightrightarrows X$ defined by $id_X(x) = \{x\}$ is an m-homeomorphism.

Proof. It is known that id_X is continuous. Also $id_X(x) \cap id_X(x_0) = \emptyset$ for all $x, x_0 \in X$ such that $x \neq x_0$ and $R(id_X) = X$. Moreover, we have

$$\begin{aligned}
(id_X)^- : X &\rightrightarrows X \\
x &\mapsto \left\{x_0 \mid id_X(x_0) \cap \{x\} \neq \emptyset\right\} \\
&= \left\{x_0 \in X \mid \{x_0\} \cap \{x\} \neq \emptyset\right\} \\
&= \{x\}.
\end{aligned}$$

So $(id_X)^- = id_X$. Thus $(id_X)^-$ is continuous. Therefore id_X is an m-homeomorphism. □

Lemma 3.4. *If $H : X \rightrightarrows Z$ and $K : Y \rightrightarrows W$ are m-homeomorphisms, then $H \times K$ is an m-homeomorphism.*

Proof. First, since H and K are m-homeomorphism, $H \times K$ is continuous. Now, let us assume that $(x, y) \neq (x', y')$ for $(x, y), (x', y') \in X \times Y$. Since $H(x) \cap H(x') = \emptyset$ and $K(y) \cap K(y') = \emptyset$, we can get that $H \times K(x, y) \cap H \times K(x', y') = \emptyset$. Also, since $R(H) = H(X) = Z$ and $R(K) = K(Y) = W$,

$$R(H \times K) = \bigcup_{(x,y) \in X \times Y} H(x) \times K(y) = H(X) \times K(Y) = Z \times W.$$

So $H \times K$ is one-to-one and surjective. Finally, we must show that $(H \times K)^-$ is continuous. We already know that $(H \times K)^- = H^- \times K^-$. Since H^- and K^- are continuous multi-valued functions, $(H \times K)^-$ is continuous. Therefore, $H \times K$ is an m-homeomorphism. □

Theorem 3.5. *Let X and Y be topological spaces. Assume that $H : X \rightrightarrows Y$ is an m-homeomorphism. Then $H|_V : V \rightrightarrows H(V)$ is an m-homeomorphism for $V \subset X$ is open*

Proof. Let $H : X \rightrightarrows Y$ be an m-homeomorphism. So H is a continuous function. In this case $H|_V : V \rightrightarrows H(V)$ is also continuous. We know that $H(x) \cap H(x') = \emptyset$ for all $x, x' \in X$ such that $x \neq x'$ and $H|_V(v) = H(v)$ for all $v \in V$. Then $H|_V(v) \cap H|_V(v') = \emptyset$ for all $v, v' \in V$ such that $v \neq v'$. Furthermore, since $H|_V(V) = H(V)$, we can get $R(H|_V) = H(V)$. Finally, we must show that $(H|_V)^- : H(V) \rightrightarrows V$ is continuous. If H is an m-homeomorphism, $H^- : Y \rightrightarrows X$ is continuous. So for all $y \in Y$ and for any open set $X_y \subset X$ such that $H^-(y) \cap X_y \neq \emptyset$, there exists a neighborhood $Y_y \subset Y$ such that $H^-(y') \cap X_y \neq \emptyset$ for all $y' \in Y_y$. Let $U = Y_y \cap H(V)$. Since Y_y is open in Y, U is open in $H(V)$. If $y \in U$, then $y \in Y_y$ and $y \in H(V)$. Then

$$\begin{aligned}
(H|_V)^-(y) &= \left\{v \in V \mid H|_V(v) \cap \{y\} \neq \emptyset\right\} \\
&= \left\{v \in V \mid H(v) \cap \{y\} \neq \emptyset\right\} \\
&= H^-(y).
\end{aligned}$$

So, $(H|_V)^-(y) \cap (X_y \cap V) \neq \emptyset$ for all $y \in U$. Thus $(H|_V)^-$ is lower semi-continuous. Similarly, we can prove that $(H|_V)^-$ is upper semi-continuous. Consequently, $H|_V$ is an m-homeomorphism. □

Lemma 3.6. *Let $H : X \rightrightarrows Y$ be an m-homeomorphism. Then H is an open multi-valued function.*

Proof. Assume that $V \subset X$ is an open subset. H^- is continuous, so $(H^-)^-(V) \subset Y$ is open. Since $(H^-)^- = H$, H is an open multi-valued function. □

Proposition 3.7. *et us assume that X is a discrete topological space and Y is any topological space. There is an m-homeomorphism between grF and X if $F : X \rightrightarrows Y$ is a continuous multi-valued function.*

Proof. Define a multi-valued function H such that

$$\begin{aligned} H : X &\rightrightarrows grF \\ x &\mapsto \big(x, F(x)\big) \end{aligned}$$

Assume that $x, x^{'} \in X$ such that $x \neq x^{'}$.

$$\begin{aligned} x \neq x^{'} &= \big(x, F(x)\big) \cap \big(x^{'}, F(x^{'})\big) = \emptyset \\ &= H(x) \cap H(x^{'}) = \emptyset. \end{aligned}$$

So H is one-to-one. If $(x, y) \in grF$, then we can conclude that $(x, y) \in R(H)$, and vice versa. So H is onto. Let $x \in X$ and $H(x) \cap V \neq \emptyset$. Because of the discretness of X, there exists a neighborhood U of x such that for all $x_0 \in U$, $H(x_0) \cap V \neq \emptyset$. So H is lower semi-continuous and also it can be shown that H is upper semi-continuous. Lastly, we must show that H^- is continuous.

$$\begin{aligned} H^- : grF &\rightrightarrows X \\ (x, y) &\mapsto \{x^{'} \in X \mid H(x^{'}) \cap \{(x, y)\} \neq \emptyset\}. \end{aligned}$$

In the light of the definiton of H, we get $H^-(x, y) = \{x\}$. So $H^- = proj_X|_{grF}$. Since $proj_X : X \times Y \rightrightarrows X$, $(x, y) \mapsto \{x\}$ is a continuous multi-valued function, H^- is continuous. Thus, H is an m-homeomorphism between grF and X. □

4. Locally *m*-pathwise connectedness

In this section, locally pathwise connectedness will be generalized to multivalued functions. However, we must first give a few definitions. First of all, to give the definition of m-path [28], the *m-path* is a continuous multi-valued function such that $P : I \rightrightarrows A$ for any topological space A. Second and lastly, the *m-pathwise component of A* is a maximal m-pathwise connected subset of A.

Definition 4.1. If there is an m-pathwise connected neighborhood G of a such that $G \subset U$ for every neighborhood U of a, then A is called locally m-pathwise connected at $a \in A$. If A is locally m-pathwise connected at every $a \in A$, then it is called locally m-pathwise connected.

Proposition 4.2. *Let A be a topological space. Then m-pathwise components of open subsets of A are open if and only if A is locally m-pathwise connected.*

Proof. Let m-pathwise components of open subset U be open. Assume that $a \in U$ is a neighborhood of $a \in A$ and C is an m-pathwise component of U such that $a \in C$. C is open for acceptance. So A is locally m-pathwise connected. Let A be a locally m-pathwise connected space and U be an open. Assume that $C \subset U$ is an m-pathwise component and $a \in C$. Since A is locally m-pathwise connected, there is a neighborhood G of a such that $G \subset U$. Then we get $G \subset C$. We chose $a \in C$ arbitrarily at first. So C is open. □

5. Some properties of the multi-valued function in homotopy theory

In this section, some properties of a multi-valued function will be given with respect to the definition of multi-homotopy [28]; see [5], [6], [17], and [18] for the works of digital H-spaces in digital homotopy.

Definition 5.1. ([28]) Assume that $F, G : X \rightrightarrows Y$ be multi-valued functions. Then F is called m-homotopic (multi-homotopic) to G if there exists a multi-valued function $H : X \times I \rightrightarrows Y$ such that it is continuous, $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$. We write $F \simeq_m G$.

Lemma 5.2. ([13]) *The m-homotopy relation is an equivalence relation.*

Theorem 5.3. ([13]) *Let us assume that X, Y, and Z are compact spaces. Suppose that F and G are two continuous multi-valued functions from X to Y. If $F \simeq_m G$ and $H : Y \rightrightarrows Z$ is a continuous multi-valued function, then $H \circ F \simeq_m H \circ G$.*

Theorem 5.4. ([13](*Let us assume that X, Y, and Z are compact spaces. Suppose that F and G are two continuous multi-valued functions from X to Y. If $F \simeq_m G$ and $H : Z \rightrightarrows X$ is a continuous multi-valued function, then $F \circ H \simeq_m G \circ H$.*

Lemma 5.5. *Let $F_0 : X \rightrightarrows Y$, $x \mapsto Y_0$; $F_1 : X \rightrightarrows Y$, $x \mapsto Y_1$ be two constant multi-valued maps and $Y_0, Y_1 \subset Y$ are closed. If $F_0 \simeq_m F_1$, then Y_0 and Y_1 are in the same m-pathwise component of Y.*

Proof. Assume that $F_0 \simeq_m F_1$. In this case, there exists a multi-valued function $H : X \times I \rightrightarrows Y$ such that it is continuous, $H(x, 0) = F_0(x)$ and $H(x, 1) = F_1(x)$. $H|_{\{x_0\}\times I}$ is continuous for $x_0 \in X$. Then we have $H|_{\{x_0\}\times I}(x_0, 0) = F_0(x_0) = Y_0$ and $H|_{\{x_0\}\times I}(x_0, 1) = F_1(x_0) = Y_1$. So $H|_{\{x_0\}\times I}$ is an m-path between Y_0 and Y_1. Consequently, Y_0 and Y_1 are in the same m-pathwise component of Y. □

Strother [28] has given a multi-valued version of null homotopy for single-valued functions.

Definition 5.6. ([28]) If $F : X \rightrightarrows Y$ is m-homotopic to a constant multi-valued function C, then we say that F is null m-homotopic and we denote $F \simeq_m C$.

Strother [28] has defined the product $F * G$ of F and G.

Definition 5.7. ([28]) For each space Y, a closed subset Y_0 of Y, and a positive integer n, we define

$$MQ(n, Y, Y_0) = \{F : I^n \rightrightarrows Y | F \text{ is continuous and } F(x) = Y_0 \text{ for all } x \in B^{n-1}\}.$$

The subset B^{n-1} of I^n (product of unit intervals) consisting of points $(x_1, \cdots, x_n)$ for which some coordinate is zero or one, and is called the boundary of I^n.

Definition 5.8. ([28](Let F and G be elements of $MQ(n, Y, Y_0)$. Define $H = F * G$ by

$$H(x_1, \cdots, x_n) = \begin{cases} F(2x_1, \cdots, x_n), & \text{if } 0 \le x_1 \le \frac{1}{2} \\ G(2x_1 - 1, \cdots, x_n), & \text{if } \frac{1}{2} \le x_1 \le 1. \end{cases}$$

Denote the class of functions m-homotopic to F relative to (B^{n-1}, Y_0) by $[F]$. Define $[F] * [G]$ to be $[F * G]$. Moreover, the function H is well-defined and continuous by Lemma 2.4.

Theorem 5.9. ([28]) *Let Y be a compact Hausdorff space. Then the m-homotopy classes of the continuous functions in $MQ(n, Y, Y_0)$ form a group $M\Pi_n(Y, Y_0)$ called the n-th m-homotopy group of the space Y with base set Y_0. For $x \in B^{n-1}$, the zero element in $M\Pi_n(Y, Y_0)$ is the constant function $F(x) = Y_0$ and the inverse of a function F is defined by $F^{-1}(x_1, \cdots, x_n) = F(1 - x_1, \cdots, x_n)$, for $(x_1, \cdots, x_n) \in B^{n-1}$.*

From the definition of a product operation and [28, Theorem 5], we give the following property about product operation in [13]. Suppose that $F, G \in MQ(n, Y, Y_0)$. Then

$$G * F^{-1}(x) = \begin{cases} G(2x), & 0 \leq x \leq \frac{1}{2} \\ F(2-2x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Moreover, when $x = \frac{1}{2}$, then $G(1) = Y_0 = F(1)$. On the other hand, we can get that $(F * G)^{-1} = G^{-1} * F^{-1}$.

Theorem 5.10. *Let $X = \{x_0\}$ be a one-point space and Y any topological space. If Y is m-pathwise connected, then the set of all m-homotopy classes of continuous multi-valued functions $F : X \rightrightarrows Y$ has only one element.*

Proof. Since Y is m-pathwise connected, we can define m-path $F : I \rightrightarrows Y$, $t \mapsto F(t)$ and for all $Y_0, Y_1 \subset Y$ are closed, $F(0) = Y_0$ and $F(1) = Y_1$. Assume that $F_0 : X \rightrightarrows Y$, $x_0 \mapsto Y_0$ and $F_1 : X \rightrightarrows Y$, $x_0 \mapsto Y_1$. Define a function H by

$$\begin{aligned} H : X \times I &\rightrightarrows Y \\ (x, t) &\mapsto H(x, t) = F(t). \end{aligned}$$

H is continuous because F is continuous. Also, $H(x_0, 0) = F(0) = Y_0 = F_0(x)$ and $H(x_0, 1) = F(1) = Y_1 = F_1(x)$. So $F_0 \simeq_m F_1$. This means that the set of all m-homotopy classes of continuous multi-valued functions from X to Y has only one element. □

Lemma 5.11. *Let X be an indiscrete space. Then $M\Pi_n(X, X_0)$ has only one element.*

Proof. Let $F \in MQ(n, X, X_0)$ and $C : I^n \rightrightarrows X$, $s \mapsto X_0$. We define multi-valued function H as follows.

$$H : I^n \times I \to X$$
$$(s, t) \mapsto \begin{cases} X_0, & t = 0 \\ F(s), & t \in (0, 1]. \end{cases}$$

Then $H(s, 0) = C(s)$ and $H(s, 1) = F(s)$. Since X has the indiscrete topology, H is continuous. So $C \simeq_m F$. F is chosen arbitrarily at first. Therefore, $M\Pi_n(X, X_0)$ has only one element. □

Definition 5.12. ([13]) Let $K : (Y, Y_0) \rightrightarrows (X, X_0)$ be a continuous multi-valued function and $F \in M\Pi_n(Y, Y_0)$. Define a function as follows:

$$\begin{aligned} K_* : M\Pi_n(Y, Y_0) &\to M\Pi_n(X, X_0) \\ [F] &\mapsto K_*[F] = [K \circ F]. \end{aligned}$$

In this case, K_* is said to be the m-homomorphism induced by K. Moreover, K_* is well-defined and $K_*([F] * [H]) = K_*([F]) * K_*([H])$.

Proposition 5.13. ([13]) *Let X, Y, and Z be compact spaces. $(G \circ F)_* = G_* \circ F_*$, if $F : (X, X_0) \rightrightarrows (Y, Y_0)$ and $G : (Y, Y_0) \rightrightarrows (Z, Z_0)$ are the continuous multi-valued functions. Moreover, id_* is the identical m-homomorphism.*

Definition 5.14. ([13]) Let us assume that X and Y are compact spaces. Suppose that $F : X \rightrightarrows Y$ is a continuous multi-valued function. F is an m-homotopy equivalence if there is a continuous multi-valued function $G : Y \rightrightarrows X$ with $G \circ F \simeq_m \mathrm{id}_X$ and $F \circ G \simeq_m \mathrm{id}_Y$. Two spaces X and Y have the same m-homotopy type if there is an m-homotopy equivalence and it is denoted as $X \simeq_m Y$.

Lemma 5.15. ([13]) *The same m-homotopy type relation on compact spaces is an equivalence relation.*

Definition 5.16. ([13]) A space X is m-contractible if an identical function id_X is null m-homotopic.

Lemma 5.17. *If X is m-contractible, compact topological space and $x_0 \in X$, then X and $\{x_0\}$ have the same m-homotopy type.*

Proof. Define the functions $F : X \rightrightarrows \{x_0\}$, $x \mapsto \{x_0\}$ and $G : \{x_0\} \rightrightarrows X$, $x \mapsto \{x_0\}$. These functions are continuous. We know that if X is m-contractible, then $id_X \simeq C$ such that id_X is an identical multi-valued function and $C : X \rightrightarrows X$, $x \mapsto \{x_0\}$. Since $G \circ F = C$ and $id_X \simeq C$, $G \circ F \simeq_m id_X$. Also $F \circ G = id_X$. Therefore $X \simeq_m \{x_0\}$. □

Lemma 5.18. *Let X and Y be compact spaces. We can say that X and Y have the same m-homotopy type if X and Y are given as m-contractible.*

Proof. Given that $x_0 \in X$ and $y_0 \in Y$, we can define the functions $F : \{x_0\} \rightrightarrows \{y_0\}$, $x \mapsto \{y_0\}$ and $G : \{y_0\} \rightrightarrows \{x_0\}$, $y \mapsto \{x_0\}$. These functions are continuous. $F \circ G = id_{\{y_0\}}$ and $G \circ F = id_{\{x_0\}}$. So $\{x_0\}$ and $\{y_0\}$ have the same m-homotopy type. Since X is m-contractible, X and $\{x_0\}$ have the same m-homotopy type. Similarly, since Y is m-contractible, Y and $\{y_0\}$ have the same m-homotopy type. In the light of 5.15, $X \simeq_m Y$. □

Definition 5.19. ([13]) Let A be a closed subset of the compact topological space X. Then, A is an m-retract of X if there is a continuous multi-valued function $R : X \rightrightarrows A$ with $R(a) = \{a\}$, for all $a \in A$. In this case, R is called an m-retraction. Equivalently, a continuous multi-valued function R such that $R \circ \mathcal{I} = id_A$ is called an m-retraction, where $\mathcal{I} : A \rightrightarrows X$ is an m-inclusion function.

Lemma 5.20. *Let us assume that A is a closed subset of the compact topological space X. Suppose that $R : X \rightrightarrows A$ is an m-retraction and $A_0 \subset A$ is closed. Then*

$$\begin{array}{rcl} R_* : M\Pi_n(X, A_0) & \rightarrow & M\Pi_n(A, A_0) \\ [F] & \mapsto & R_*([F]) = [R \circ F] \end{array}$$

is surjective.

Proof. Assume that $[F] \in M\Pi_n(A, A_0)$ and $I : A \rightrightarrows X$ is an m-inclusion. Then $I \circ F$ is an m-path in X such that $I \circ F(x) = A_0$ for all $x \in B^{n-1}$. Let $H = I \circ F$.

$$R_*([H]) = [R \circ H] = [R \circ (I \circ F)] = [(R \circ I) \circ F] = [id_A \circ F] = [F]$$

For all $[F] \in M\Pi_n(A, A_0)$, we can find $H \in M\Pi_n(X, A_0)$. Therefore R_* is surjective. □

Definition 5.21. ([13]) Let X be a compact topological space and A be a closed subspace of X. If there exists an m-homotopy function $H : X \times I \rightrightarrows X$ such that for all $x \in X$ and $a \in A$,

$$\begin{aligned} &H(x, 0) = \{x\}, \\ &H(x, 1) \in A \text{ and} \\ &H(a, 1) = \{a\}, \end{aligned}$$

then A is called a deformation m-retract of X. Equivalent to this definition, if there exists an m-retraction $R : X \rightrightarrows A$ such that $R \circ \mathcal{I} = id_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$, where $\mathcal{I} : A \rightrightarrows X$ is an m-inclusion function, then a subspace A of X is called a deformation m-retract of X.

Lemma 5.22. *Assume that X is a compact and m-contractible space. Then $\{x_0\}$ is a deformation m-retract of X for all $x_0 \in X$.*

Proof. First, assume that X is an m-contractible space and $x_0 \in X$. If X is m-contractible, $id_X \simeq_m C$ such that $C(x) = \{x_0\}$. Now let $I : \{x_0\} \rightrightarrows X$, $x \mapsto \{x_0\}$ and $R : X \rightrightarrows \{x_0\}$, $x \mapsto \{x_0\}$. Then $R \circ I = id_{\{x_0\}}(x)$ and $I \circ R = C(x)$. Thus, since x_0 is an arbitrary point of X, $\{x_0\}$ is a deformation m-retract of X for all $x_0 \in X$. □

Lemma 5.23. *Suppose that A and B are closed subsets of a compact topological space X. If A and B are deformation m-retract of X, then A and B have the same m-homotopy type.*

Proof. If A and B are deformation m-retracts of X, then there exists $R_0 : X \rightrightarrows A$ such that $R_0 \circ \mathcal{I}_0 = id_A$ and $\mathcal{I}_0 \circ R_0 \simeq_m id_A$, where $\mathcal{I}_0 : A \rightrightarrows X$ is the m-inclusion function and, there exists $R_1 : X \rightrightarrows B$ such that $R_1 \circ \mathcal{I}_1 = id_B$ and $\mathcal{I}_1 \circ R_1 \simeq_m id_B$, where $\mathcal{I}_1 : B \rightrightarrows X$ is the m-inclusion function. Then, we have $(R_1 \circ \mathcal{I}_0) : A \rightrightarrows B$ and $(R_0 \circ \mathcal{I}_1) : B \rightrightarrows A$ are continuous.

$$\begin{aligned}(R_1 \circ \mathcal{I}_0) \circ (R_0 \circ \mathcal{I}_1) &\simeq_m id_B\\ (R_0 \circ \mathcal{I}_1) \circ (R_1 \circ \mathcal{I}_0) &\simeq_m id_A.\end{aligned}$$

Thus $A \simeq_m B$. □

Definition 5.24. Let X be an m-pathwise connected space. If $M\Pi_n(X, X_0) = \{0\}$ for every closed subset $X_0 \subset X$, then X is called m-simply connected.

Lemma 5.25. *Let X be an m-simply connected space and $X_0, X_1 \subset X$ be closed. If $F : I \rightrightarrows X$ and $G : I \rightrightarrows X$ are m-paths such that $F(0) = X_0 = G(0)$ and $F(1) = X_1 = G(1)$, then $F \simeq_m G$.*

Proof. We have $F : I \rightrightarrows X$ and $G : I \rightrightarrows X$ are m-paths such that $F(0) = X_0 = G(0)$ and $F(1) = X_1 = G(1)$. So $F * G^-$ is an m-path in X such that $F * G^-(0) = X_0$ and $F * G^-(1) = X_0$. Since X is m-simply connected, $F * G^- \simeq_m C_{X_0}$. Then

$$F \simeq_m F * C_{X_1} \simeq_m F * (F^- * G) \simeq_m C_{X_0} * G \simeq_m G.$$

□

6. m-covering spaces

In this section, we apply definitions and properties in covering spaces to multi-valued functions.

Definition 6.1. Let $P : E \rightrightarrows B$ be a continuous multi-valued function such that E and B are topological spaces. Let us consider U as an open set in B. If $P^-(U)$ is a disjoint union of open sets S_i in E and $P|_{S_i} : S_i \rightrightarrows U$ is an m-homeomorphism for every i, then we say that U is evenly m-covered by P. In this case, S_i is called m-sheets.

Definition 6.2. Let E be an m-pathwise connected space and B be a topological space. Let us consider $P : E \rightrightarrows B$ as a continuous multi-valued function. If b has an open neighborhood $U = U_b$ that is evenly m-covered by P, for every $b \in B$, then an ordered pair (E, P) is an m-covering space of B. In this case, P is called the m-covering projection, and $U = U_b$ is called m-admissible.

Lemma 6.3. *Let (E, P) be an m-covering space of B. Then $P : E \rightrightarrows B$ is a surjection.*

Proof. If $b \in B$ and $U_b \subset B$ is m-admissible; then there exist m-homeomorphisms $P|_{S_i} : S_i \rightrightarrows U_b$. So we can find $e \in S_i$ such that $b \in P|_{S_i}(e) = P(e)$. We chose $b \in B$ arbitrarily at first. Therefore $R(P) = B$. □

Lemma 6.4. *Let E and B be compact spaces. Let us consider $P : E \rightrightarrows B$ as a closed, continuous multi-valued function. If (E, P) is an m-covering space of B, then B is m-pathwise connected.*

Proof. Assume that (E, P) is an m-covering space of B. In this case, E is m-pathwise connected, and P is a surjection. So $P(E) = B$. Since E is m-pathwise connected, B is m-pathwise connected. □

Example 6.5. Let us assume that E and B are m-pathwise connected topological spaces. If $P : E \rightrightarrows B$ is an m-homeomorphism, then P is an m-covering projection.

Proof. Suppose that $P : E \rightrightarrows B$ is an m-homeomorphism. Since P is an m-homeomorphism, P is continuous. So $P^-(U)$ is open for all $U \subset B$ is open. Thus, it is concluded that $P^-(U)$ is a union of disjoint open sets, and $P|_{P^-(U)}$ is an m-homeomorphism. E is m-pathwise connected and P is continuous by hypothesis. Thus P is an m-covering projection. □

Example 6.6. Assume that B is an m-pathwise connected space. Then the identical multi-valued function $id_B : B \rightrightarrows B$ is an m-covering projection.

Proof. Since id_B is continuous, $(id_B)^{-}(U)$ is open. So $(id_B)^{-}(U)$ is a disjoint union of open sets in B. We know that id_B is an m-homeomorphism by Example 3.3. This means $id_B|_{(id_B)^{-}(U)}$ is also m-homeomorphism. Namely, U is evenly m-covered by id_B. Therefore $id_B : B \rightrightarrows B$ is an m-covering projection. □

For the version of the following theorem for single-valued functions, see [1].

Theorem 6.7. *Let (E_1, P_1) and (E_2, P_2) be m-covering space of B_1 and B_2, respectively. Define the multi-valued function $P_1 \times P_2$ as follows.*

$$\begin{aligned} P_1 \times P_2 : E_1 \times E_2 &\rightrightarrows B_1 \times B_2 \\ (e_1, e_2) &\mapsto P_1(e_1) \times P_2(e_2) \end{aligned}$$

Then $(E_1 \times E_2, P_1 \times P_2)$ is an m-covering space of $B_1 \times B_2$.

Proof. If (E_1, P_1) is an m-covering space of B_1, then E_1 is m-pathwise connected. In the same way, we can say that E_2 is m-pathwise connected. So $E_1 \times E_2$ is m-pathwise connected. If P_1 and P_2 are m-covering projections, then $P_1 \times P_2$ is continuous. We must show that each $(b_1, b_2) \in B_1 \times B_2$ has an open neighborhood U such that U is evenly m-covered by $P_1 \times P_2$. Assume that U is the neighborhood for $(b_1, b_2) \in B_1 \times B_2$. Let U be an open set of product topology $B_1 \times B_2$. In this case, there are two open sets U_1 and U_2 such that $U = U_1 \times U_2$. Since U_1 is evenly m-covered by P_1, $(P_1)^{-}(U_1)$ is a disjoint union of open sets $(S_1)_i$ in E_1. Since U_2 is evenly m-covered by P_2, $(P_2)^{-}(U_2)$ is a disjoint union of open sets $(S_2)_j$ in E_2. So $(P_1)^{-}(U_1) \times (P_2)^{-}(U_2) = (P_1 \times P_2)^{-}(U_1 \times U_2)$ can be written as disjoint union of $(S_1)_i \times (S_2)_j$. For every i and j, $P_1|_{((S_1)_i)}$ and $P_2|_{((S_2)_j)}$ are m-homeomorphisms. Then $P_1|_{((S_1)_i)} \times P_2|_{((S_2)_j)} = P_1 \times P_2|_{(S_1)_i \times (S_2)_j}$ is an m-homeomorphism. □

Theorem 6.8. *If E is a locally m-pathwise connected space, then every one-to-one m-covering projection $P : E \rightrightarrows B$ is an open multi-valued function.*

Proof. If $P : E \rightrightarrows B$ is an m-covering projection and G is open set E, then we claim that $P(G)$ is open in B. Let $b \in P(G)$ and $e \in P^{-}(\{b\})$ and U be the m-admissible neighborhood for b. Since $e \in P^{-}(\{b\})$, we can conclude that $e \in G$. Also, since $P(e) \cap \{b\} \neq \emptyset$, $b \in P(e)$. Assume that W is the m-path component of $P^{-}(U)$ and $e \in W$. We know that B is locally m-path connected. So W is open in E by proposition 4.2. Since $P|_W : W \rightrightarrows U$ is an m-homeomorphism, $P|_W(W \cap G)$ is an open set in B. So $b \in P|_W(W \cap G) \subset P(G)$. Since $b \in P(G)$ is chosen arbitrarily, it follows that there is an open set for all $b \in P(G)$ such that the open set is a subset of $P(G)$ and hence $P(V)$ is an open set. Consequently, P is an open multi-valued function. □

Proposition 6.9. *Let (E, P) be an m-covering space of B. If P is a one-to-one multi-valued map, then the induced m-homomorphism of P is an m-monomorphism.*

Proof. Let E_0 and B_0 be closed subsets of E and B, respectively. The induced m-homomorphism P_* is defined as follows.

$$\begin{aligned} P_* : M\Pi_n(E, E_0) &\rightarrow M\Pi_n(B, B_0) \\ [F] &\mapsto [P \circ F]. \end{aligned}$$

Let $[F] \in kerP_*$. Since $P_*([F]) = [C]$, $F \simeq_m C_{E_0}$. Then P_* is an m-monomorphism. □

Definition 6.10. Let (E, P) be the m-covering space of B. If $P_*(M\Pi_n(E, E_0))$ is a normal subgroup of $M\Pi_n(B, B_0)$ for every closed subset B_0, (E, P) is called a regular m-covering space.

Lemma 6.11. *Let (E, P) be an m-covering space of B. If $B_0 \subset B$ is closed and $M\Pi_n(B, B_0)$ is an Abelian group, then (E, P) is a regular m-covering space of B.*

Proof. Let P be an m-covering projection. So P_* is an m-homomorphism. In this case $P_*(M\Pi_n(E, E_0))$ is a subgroup of $M\Pi_n(B, B_0)$. Since $M\Pi_n(B, B_0)$ is an Abelian group, $P_*(M\Pi_n(E, E_0))$ is a normal subgroup of $M\Pi_n(B, B_0)$. Hence, (E, P) is a regular m-covering space. □

7. Conclusion

Multi-valued functions can be thought of as a generalization of single-valued functions. For this reason, many definitions and properties that apply to single-valued functions can be generalized to multi-valued functions. In this paper, we give multi-valued versions of properties related to the homeomorphism notion in general topology and lemmas and theorems related to the covering space in algebraic topology.

By using algebraic topology methods and tools, we aim to gain more knowledge about the homotopy theory of multivalued functions. For this purpose, we are trying to adapt the definitions and properties that already exist in the field of algebraic topology to multivalued functions. We are investigating whether there are any additional conditions required during these applications. As a result, our main goal is to advance research in algebraic topology about multivalued functions.

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References

[1] M.R. Adhikari, Basic Algebraic Topology and its Applications, Springer India (2016) 615.
[2] C.M. Anisiu, Point-to-set-mappings. Continuity, Babes-Bolyai University, Faculty of Mathematics, Research Seminaries, Preprint 3 (1981) 1–100.
[3] C.J.R. Borges, A study of multivalued functions, Pacific J. Math. 23 (1967) 451–461.
[4] G. Choquet, Convergences, Annales Univ. Grenoble 23 (1947-1948) 55–112.
[5] O. Ege, I. Karaca, Digital H-spaces, 3rd International Symposium on Computing in Science and Engineering, ISCSE (2013) 133–138.
[6] O. Ege, I. Karaca, Some properties of digital H-spaces, Turkish Journal of Electrical Engineering and Computer Sciences 24 (2016) 1930–1941.
[7] A. Geletu, Introduction to Topological Spaces and Set-Valued Maps, Lectures Notes (2006).
[8] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, Springer Science and Business Media (1999).
[9] H. Hahn, Reele Funktionen, Akademische Verlagagesellschaft, Leipzig (1932).
[10] L.S. Hill, Properties of certain aggregate functions, Amer. J. Math. 49 (1927) 419–432.
[11] W. Hurewicz, Über stetige blinder von punktmengen, K. Akad. Wet. Amsterdam Proc. 29 (1926) 1014–1017.
[12] S. Kakutani, A generalization of Brouwer's fixed point theorem, Duke Math. J. 8 (1941) 457–459.
[13] I. Karaca, H.S. Denizalti, G. Temizel, On classification of multi-valued functions using multi-homotopy, J. Int. Math. Virtual Inst. 11 (2021) 161–188.
[14] I. Karaca, M. Özkan, Homotopy extension property for multi-valued functions, Math. Moravica, Vol. 27, No. 1 (2023), 1–12.
[15] A.H. Kruse, Introduction to the Theory of Block Assemblages and Related Topics in Topology, University of Kansas (1956).
[16] K. Kuratowski, Les fonctions semi-continues dans l'espace des ensembles fermes, Fund. Math. 18 (1932) 148–159.
[17] D.-W. Lee, Near-rings on digital Hopf groups, Appl. Algebra Engrg. Comm. Comput. 29 (2018) 261–282.
[18] D.-W. Lee, Digital H-spaces and actions in the pointed digital homotopy category, Appl. Algebra Engrg. Comm. Comput. 31 (2020) 149–169.
[19] E. Micheal, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.
[20] J.R. Munkres, Topology, Springer-Verlag, New York (1984).
[21] M. Ozkan, I.Karaca, Multi-(co)homology, Preprint (2022).
[22] V.I. Ponomarev, A new space of closed sets and multivalued continuous mappings of bicompacta, Amer. Math. Soc. Transl. 38 (1964) 95–118.
[23] V.I. Ponomarev, Properties of topological spaces preserved under multivalued continuous mappings, Amer. Math. Soc. Transl. 38 (1964) 119–140.
[24] C.J. Rhee, Homotopy functors determined by set-valued maps, Math. Z. 113 (1970) 154–158.
[25] E.H. Spanier, Algebraic Topolgy, Springer-Verlag, New York XIV (1966).
[26] R.E. Smithson, Some general properties of multi-valued functions, Pacific J. Math. 15 (1965) 681–703.
[27] W.L. Strother, On an open question concerning fixed points, Proc. Amer. Math. Soc. 4 (1953) 988–993.
[28] W.L. Strother, Multi-homotopy, Duke Math. J. 22 (1955) 281–286.
[29] W.L. Strother, Continuous multi-valued functions, Bol. Soc. S. Paulo 10 (1955) 87–120.
[30] A.W. Wilson, On the structure of a continuum limited and irreducible between two points, Amer. J. Math. 48 (1926) 147–168.