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On Loop Functions in the Theory of Topological LA-Semigroups

Hakeem A. Othman[a]

[a]*Department of Mathematics, AL-Qunfudhah University college, Umm Al-Qura University, KSA.*
[a] *Department of Mathematics, Rada'a College of Education and Science, Albaydha University, Albaydha, Yemen*

Abstract. In theory of topological LA-semigroups, we introduce the concepts of LA-loop functions for LA-lifting functions in theory of LA-fibrations. We prove that the LA-lifting function and LA-loop function for any LA-fibration are not necessarily unique. We also restrict the LA-loop function on any idempotent point in any LA-fiber space to get another LA-map which is called a LA-loop restriction.

1. Introduction

Many concepts of Hurewicz fibration, [3], have been important tools in the study of maps in homotopy theory for topological spaces. Cerin, [1] extended this notion into theory of topological semigroups by giving the concepts $S_{\aleph}-$homotopy classes and $S_{\aleph}-$fibration classes.

The concept of left almost semigroups (simply, LA-semigroups) is introduced in 1972 by Kazim and Naseeruddin, [5], which is considered as an algebraic structure between a groupoid and a commutative semigroup. A groupoid (L, a) is called a LA-semigroup the operation $c : L\times L \to L$ from $L\times L$ into L satisfies a left invertive law, i.e., $c(u, c(v, w)) = c(c(u, v), w)$ for all $u, v, w \in L$. Many authors in theory of LA-semigroups gave useful results such as Mushtaq [8] and others ([2, 4, 9, 10]).

In this paper, in Section 2, we give the concepts of LA-homotopy relation and LA-fibration as well as we extend the concept of lifting functions for Hurewicz fibrations into analogical structure in theory of LA-fibrations for topological LA-semigroups. Section 3 gives the concepts of LA-loop functions for LA-fibrations and emphasize on the uniqueness property for the LA- lifting function and LA-loop function for any LA-fibration. That is, we prove that the LA-lifting function and LA-loop function for any LA-fibration are not necessarily unique. Then we prove that any two LA-loop functions for LA-fibration are LA-homotopic. In Section 5, under the notion of the LA-loop function, we restrict the LA-loop function on any idempotent point in the LA-fiber space J_e to get another LA-map which is called a LA-loop restriction.

Here we recall some definitions and theorems which will be used in our work. The set of all continuous functions of topological space L into a space L' is denoted by L'^L. We will use the compact-open topology with the space L'^L which has a subbase

$$\mathcal{B} = \{\mathcal{W}(K, U) : K \text{ is compact set in } L \text{ and } U \text{ is an open set in } L'\},$$

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Email address: haoali@uqu.edu.sa & hakim_albdoie@yahoo.com (Hakeem A. Othman)

where $\mathcal{W}(K,U) = \{f \in L'^L : f(K) \subseteq U\}$. $P(L)$ denotes the space of all continuous functions (called paths) of an interval $I = [0,1]$ with the compact-open topology into L and by $\widetilde{x}$ we mean the constant path at x for all $x \in L$. For every a space L and $x \in L$, by $\mathcal{L}(L,x)$ we mean the subspace of $P(L)$ which is the set of all loops based at x in L.

Theorem 1.1. ([6]) *Let L be a topological space. The subspace $\mathcal{L}(L,x)$ is a closed subspaces of $P(L)$ for every $x \in L$.*

Theorem 1.2. ([7]) *Let L be a compact space. Then for any metrizable space L', the metrizable space L'^L coincides with the compact-open topological space on L'^L.*

Theorem 1.3. ([11]) *If $F : L \times L' \to L''$ is a continuous function then $H : L \to L''^{L'}$ given by $H(x)(y) = F(x,y)$ for all $x \in L, y \in L'$ is continuous. If $H : L \to L''^{L'}$ is continuous and L' is locally compact and regular space then the function $F : L \times L' \to L''$ given by $F(x,y) = H(x)(y)$ for all $x \in L, y \in L'$ is continuous.*

2. LA-regular lifting functions

A *topological LA-semigroup* (L,a) is a a topological space L and Al-semigroup (L,a) such that the operation $a : L \times L \to L$ is a continuous function. A pair (J,c) is called a *LA-subspace* of topological LA-semigroup (L,a) if J is a subspace of L and $a(u,v) = c(u,v)$ for all $u,v \in J$. For any topological LA-semigroup (L,a), by $\mathcal{I}(L,a)$ we mean the idempotent set of LA-semigroup (L,a), that is, $\mathcal{I}(L,a) = \{x \in L : a(x,x) = x\}$.

For any topological LA-semigroup (L,a), it is clear that the pair $(P(L),\widehat{a})$ is a topological LA-semigroup such $\widehat{a} : P(L) \times P(L) \to P(L)$ is a function given by $\widehat{a}(\beta_1,\beta_2)(t) = a(\beta_1(t),\beta_2(t))$ for all $\beta_1,\beta_2 \in P(L), t \in I$. It is clear also to show that the pair $(\mathcal{L}(L,x),\widehat{a})$ is LA-subspace of $(P(L),\widehat{a})$ for every topological LA-semigroup (L,a) and $x \in L$.

The function $f : (L,a) \to (L',b)$ of two topological LA-semigroups (L,a) and (L',b) is called *a LA-map* if the function f is a continuous and $f(a(u,v)) = b(f(u),f(v))$ for all $u,v \in L$. The identity $id_L : (L,a) \to (L,a)$ of topological LA-semigroup (L,a) is a LA-map and the composition of two LA-maps is a LA-map. Let (L,a) and (L',b) be two topological LA-semigroups. The two LA-maps $h,h' : (L,a) \to (L',b)$ are called *LA-homotopic* if there is a LA-map $\mathcal{H} : (L,a) \to (P(L'),\widehat{b})$ such that $\mathcal{H}(x)(0) = h(x)$ and $\mathcal{H}(x)(1) = h'(x)$ for all $x \in L$. The LA-map $\mathcal{H}$ is called a *LA-homotopy* from h into h' and write $h \simeq_{LA} h'$. Similar of the proof Theorem(2.10 in [1] we show that the relation $\simeq_{LA}$ is an equivalence relation on the set of all LA-maps of a topological LA-semigroup (L,a) into a topological LA-semigroup (L',b).

Definition 2.1. The LA-map $f : (L,a) \to (L',a')$ is called a *LA-fibration* if for every topological LA-semigroup (U,b), a LA-map $k : (U,b) \to (L,a)$ and a LA-homotopy $\mathcal{F} : (U,b) \to (P(L'),\widehat{a'})$ with $\mathcal{F}_0 = f \circ k$, there is a LA-homotopy $\mathcal{K} : (U,b) \to (P(L),\widehat{a})$ such that $\mathcal{K}_0 = k$ and $f[\mathcal{K}(u)(t)] = \mathcal{F}(u)(t)$ for all $u \in U, t \in I$.

Let $f : (L,a) \to (L',a')$ be a LA-map and

$$\delta(f) = \{(x,\omega) \in L \times P(L') : f(x) = \omega(0)\}$$

be a subset of $L \times P(L')$. Then one can easy show that the pair $(\delta(f), a \times \widehat{a'})$ is a LA-subspace of a topological LA-semigroup $(L \times P(L'), a \times \widehat{a'})$, where the operation

$$a \times \widehat{a'} : (L \times P(L')) \times (L \times P(L')) \to L \times P(L')$$

defined by

$$(a \times \widehat{a'})[(x_1,\omega_1),(x_2,\omega_2)] = [a(x_1,x_2),\widehat{a'}(\omega_1,\omega_2)],$$

for all $x_1, x_2 \in L, \omega_1, \omega_2 \in P(L')$. Note that for all $(x_1, \omega_1), (x_2, \omega_2) \in \delta(f)$,

$$[\widehat{a'}(\omega_1, \omega_2)](0) = a'(\omega_1(0), \omega_2(0)) = a'(f(x_1), f(x_2)) \qquad = f(a(x_1, x_2)),$$

that is,

$$(a \times \widehat{a'})[(x_1, \omega_1), (x_2, \omega_2)] = [a(x_1, x_2), \widehat{a'}(\omega_1, \omega_2)] \in \delta(f).$$

Definition 2.2. Let $f : (L, a) \to (L', a')$ be a LA-map. Then the function

$$\lambda_f : (\delta(f), a \times \widehat{a'}) \to (P(L), \widehat{a})$$

is said to be a *LA-lifting function* of f if λ_f is a LA-map, $\lambda_f(x, \omega)(0) = x$ and $f[\lambda_f(x, \omega)] = \omega$ for all $(x, \omega) \in \delta(f)$.

We say that the LA-lifting function λ_f is said to be a *LA-regular lifting function* or has *regularity property* if $\lambda_f(x, f \circ \widetilde{x}) = \widetilde{x}$ for all $x \in L$. The LA-map $f : (L, a) \to (L', a')$ is said to be a *LA-regular fibration* or has *regularity property* if it has LA-lifting function with regularity property.

Example 2.3. Let (L, a) and (L', a') be two topological LA-semigroups. It is clear that the usual product $(L' \times L, a' \times a)$ is also a topological LA-semigroup and the usual first projection $\mathcal{P}_1 : (L' \times L, a' \times a) \to (L', a')$ defined by $\mathcal{P}_1(x', x) = x'$ for all $(x', x) \in L' \times L$ is a LA-map and LA-fibration. Define the LA-map

$$\lambda_{\mathcal{P}_1} : (\delta(\mathcal{P}_1), (a' \times a) \times \widehat{a'}) \to (P(L' \times L), \widehat{a' \times a})$$

by $\lambda_{\mathcal{P}_1}[(x', x), \omega](t) = (\omega(t), x)$ for all $t \in I, ((x', x), \omega) \in \delta(\mathcal{P}_1)$. Note that for $((x', x), \omega) \in \delta(\mathcal{P}_1)$,

$$\lambda_{\mathcal{P}_1}[(x', x), \omega](0) = (\omega(0), x) = (\mathcal{P}_1(x', x), x) = (x', x);$$

and for every $((x', x), \omega) \in \delta(\mathcal{P}_1)$,

$$\mathcal{P}_1[\lambda_{\mathcal{P}_1}((x', x), \omega)](t) = \mathcal{P}_1(\omega(t), x) = \omega(t),$$

for all $t \in I$. Hence $\lambda_{\mathcal{P}_1}$ is a LA-lifting function for $\mathcal{P}_1$. Note that for $(x', x) \in L' \times L$,

$$\begin{aligned} \lambda_{\mathcal{P}_1}[(x', x), \mathcal{P}_1 \circ \widetilde{(x', x)}](t) &= [(\mathcal{P}_1 \circ \widetilde{(x', x)})(t), x] = (\mathcal{P}_1(x', x), x) \\ &= (x', x) = \widetilde{(x', x)}(t), \end{aligned}$$

for all $t \in I$. Hence $\lambda_{\mathcal{P}_1}$ is a LA-regular lifting function for $\mathcal{P}_1$.

Theorem 2.4. *The LA-map $f : (L, a) \to (L', a')$ is a LA-fibration if and only if it has LA-lifting function.*

Proof. Suppose that a LA-map $f : (L, a) \to (L', a')$ has a LA-lifting function

$$\lambda_f : (\delta(f), a \times \widehat{a'}) \to (P(L), \widehat{a}).$$

Let (U, b) be any topological LA-semigroup, $k : (U, b) \to (L, a)$ be any LA-map and $\mathcal{F} : (U, b) \to (P(L'), \widehat{a'})$ be a LA-homotopy with $\mathcal{F}_0 = f \circ k$. For every $u \in U$, let ω_u be a path: $t \to \mathcal{F}(u)(t)$. Then by lemma above, the function $\mathcal{H} : (U, b) \to (P(L'), \widehat{a'})$ defined by $\mathcal{H}(u) = \omega_u$ for all $u \in U$, is a LA-map. Hence define the LA-homotopy $\mathcal{K} : (U, b) \to (P(L), \widehat{a})$ by

$$\mathcal{K}(u)(t) = \lambda_f[k(u), \omega_u](t)$$

for all $u \in U, t \in I$. Hence $\mathcal{K}_0 = k$ and $f \circ \mathcal{K} = \mathcal{F}$. Therefore f is LA-fibration.

Conversely, let f be a LA-fibration and $(U, b) = (\delta(f), a \times \widehat{a'})$. Define LA-homotopy $\mathcal{F} : (U, b) \to (P(L'), \widehat{a'})$ by $\mathcal{F}(x, \omega)(t) = \omega(t)$ for all $(x, \omega) \in \delta(f), t \in I$ and a LA-map $k : (U, b) \to (L, a)$ by $k(x, \omega) = x$ for all $(x, \omega) \in \delta(f)$. Since $\mathcal{F}_0 = f \circ k$, then there is a LA-homotopy $\mathcal{K} : (U, b) \to (P(L), \widehat{a})$ such that $\mathcal{K}_0 = k$ and $f[\mathcal{K}(u)(t)] = \mathcal{F}(u)(t)$ for all $u \in U, t \in I$. Hence define the LA-lifting function $\lambda_f : (\delta(f), a \times \widehat{a'}) \to (P(L), \widehat{a})$ for f by

$$\lambda_f(x, \omega)(t) = \mathcal{K}(x, \omega)(t),$$

for all $(x, \omega) \in \delta(f), t \in I$. $\square$

3. The LA-loop functions

Let $f : (L, a) \to (L', a')$ be a LA-fibration with $\mathcal{I}(L', a') \neq \emptyset$ and $e \in \mathcal{I}(L', a')$ be a chosen base idempotent point in (L', a'). From now we will mean by the symbol J_e the set $J_e = f^{-1}(e)$. Consider J_e a topological space with the relative topology of L. Since e is an idempotent point in (L', a'), we observe that for all $x_1, x_2 \in J_e$,

$$f(a(x_1, x_2)) = a'(f(x_1), f(x_2)) = a'(e, e) = e.$$

That is, $a(x_1, x_2) \in J_e$ and (J_e, a) will be a LA-subspace of (L, a) and is called a *LA-fiber space over e*.

Definition 3.1. Let $f : (L, a) \to (L', a')$ be a LA-fibration with $\mathcal{I}(L', a') \neq \emptyset$. Let $e \in \mathcal{I}(L', a')$ be a chosen base idempotent point in (L', a'). The *LA-loop map or function of LA-fibration f* which induced by a LA-lifting function λ_f is a function

$$LO_{\lambda_f} : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$$

which is defined by

$$LO_{\lambda_f}(\alpha, x) = \lambda_f(x, \alpha)(1)$$

for all $x \in J_e, \alpha \in \mathcal{L}(L', e)$.

The following theorem shows that LA-loop functions for any LA-fibration f are well-defined LA-maps between two topological semigroups $(\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a)$ and (J_e, a).

Theorem 3.2. *Let $f : (L, a) \to (L', a')$ be a LA-fibration. Then the LA-loop function LO_{λ_f} is LA-map.*

Proof. Firstly, we prove that the LA-loop function $LO_{\lambda_f} : \mathcal{L}(L', e) \times J_e \to J_e$ is continuous. It is clear that the space $\mathcal{L}(L', e) \times J_e$ is a subspace of $\delta(f)$ and the LA-lifting function $\lambda_f : \delta(f) \to P(L)$ is a continuous. By Theorem 1.3, note that the function $\mathcal{H} : \mathcal{L}(L', e) \times J_e \times I \to L$ defined by

$$\mathcal{H}((\gamma, x), t) = \lambda_f(x, \gamma)(t) \text{ for all } (\gamma, x) \in \mathcal{L}(L', e) \times J_e, t \in I,$$

is a continuous. Hence a restriction $\mathcal{H}|_{\mathcal{L}(L',e)\times J_e\times\{1\}}$ of $\mathcal{H}$ on $\mathcal{L}(L', e) \times J_e \times \{1\}$ is also continuous. Since

$$f[LO_{\lambda_f}(\gamma, x)] = f[\lambda_f(x, \gamma)(1)] = \gamma(1) = e$$

for all $(\gamma, x) \in \mathcal{L}(L', e) \times J_e$, then $LO_{\lambda_f}(\gamma, x) \in J_e$. Hence note that $LO_{\lambda_f} = \mathcal{H}|_{\mathcal{L}(L',e)\times J_e\times\{1\}}$ is a well defined as mapping : $(\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$. Note that

$$\begin{aligned} LO_{\lambda_f}\{(\widehat{a'} \times a)[(\gamma_1, x_1), (\gamma_2, x_2)]\} &= \lambda_f\{(\widehat{a'} \times a)[(x_1, \gamma_1), (x_2, \gamma_2)]\}(1) \\ &= \widehat{a}[\lambda_f(x_1, \gamma_1), \lambda_f(x_2, \gamma_2)](1) \\ &= a[\lambda_f(x_1, \gamma_1)(1), \lambda_f(x_2, \gamma_2)(1)] \\ &= a[LO_{\lambda_f}(\gamma_1, x_1), LO_{\lambda_f}(\gamma_2, x_2)] \end{aligned}$$

for all $(\gamma_1, x_1), (\gamma_2, x_2) \in \mathcal{L}(L', e) \times J_e$. Then LO_{λ_f} is a LA-map. □

The symbol $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ means the LA-regular fibration $f : (L, a) \to (L', a')$ with a LA-loop function

$$LO_{\lambda_f} : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to (J_e, a),$$

induced by the LA-lifting function $\lambda_f : (\delta(f), a \times \widehat{a'}) \to P(L, a)$ and with the LA-fiber space $J_e = f^{-1}(e)$, where $e \in \mathcal{I}(L', a') \neq \emptyset$ is a chosen base idempotent point in (L', a').

Remark 3.3. In any LA-fibration $f:(L,a)_{J_e} \to (L',a')_{LO_{\lambda_f}}$, by the LA-regularity of LA-lifting function λ_f, the LA-loop function LO_{λ_f} has the following property:

$$LO_{\lambda_f}(\widetilde{e},x) = \lambda_f(x,\widetilde{e})(1) = \lambda_f(x, fa'\widetilde{x})(1) = \widetilde{x}(1) = x$$

for all $x \in J_e$.

Example 3.4. In Example(2.3), the first LA-fibration $\mathcal{P}_1 : (L' \times L, a' \times a) \to (L', a')$ has a LA-regular lifting function

$$\lambda_{\mathcal{P}_1} : (\triangle\mathcal{P}_1, (a' \times a) \times \widehat{a'}) \to P(L' \times L, a' \times a)$$

defined by

$$\lambda_{\mathcal{P}_1}[(b,x),\gamma](t) = (\gamma(t),x) \text{ for all } t \in I, [(b,x),\gamma] \in \delta(\mathcal{P}_1).$$

Then the LA-loop function $LO_{\lambda_{\mathcal{P}_1}} : (\mathcal{L}(L',e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$ for LA-fibration $\mathcal{P}_1$ induced by $\lambda_{\mathcal{P}_1}$ defined by

$$LO_{\lambda_{\mathcal{P}_1}}(\gamma,x) = x \text{ for all } x \in J_e, \gamma \in \mathcal{L}(L',e).$$

Example 3.5. Let (L',a') be topological semigroup with $\mathcal{I}(L',a') \neq \emptyset$. Let $e \in \mathcal{I}(L',a')$ be a base idempotent point in (L',a'). The LA-fibration $f : (P(L'),\widehat{a'}) \to (L',a')$ defined by $f(\gamma) = \gamma(1)$ for all $\gamma \in P(L')$ has LA-regular lifting function $\lambda_f : (\delta(f), \widehat{a'} \times \widehat{a'}) \to (P(P(L')), \widehat{\widehat{a'}})$ defined by

$$[\lambda_f(\gamma,\omega)(s)](t) = \begin{cases} \gamma(2t/2-s), & \text{for } \ 0 \leq t \leq 1-s/2, \\ \omega(2t+s-2), & \text{for } \ 1-s/2 \leq t \leq 1, \end{cases}$$

for all $s \in I$ and $(\gamma,\omega) \in \delta(f)$. An LA-loop function

$$LO_{\lambda_f} : (\mathcal{L}(L',e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$$

for the LA-fibration induced by λ_f defined by

$$LO_{\lambda_f}(\omega,\gamma)(t) = \begin{cases} \gamma(2t), & \text{for } \ 0 \leq t \leq 1/2, \\ \omega(2t-1), & \text{for } \ 1/2 \leq t \leq 1. \end{cases}$$

Since λ_f is not LA-regular, we get that $LO_{\lambda_f}(\widetilde{e},\gamma) \neq \gamma$.

Definition 3.6. A topological LA-semigroup (L,a) is called a *LA-absolute retract* for normal topological LA-semigroups (E,c) if for every closed LA-subspace (J,c) of (E,c), any LA-map $h : (J,c) \to (L,a)$ has a extension LA-map $H : (E,c) \to (L,a)$.

Let (J,a) be any LA-subspace of topological LA-semigroup (L,a) (L,a). The $(\mathcal{K}^1_A, \mathcal{K}^2_L, L')-$maps for (J,c) in (L,a) w.r.t (L',a') consists two LA-maps $\mathcal{K}^1_J : (J,a) \to (P(L'),\widehat{a'})$ and $\mathcal{K}^2_L : (L,a) \to (L',a')$ such that $\mathcal{K}^1_J(j)(0) = \mathcal{K}^2_L(j)$ for all $j \in J$.

Lemma 3.7. *Let $f : (L,a) \to (L',a')$ be a LA-regular fibration with a LA-absolute retract (L,a). Let (J,a'') be any closed LA-subspace of a normal space (L'',a''). If there is $(\mathcal{K}^1_J, \mathcal{K}^2_{L''}, L)-$maps such that $f[\mathcal{K}^1_J(j)(t)] = f[\mathcal{K}^1_J(j)(0)]$ for all $j \in J, t \in I$ then $\mathcal{K}^1_J$ extends to the LA-map $\mathcal{H} : (L'',a'') \to (P(L),\widehat{a})$ with $\mathcal{H}_0 = \mathcal{K}^2_{L''}$ and $f \circ \mathcal{H}(x) = f[\widetilde{\mathcal{H}(x)}(0)]$ for all $x \in L''$.*

Proof. Since (J, a'') is a closed LA-subspace of a normal space (L'', a'') and (L, a) is a LA-absolute retract, then the LA-map $\mathcal{K}_J^1$ ca be extended to LA-map $\mathcal{K} : (L'', a'') \to (P(L), \widehat{a})$ such that $\mathcal{K}_0 = \mathcal{K}_{L''}^2$. For every $\omega \in (P(L), \widehat{a})$ and $r \in I$, define the path ω_r in $(P(L), \widehat{a})$ by $\omega_r(t) = \omega[r(1-t)]$ for $t \in I$. Then by Lemma 3.4,

$$(P(L), \widehat{a}) \to P(P(L), \widehat{a}) : (\omega)(r) \to \omega_r$$

is a LA-map. Hence we ca define the LA-map $\mathcal{H} : (L'', a'') \to (P(L), \widehat{a})$ by

$$\mathcal{H}(u)(t) = \lambda_f[\mathcal{K}(u)(t), f \circ \mathcal{K}(u)_t](1)$$

for all $u \in L'', t \in I$. We prove that $\mathcal{H}_0 = \mathcal{K}_{L''}^2$. Since λ_f is regular and $\mathcal{K}_J^1$ extends to $\mathcal{K}$, then for $u \in L''$,

$$\mathcal{H}(u)(0) = \lambda_f(\mathcal{K}(u)(0), f \circ \mathcal{K}(u)_0)(1) = [\widetilde{\mathcal{K}(u)(0)}](1) = \mathcal{K}(u)(0) = \mathcal{K}_{L''}^2(u).$$

Now we prove that $\mathcal{H}$ is a extension of $\mathcal{K}_J^1$. Since $\mathcal{K}$ is a extension for $\mathcal{K}_J^1$, then

$$\begin{aligned} f[\mathcal{K}(j)_r(t)] &= f[\mathcal{K}(j)(r(1-t))] = f[\mathcal{K}_J^1(j)(r(1-t))] \\ &= f[\mathcal{K}_J^1(j)(0)] = f[\mathcal{K}_J^1(j)(r)] = f[\mathcal{K}(j)(r)], \end{aligned}$$

for all $j \in J$ and $r, t \in I$. Hence by the LA-regularity of λ_f we get that

$$\mathcal{H}(j)(r) = \mathcal{K}(j)(r) = \mathcal{K}_J^1(j)(r)$$

for all $r \in I, j \in J$. Hence $\mathcal{H}$ is a extension for $\mathcal{K}_J^1$. Finally

$$f[\mathcal{H}(u)(t)] = f[\lambda_f(\mathcal{K}(u)(t), f \circ \mathcal{K}(u)_t)(1)] = f[\mathcal{K}(u)(0)] = f[\mathcal{K}_{L''}^2(u)] = f[\mathcal{H}(u)(0)]$$

for all $u \in L'', t \in I$. $\square$

Lemma 3.8. *Let $f : (L, a) \to (L', a')$ be a LA-regular fibration with a LA-absolute retract (L, a) and (J, a'') be any closed LA-subspace of a normal space (L'', a''). Let $g_1, g_2 : (J, a'') \to (L, a)$ and $\mathcal{R} : (J, a'') \to (P(L), \widehat{a})$ be three LA-maps such that $\mathcal{R}_0 = g_1$, $\mathcal{R}_1 = g_2$ $f \circ \mathcal{R}(j) = f[\widetilde{\mathcal{R}(j)}(0)]$ for all $j \in J$. If g_1 has a extension LA-map $\mathcal{G}_1$ to all of (L'', a''), then g_2 has a extension LA-map $\mathcal{G}_2$ to all of (L'', a''). Also there is a LA-homotopy $\mathcal{H} : (L'', a'') \to (P(L), \widehat{a})$ between K_1 and K_2 such that $\mathcal{H}$ is a extension of $\mathcal{R}$ and $f \circ \mathcal{H}(x) = f[\widetilde{\mathcal{H}(x)}(0)]$ for all $x \in L''$.*

Proof. Let $\mathcal{G}_1$ be an extension LA-map of g_1 to all of (L'', a''), that is,

$$\mathcal{G}_1(j) = g_1(j) = \mathcal{R}(j)(0)$$

for all $j \in J$. Then there exists $(\mathcal{R}, \mathcal{G}_1, L)-$maps with

$$f \circ \mathcal{R}(j) = f[\widetilde{\mathcal{R}(j)}(0)]$$

for all $j \in J$. By theorem above $\mathcal{R}$ extends to LA-map $\mathcal{H} : (L'', a'') \to (P(L), \widehat{a})$ with $\mathcal{H}_0 = \mathcal{G}_1$ and $f \circ \mathcal{H}(x)(t) = f[\widetilde{\mathcal{H}(x)}(0)]$ for all $x \in L''$. Define the LA-map $\mathcal{G}_2 : (L'', a'') \to (L, a)$ by $\mathcal{G}_2(x) = \mathcal{H}_1(x)$ for all $x \in L''$. Then $\mathcal{G}_1$ is LA-homotopic to $\mathcal{G}_2$ by $\mathcal{H}$ and

$$\mathcal{G}_2(j) = \mathcal{H}(j)(1) = g_2(j)$$

for all $j \in J$. That is, $\mathcal{G}_2$ is a extension of g_2. $\square$

Theorem 3.9. *Let $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ be a LA-fibration with metrizable compact spaces L and L'. Let $LO : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$ be a LA-map such that*

$$LO_{\lambda_f} \simeq_{LA} LO \quad \text{and} \quad LO(\widetilde{e}, x) = x \text{ for all } x \in J_e.$$

If (L, a) is a LA-absolute retract, then there is at least one LA-regular lifting function L'_f for f that induces LO.

Proof. By $LO_{\lambda_f} \simeq_{LA} LO$ there is a LA-homotopy

$$\mathcal{R} : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to P(J_e, a)$$

such that $\mathcal{R}(\gamma, x)(0) = LO_{\lambda_f}(\gamma, x)$ and $\mathcal{R}(\gamma, x)(1) = LO(\gamma, x)$ for all $x \in J_e, \gamma \in \mathcal{L}(L', e)$. LA-map LO_{λ_f} can extend to a LA-map

$$LO'_{\lambda_f} : (P(L', e) \times J_e, \widehat{a'} \times a) \to (L, a)$$

defined by

$$LO'_{\lambda_f}(\gamma, x) = \lambda_f(x, \gamma)(1) \text{ for all } x \in J_e, \gamma \in P(L', e)$$

with the property $f[LO'_{\lambda_f}(\gamma, x)] = \gamma(1)$ for all $x \in J_e, \gamma \in P(L', e)$. Then we apply Lemma(3.8) on the LA-fibration f as follows: Since L and L' are metrizable spaces, then by Theorem(1.2), we get that $P(L') \times L$ is normal space and by Theorem(1.1), $\mathcal{L}(L', e)$ is a closed in $P(L')$ and J_e is closed in L. That is, $\mathcal{L}(L', e) \times J_e$ is closed in $P(L') \times L$. Then LO can be extended to a LA-map $LO' : (P(L', e) \times J_e, \widehat{a'} \times a) \to (L, a)$ with

$$f[LO'(\gamma, x)] = \gamma(1) \quad \text{for all } \ x \in J_e, \gamma \in \mathcal{L}(L', e).$$

For $\gamma \in P(L', a')$ and $s \in I$, define two paths $\gamma_s, \gamma'_s \in P(L', a')$ by

$$\gamma_s(t) = \gamma(st) \text{ and } \gamma'_s(t) = \gamma(s + (1 - s)t)$$

for all $t \in I$. Note that the two functions

$$P(L', a') \to P(P(L'), \widehat{a'}) : (\gamma)(s) \to \gamma_s \quad \text{and} \quad (\gamma)(s) \to \gamma'_s$$

are LA-maps. Hence define a LA-homotopy

$$\mathcal{H}' : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to P(J_e, a)$$

by

$$\mathcal{H}'(\gamma, x)(t) = \lambda_f[LO'(\gamma_t, x), \gamma^t](1)$$

for all $t \in I, x \in J_e, \gamma \in \mathcal{L}(L', e)$. By the hypothesis and the LA-regularity for λ_f we get that

$$\begin{aligned}
\mathcal{H}'(\gamma, x)(0) &= \lambda_f[LO'(\gamma_0, x), \gamma^0](1) \\
&= \lambda_f[LO'(\widetilde{e}, x), \gamma](1) \\
&= \lambda_f(x, \gamma)(1) = LO_{\lambda_f}(\gamma, x)
\end{aligned}$$

for all $x \in J_e, \gamma \in \mathcal{L}(L', e)$,

$$\begin{aligned}
\mathcal{H}'(\gamma, x)(1) &= \lambda_f[LO'(\gamma_1, x), \gamma^1](1) \\
&= \lambda_f[LO'(\gamma, x), \widetilde{e}](1) \\
&= LO'(\gamma, x) = LO(\gamma, x)
\end{aligned}$$

for all $x \in J_e, \gamma \in \mathcal{L}(L', e)$ and

$$\begin{aligned}
\mathcal{H}'(\widetilde{e}, x)(t) &= \lambda_f[LO'((\widetilde{e})_t, x), (\widetilde{e})^t](1) \\
&= \lambda_f[LO'(\widetilde{e}, x), \widetilde{e}](1) \\
&= \lambda_f(x, \widetilde{e})(1) = x
\end{aligned}$$

for all $x \in J_e$.

Let $A_1 = [\mathcal{L}(L', e) \times J_e]$, $A_2 = [(\widetilde{L'} \times L) \cap \delta'(f)]$, and $A = A_1 \cup A_2$, where $\widetilde{L'} = \{\widetilde{x'} : x' \in L'\}$ and $\delta'(f) = \{(\gamma, x) \in P(L') \times L : \gamma(0) = f(x)\}$. Now we apply Lemma(3.7) on the LA-fibration f as follows: Note that A is a closed subspace of $\delta'(f)$ and $\delta'(f)$ is subspace of normal space $P(L') \times L$; We have $(G^1_A, G^2_{\delta'(f)}, L)$−maps given by

$$G^2_{\delta'(f)}(\gamma, x) = \lambda_f(x, \gamma)(1)$$

for all $(\gamma, x) \in \delta'(f)$ and

$$G^1_A(\gamma, x)(t) = \begin{cases} \mathcal{H}'(\gamma, x)(t), & \text{for} \quad ((\gamma, x), t) \in A_1 \times I, \\ x, & \text{for} \quad ((\gamma, x), t) \in A_2 \times I; \end{cases}$$

Note that for $(\gamma, x) \in A$,

$$\begin{aligned} f[G^1_A((\gamma, x))(t)] &= \begin{cases} f[\mathcal{H}'(\gamma, x)(t)], & \text{for} \quad ((\gamma, x), t) \in A_1 \times I, \\ f(x), & \text{for} \quad ((\gamma, x), t) \in A_2 \times I; \end{cases} \\ &= \begin{cases} f(x), & \text{for} \quad ((\gamma, x), t) \in A_1 \times I, \\ f(x), & \text{for} \quad ((\gamma, x), t) \in A_2 \times I; \end{cases} \\ &= f[G^1_A(\gamma, x)(0)]; \end{aligned}$$

By the hypothesis, (L, a) is a LA-absolute retract, then G^1_A extends to LA-map $\mathcal{H} : (\delta'(f), \widehat{a'} \times a) \to P(L, a)$ with $\mathcal{H}_0 = G^2_{\delta'(f)}$ and $f \circ \mathcal{H}(\gamma, x)] = f[\mathcal{H}(\widetilde{\gamma, x})(0)]$ for all $(\gamma, x) \in \delta'(f)$.

Now define a LA-map $L'_f : (\delta(f), a \times \widehat{a'}) \to P(L, a)$ by

$$L'_f(x, \gamma)(t) = \mathcal{H}(\gamma_t, x)(1) \quad \text{for all} \;\; (x, \gamma) \in \delta(f), t \in I.$$

Note that: For $(x, \gamma) \in \delta(f)$, we have

$$L'_f(x, \gamma)(0) = \mathcal{H}(\gamma_0, x)(1) = \mathcal{H}(\widetilde{\gamma(0)}, x)(1) = G^1(\widetilde{\gamma(0)}, x)(1) = x;$$

For $(x, \gamma) \in \delta(f)$ and $t \in I$, we have

$$\begin{aligned} [f \circ L'_f(x, \gamma)](t) &= f[\mathcal{H}(\gamma_t, x)(1)] = f[\mathcal{H}(\gamma_t, x)(0)] = f[\lambda_f(x, \gamma_t)(1)] \\ &= \gamma_t(1) = \gamma(t); \end{aligned}$$

For $x \in L$,

$$L'_f(x, fa'\widetilde{x})(t) = \mathcal{H}(\widetilde{f(x)}_t, x)(1) = G^1_A(\widetilde{f(x)}_t, x)(1) = x.$$

Hence the function L'_f is LA-regular lifting function of f. Then

$$\begin{aligned} L'_f(x, \gamma)(1) &= \mathcal{H}(\gamma_1, x)(1) = \mathcal{H}(\gamma, x)(1) = G^1_A(\gamma, x)(1) \\ &= \mathcal{H}'(\gamma, x)(1) = LO(\gamma, x). \end{aligned}$$

for all $(\gamma, x) \in \mathcal{L}(L', e) \times J_e$. That is, LO is a LA-loop function of f inducing by L'_f. □

Corollary 3.10. *Let $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ be a LA-fibration with metrizable compact spaces L and L'. If (L, a) is LA-absolute retract then the LA-lifting function λ_f and LA-loop function LO_{λ_f} for f are not necessarily unique.*

Proof. For $\gamma \in P(L', a')$ and $s \in I$, define two paths $\gamma_s, \gamma'_s \in P(L', a')$ by

$$\gamma_s(t) = \gamma(st) \quad \text{and} \quad \gamma'_s(t) = \gamma[s + (1-s)t]$$

for all $t \in I$. Note that the two functions

$$P(L', a') \to P(P(L'), \widehat{a'}) : (\gamma)(s) \to \gamma_s \quad \text{and} \quad (\gamma)(s) \to \gamma'_s$$

are LA-maps. Hence define a LA-map $\mathcal{F} : (\delta(f), a \times \widehat{a'}) \to P(L, a)$ by

$$\mathcal{F}(x, \gamma)(t) = \lambda_f[\lambda_f(x, \gamma_t)(1), \gamma'_t](1)$$

for all $t \in I, (x, \gamma) \in \delta(f)$. Now define a LA-map $LO : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to (J_e, a)$ by

$$LO(\gamma, x) = \mathcal{F}(x, \gamma)(1/2)$$

for all $(\gamma, x) \in \mathcal{L}(L', e) \times J_e$. Note that for $(\gamma, x) \in \mathcal{L}(L', e) \times J_e$,

$$\begin{aligned}
\mathcal{F}(x, \gamma)(0) &= \lambda_f[\lambda_f(x, \gamma_0)(1), \gamma'_0](1) \\
&= \lambda_f[\lambda_f(x, \widetilde{\gamma(0)})(1), \gamma](1) \\
&= \lambda_f[\lambda_f(x, \widetilde{fa'x})(1), \gamma](1) \\
&= \lambda_f(x, \gamma)(1) \\
&= LO_{\lambda_f}(\gamma, x),
\end{aligned}$$

that is, $LO_{\lambda_f} \simeq_{LA} LO$ and for $x \in J_e$,

$$\begin{aligned}
LO(\widetilde{e}, x) &= \mathcal{F}(x, \widetilde{e})(1/2) = \lambda_f[\lambda_f(x, \widetilde{e}_{1/2})(1), \widetilde{e}'_{1/2}](1) \\
&= \lambda_f[\lambda_f(x, \widetilde{e})(1), \widetilde{e}](1) = x.
\end{aligned}$$

Hence by Theorem(3.9), we can get a LA-lifting function $L \neq \lambda_f$of f given by the form in the proof of Theorem(3.9). That is, the LA-loop function and LA-lifting function for f are not necessarily unique. □

We proved in the above corollary that the LA-loop function is not necessarily unique and in the following theorem we clarify that the LA-loop function for any LA-fibration is uniquely determined up to a LA-homotopy class.

Theorem 3.11. *If LA-fibration $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ has two LA-lifting functions λ_f and λ'_f, then LO_{λ_f} and $LO_{\lambda'_f}$ are LA-homotopic.*

Proof. For $\gamma \in P(L', a')$ and $s \in I$, define two paths γ_s and γ'_s in $P(L', a')$ by

$$\gamma_s(t) = \gamma(st) \quad \text{and} \quad \gamma'_s(t) = \gamma(s + (1-s)t)$$

for all $t \in I$. Note that the two functions

$$P(L', a') \to P(P(L'), \widehat{a'}) : (\gamma)(s) \to \gamma_s \quad \text{and} \quad (\gamma)(s) \to \gamma'_s$$

are LA-maps. Hence define a LA-homotopy

$$\mathcal{H} : (\mathcal{L}(L', e) \times J_e, \widehat{a'} \times a) \to P(J_e, a)$$

by

$$\mathcal{H}(\gamma, x)(t) = \lambda_f[\lambda'_f(x, \gamma_t)(1), \gamma^t](1)$$

for all $t \in I, x \in J_e, \gamma \in \mathcal{L}(L', e)$. Since λ_f and λ'_f are regulars, then

$$\begin{aligned} \mathcal{H}(\gamma, x)(0) &= \lambda_f[\lambda'_f(x, \gamma_0)(1), \gamma^0](1) = \lambda_f[\lambda'_f(x, \widehat{e})(1), \gamma](1) \\ &= \lambda_f(x, \gamma)(1) = LO_{\lambda_f}(\gamma, x), \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}(\gamma, x)(1) &= \lambda_f[\lambda'_f(x, \gamma_1)(1), \gamma^1](1) = \lambda_f[\lambda'_f(x, \gamma)(1), \widehat{e}](1) \\ &= \lambda'_f(x, \gamma)(1) = LO_{\lambda'_f}(\gamma, x), \end{aligned}$$

for all $x \in J_e$, $\gamma \in \mathcal{L}(L', e)$. Hence LO_{λ_f} and $LO_{\lambda'_f}$ are LA-homotopic. □

4. LA-loop restriction functions

A topological LA-semigroup (L, a) is called a *pathwise LA-connected* if for each $x_1, x_2 \in L$, there is topological LA-semigroup (L', a') with $\mathcal{I}(L', a') \neq \emptyset$ and a LA-homotopy $H : (L', a') \to (P(L), \widehat{a})$ such that $H(x)(0) = x_1$ and $H(x)(1) = x_2$ for all $x \in L'$. It is clear that any pathwise LA-connected semigroup (L, a), a space L is pathwise connected.

Definition 4.1. Let $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ be a LA-fibration. For $e_o \in J_e$, the map $\mathcal{R} : (\mathcal{L}(L', e), \widehat{a'}) \to (J_e, a)$ defined by $\mathcal{R}(\gamma) = LO_{\lambda_f}(\gamma, e_o)$ for all $\gamma \in \mathcal{L}(L', e)$ is called a *LA-loop restriction for the LA-fibration f* if $e_o \in \mathcal{I}(J_e, a)$ and we denote it by $\mathcal{RE}^{e_o}$.

Example 4.2. In Example(3.4), if we take $a = a' = \pi_i$. Then the idempotent set of $(L' \times L, a' \times a)$ is $L' \times L$. Hence for every $e_o \in J_e$, we give the LA-loop restriction $\mathcal{RE}_1^{e_o} : (\mathcal{L}(L', e), \widehat{a'}) \to (J_e, a)$ for the LA-fibration $\mathcal{P}_1$ by $\mathcal{RE}_1^{e_o}(\gamma) = e_o$ for all $\gamma \in \mathcal{L}(L', e)$.

The following theorem clarifies that the LA-loop restriction for any LA-fibration is a well-defined LA-map

Theorem 4.3. *Let $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ be a LA-fibration. Then every LA-loop restriction for the LA-fibration f is LA-map.*

Proof. Let $\mathcal{RE}^{e_o} : (\mathcal{L}(L', e), \widehat{a'}) \to (J_e, a)$ be a LA-loop restriction for the LA-fibration f. We observe that a LA-loop restriction $\mathcal{RE}^{e_o}$ is restriction of the LA-loop function LO_{λ_f} on $\mathcal{L}(L', e) \times \{e_o\}$. That is,

$$\mathcal{RE}^{e_o} = LO_{\lambda_f} \mid_{\mathcal{L}(L',e)\times\{e_o\}} .$$

Hence the LA-loop restriction $\mathcal{RE}^{e_o}$ is continuous. Since $e_o \in \mathcal{I}(J_e, a)$, then for $\gamma_1, \gamma_2 \in \mathcal{L}(L', e)$,

$$\begin{aligned} \mathcal{RE}^{e_o}[\widehat{a'}(\gamma_1, \gamma_2)] &= LO_{\lambda_f}[\widehat{a'}(\gamma_1, \gamma_2), e_o] \\ &= LO_{\lambda_f}[\widehat{a'}(\gamma_1, \gamma_2), a(e_o, e_o)] \\ &= LO_{\lambda_f}\{(\widehat{a'} \times a)[(\gamma_1, e_o), (\gamma_2, e_o)]\} \\ &= a[LO_{\lambda_f}(\gamma_1, e_o), LO_{\lambda_f}(\gamma_2, e_o)] \\ &= a[\mathcal{RE}^{e_o}(\gamma_1), \mathcal{RE}^{e_o}(\gamma_2)]. \end{aligned}$$

Hence $\mathcal{RE}^{e_o}$ is a LA-map. □

Theorem 4.4. *Let $f : (L, a)_{J_e} \to (L', a')_{LO_{\lambda_f}}$ be a LA-fibration with LA-loop restrictions $\mathcal{RE}^{e_1}, \mathcal{RE}^{e_2} : (\mathcal{L}(L', e), \widehat{a'}) \to (J_e, a)$. If (J_e, a) is a pathwise LA-connected semigroup then $\mathcal{RE}^{e_1}$ and $\mathcal{RE}^{e_2}$ are LA-homotopic.*

Proof. Since (J_e, a) is pathwise LA-connected semigroup, then there is a topological LA-semigroup (L'', c) with $\mathcal{I}(L'', c) \neq \emptyset$ and a LA-homotopy $G : (L'', c) \to (P(J_e), \widehat{a})$ such that

$$G(x)(0) = e_1 \quad \text{and} \quad G(x)(1) = e_2$$

for all $x \in L''$. Define the LA-homotopy $\mathcal{H} : (\mathcal{L}(L', e), \widehat{a'}) \to (P(J_e), \widehat{a})$ by

$$\mathcal{H}(\gamma)(t) = LO_{\lambda_f}[\gamma, G(e_o)(t)]$$

for all $\gamma \in \mathcal{L}(L', e), t \in I$, where $e_o \in \mathcal{I}(L'', c)$ be a fixed point. Then we have

$$\mathcal{H}(\gamma)(0) = LO_{\lambda_f}[\gamma, G(e_o)(0)] = LO_{\lambda_f}(\gamma, e_1) = \mathcal{RE}^{e_1}(\gamma),$$

and

$$\mathcal{H}(\gamma)(1) = LO_{\lambda_f}[\gamma, G(e_o)(1)] = LO_{\lambda_f}(\gamma, e_2) = \mathcal{RE}^{e_2}(\gamma),$$

for all $\gamma \in \mathcal{L}(L', e)$. Note that for $\gamma_1, \gamma_2 \in \mathcal{L}(L', e)$ and for all $t \in I$,

$$\begin{aligned}
\mathcal{H}[\widehat{a'}(\gamma_1, \gamma_2)](t) &= LO_{\lambda_f}[\widehat{a'}(\gamma_1, \gamma_2), G(e_o)(t)] \\
&= LO_{\lambda_f}[\widehat{a'}(\gamma_1, \gamma_2), G(c(e_o, e_o))(t)] \\
&= LO_{\lambda_f}\{\widehat{a'}(\gamma_1, \gamma_2), \widehat{a}[G(e_o), G(e_o)](t)\} \\
&= LO_{\lambda_f}\{\widehat{a'}(\gamma_1, \gamma_2), a[G(e_o)(t), G(e_o)(t)]\} \\
&= LO_{\lambda_f}\{(\widehat{a'} \times a)[(\gamma_1, G(e_o)(t)), (\gamma_2, G(e_o)(t))]\} \\
&= a\{LO_{\lambda_f}[\gamma_1, G(e_o)(t)], LO_{\lambda_f}[\gamma_2, G(e_o)(t)]\} \\
&= a[\mathcal{H}(\gamma_1)(t), \mathcal{H}(\gamma_2)(t)] \\
&= \widehat{a}[\mathcal{H}(\gamma_1), \mathcal{H}(\gamma_2)](t).
\end{aligned}$$

Then $\mathcal{H}$ is a LA-homotopy between $\mathcal{RE}^{e_1}$ and $\mathcal{RE}^{e_2}$. □

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