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Rings of Quotients of the Ring Consisting of Ordered Field Valued Continuous Functions with Countable Range

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Abstract. For a zero-dimensional topological space X and a totally ordered field F with interval topology, $C_c(X,F)$ denotes the ring consisting of ordered field-valued continuous functions with countable range on X. This article aims to study and investigate the rings of quotients of $C_c(X,F)$. $Q_c(X,F)$ (resp. $q_c(X,F)$), the maximal (resp. classical) ring of quotients of $C_c(X,F)$ as a modified countable analogue of $Q(X)$ (resp. $q(X)$), the maximal (resp. classical) ring of quotients of $C(X)$ are characterized. It is proved that $Q_c(S)$, the maximal ring of quotients of the subring S of $C_c(X,F)$, is a subring of $Q_c(X,F)$ if and only if every dense ideal in S has dense cozero-set in X. Also, the coincidence of rings of quotients of $C_c(X,F)$ is investigated. We show that $q_c(X,F) = C_c(X,F)$ if and only if the set of non-units and zero-divisors in $C_c(X,F)$ coincide if and only if X is almost CP_F-space. Finally, it is shown that the fixed ring of quotients and the cofinite ring of quotients of $C_c(X)$ coincide if and only if $\mathrm{Hom}(M_p^c) = C_c(X_p)$ for every $p \in X$.

1. Introduction

Unless otherwise mentioned any topological space X is zero-dimensional, any ring is commutative with identity and F is a totally ordered field with the interval topology. $C(X)$ ($C^*(X)$) denotes the ring of all real-valued continuous (bounded) functions on a space X. A ring $A(X)$ lying between $C^*(X)$ and $C(X)$ is called an intermediate ring. A class of ideals in intermediate rings of continuous functions is introduced in [5]. The subring of $C(X)$ consisting of those functions with countable (respectively, finite) image, which is denoted by $C_c(X)$ (respectively, $C^F(X)$) is an $\mathbb{R}$-subalgebra of $C(X)$. The subring $C_c^*(X)$ of $C_c(X)$ consists of bounded elements of $C_c(X)$. The rings $C_c(X)$ and $C^F(X)$ are introduced and studied in [11, 12]. It is shown in [11] that for any topological space X, there exists a zero-dimensional space Y which is a continuous image of X and $C_c(X) \cong C_c(Y)$. For more discussion on some topics related to this area, one can refer to articles [14–16, 20, 26–29]. Let F be a totally ordered field, equipped with its ordered topology and let $C(X,F)$ be the set of all F-valued continuous functions on X. This latter set becomes a commutative lattice ordered ring with identity, if the operations are defined pointwise on X. For more information in this regard, we refer the reader to articles [1, 3, 4]. For each $f \in C(X,F)$, the *zero-set* of f, denoted by $Z(f)$, is the set of zeros of f and $coz(f) = X \setminus Z(f)$ is the *cozero-set* of f and $Z(X,F)$ is the set consisting of all zero-set in X. We recall

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that a *zero-dimensional space* is a Hausdorff space with a base consisting of clopen (closed-open) sets. For a zero-dimensional topological space X and a totally ordered field F, we let

$$C_c(X,F) = \{f \in C(X,F) : f(X) \text{ is a countable subset of } F\}.$$

It is easy to check that $C_c(X,F)$ is an F-subalgebra of $C(X,F)$. This ring has been fully investigated in [2]. Given $f \in C_c(X,F)$, let $|f| : X \to F$ be defined by $|f|(x) := |f(x)| = f(x) \vee -f(x)$ for each $x \in X$. Therefore, $|-f| = |f| \geq 0$, $|f| = 0$ gives $f = 0$ and further $|f|$ is continuous, in fact, $|f| \in C_c(X,F)$. Notice that $F \supseteq \mathbb{Q}$, the set of rational numbers. So for every $f, g \in C_c(X,F)$, we have $f \vee g = \frac{f+g+|f-g|}{2} \in C_c(X,F)$, also, $f \wedge g \in C_c(X,F)$ (note, $f \wedge g = -(-f \vee -g)$). Hence, $C_c(X,F)$ is a subring as well as a sublattice of $C(X,F)$. For an ideal I of $C_c(X,F)$, we let $Z(I) = \bigcap_{f\in I} Z(f)$ and $coz(I) = X \setminus Z(I)$ which is equal to $\bigcup_{f\in I} coz(f)$.

Suppose A and B are commutative rings with identity. For an ideal I of A, we will denote by $\mathrm{Hom}_A(I)$, briefly, $\mathrm{Hom}(I)$, the set of all A-module homomorphisms of I to A. It is immediate that by the definition $r.\varphi = r\varphi$ such that $(r\varphi)(a) = r\varphi(a)$ $(a \in A)$, $\mathrm{Hom}(I)$ turns into an A-module. If I and J are two ideals in A such that $I \subseteq J$, then $\mathrm{Hom}(J) \subseteqq \mathrm{Hom}(I)$. An ideal D of A is called *dense* in A whenever $\mathrm{Ann}_A(D) = \{r \in A : rd = 0,$ for every $d \in D\} = 0$. An element $a \in A$ is called *regular* whenever its annihilator in A is 0. Clearly, $a \in A$ is regular if and only if it is a non-zero-divisor. An ideal in A is called a *regular ideal* if it contains a regular element. So a regular ideal is a dense ideal. Also, a principal ideal (a) is dense if and only if a is regular. A commutative ring is called *semiprime* (or *reduced*) if it has no nonzero nilpotent elements. In [10, Definition 1.4], $B(\supseteq A)$ is called a *ring of quotients* of A, provided that for every $b \in B$, the ideal $b^{-1}A := \{a \in A : ab \in A\}$ is dense in B, that is to say, for each $0 \neq b' \in B$ there exists $a \in A$ such that $ba \in A$ and $b'a \neq 0$.

Theorem 1.1. ([10, Theorem 1.5]) *Suppose $B \supseteq A$. If A is semiprime, then B is a ring of quotients of A if and only if $b(b^{-1}A) \neq 0$ for all nonzero $b \in B$, i.e., for $0 \neq b \in B$ there exists $a \in A$ such that $0 \neq ba \in A$.*

$Q(A)$ and $q(A)$ denote the maximal and the classical ring of quotients of A respectively. By [10, 1.2],

$$q(A) = \Big\{\frac{c}{d} : c \in A, \text{ and } d \text{ is a non-zero-divisor in } A\Big\}.$$

For more details on the structure of these rings, one can refer to [17]. Notice that $A \leqq q(A) \leqq Q(A)$. By $Q_c(X,F)$ and $q_c(X,F)$, we mean the maximal ring of quotients and the classical ring of quotients of $C_c(X,F)$. We should also remind that $Q(X)$, the maximal ring of quotients, and $q(X)$ the classical ring of quotients of $C(X)$ are fully characterized in [10], and, in fact, we are following the methods in [10] in our investigation.

We will need the next lemma which is a consequence of [17, 2.3 Exercise 3].

Lemma 1.2. *Let $B \supseteq A$ be a ring of quotients of A. Then $Q(A) = Q(B)$.*

A brief outline of this paper is as follows. In Section 2, rings of quotients of $C_c(X,F)$ and $C_c^*(X,F)$ are investigated. We show that $Q_c(X,F)$ (resp. $q_c(X,F)$) as a modified countable analogue of $Q(X)$ (resp. $q(X)$) based on rings of continuous functions with values in F are characterized. It is proved that $Q_c(S)$, the maximal ring of quotients of the subring S of $C_c(X,F)$, is a subring of $Q_c(X,F)$ if and only if every dense ideal in S has dense cozero-set in X. Section 3 deals with the coincidence of rings of quotients of the $C_c(X,F)$, in particular, we are interested in cases when some of these rings of quotients coincide with $C_c(X,F)$, itself. It is shown that for a zero-dimensional space X and a totally ordered field F, $q_c(X,F) = C_c(X,F)$ if and only if the set of non-units and zero-divisors in $C_c(X,F)$ coincide if and only if X is almost CP_F-space. At the end of the paper, the *fixed ring of quotients* and the *cofinite ring of quotients* of $C_c(X)$ are investigated. We show that the fixed ring of quotients and the cofinite ring of quotients of $C_c(X)$ coincide if and only if $\mathrm{Hom}(M_p^c) = C_c(X_p)$ for every p in X.

2. Rings of quotients of $C_c(X,F)$ and $C_c^*(X,F)$

In this section, rings of quotients of $C_c(X,F)$ and $C_c^*(X,F)$ are investigated. We show these two rings have the same maximal (resp. classical) ring of quotients. In more general, we show the maximal (resp. classical) rings of quotients of $C_c(X,F)$ and each intermediate ring $A_c(X,F)$ coincide. Based on rings of continuous functions on dense open sets (resp. dense σ-clopen sets) with values in F, the maximal (resp. classical) ring of quotients of $C_c(X,F)$ is characterized.

Notation 2.1. *If $f,g \in C_c(X,F)$ and g is a unit, then we sometimes use $\frac{f}{g}$ instead of fg^{-1}.*

Proposition 2.2. *Let $f \in C_c(X,F)$ and $f^{-1} : \mathrm{coz}(f) \to F \setminus \{0\} \subseteq F$ be defined by $f^{-1}(x) = (f(x))^{-1}$. Then f^{-1} is continuous.*

Proof. Since F is a topological field, the function $g : F \setminus \{0\} \to F \setminus \{0\}$ which $\alpha \mapsto \alpha^{-1}(= \frac{1}{\alpha})$ is continuous. The result is now obtained by the fact that $f^{-1} = g \circ (f|_{coz(f)})$. □

Corollary 2.3. *The units of $C_c(X,F)$ are precisely those $f \in C_c(X,F)$ for which $\mathrm{coz}(f) = X$, i.e., $Z(f) = \emptyset$.*

A function $f \in C_c(X,F)$ is called *bounded* if $|f(x)| \leq r$ for some $0 < r \in F$ and each $x \in X$. So each element of F is a bounded function on X. Let us put

$$C_c^*(X,F) = \Big\{f \in C_c(X,F) : f \text{ is bounded}\Big\}.$$

Then $F \subseteq C_c^*(X,F)$. Moreover, $C_c^*(X,F)$ is an F-subalgebra of $C_c(X,F)$.

Let $Z_c(X,F) = \Big\{Z(f) : f \in C_c(X,F)\Big\}$ and $Z_c^*(X,F) = \Big\{Z(g) : g \in C_c^*(X,F)\Big\}$. These two latter sets coincide. To see this, for $f \in C_c(X,F)$, let $g = |f|(1+f^2)^{-1} = \frac{|f|}{1+f^2}$. Then $g \in C_c^*(X,F)$, in fact, $0 \leq g \leq 1$ and further $Z(f) = Z(g)$. So $Z_c(X,F) \subseteq Z_c^*(X,F)$ (note, $a \leq a^2+1$ for all $a \in F$ and therefore $a(1+a^2)^{-1} = \frac{a}{a^2+1} \leq 1$).

It is stated in [2, Theorem 2.10] that X is zero-dimensional if and only if the family $Z_c(X,F)$ is a base for closed sets in X. Suppose for two subsets A and B of X, there exits $f \in C_c(X,F)$ such that $f(A) = 0$ and $f(B) = 1$. Now, if we let $g = (0 \vee f) \wedge 1$, then $g \in C_c^*(X,F)$ and we also have $g(A) = 0$ and $g(B) = 1$.

In [2, Definition 2.9], a Hausdorff space X is called *countably completely F-regular*, briefly, *CCFR space*, if given a closed set K in X and a point $x \in X \setminus K$, there exists $f \in C_c(X,F)$ such that $f(x) = 0$ and $f(K) = 1$.

Definition 2.4. Two subsets A and B of X are said to be *countably completely F-separated*, briefly, *CCF separated* (from one another) in X if there exists a function f in $C_c^*(X,F)$ such that $0 \leq f \leq 1$, $f(A) = 0$ and $f(B) = 1$.

Corollary 2.5. *Two subsets A and B of X are CCF separated in X if and only if there are disjoint zero-sets $Z, Z' \in Z_c(X,F)$ such that $A \subseteq \mathrm{int}_X Z$ and $B \subseteq \mathrm{int}_X Z'$, or equivalently, there exits $h \in C_c^*(X,F)$ such that $A \subseteq int_X Z(h)$ and $B \subseteq int_X Z(1-h)$.*

We state the next result without proof that it can be accomplished by following the arguments in [13, Theorem 3.11(a)].

Proposition 2.6. *In a CCFR space, any two disjoint closed sets, one of which is compact, are CCF separated.*

A ring $A_c(X,F)$ lying between $C_c^*(X,F)$ and $C_c(X,F)$ is called an *intermediate ring*.

Proposition 2.7. *Let $A_c(X,F)$ be an intermediate ring. Then $A_c(X,F)$ and $C_c(X,F)$ have the same classical ring of quotients and the same maximal ring of quotients.*

Proof. First, we show that $C_c(X,F)$ is a ring of quotients of $C_c^*(X,F)$. Note that $C_c(X,F)$, and therefore each subring is semiprime (i.e., $f \in C_c(X,F)$ and $f^2 = 0$, implies that $f = 0$). Also, $a \leq a^2+1$ for all $a \in F$ and therefore $a(1+a^2)^{-1} \leq 1$. Now, let $0 \neq f \in C_c(X,F)$ and $g = (1+f^2)^{-1} = \frac{1}{1+f^2}$. Then $0 < g \leq 1$ and $|f|g \leq 1$. So $0 \neq fg \in C_c^*(X,F)$, equivalently, $g \in f^{-1}C_c^*(X,F)$, so $f(f^{-1}C_c^*(X,F)) \neq 0$. Applying Theorem 1.1, we get the result. It can now be concluded that $C_c(X,F)$ is also a ring of quotients of $A_c(X,F)$ because for a chain of rings $A \leqq B \leqq C$; C is a ring of quotients of A if and only if C is a ring of quotients of B and B is a ring of quotients of A, see [10, page 8]. To show the coincidence of the classical rings of quotients, let $\frac{f}{g} \in q(A_c(X,F))$. Then

g is a non-zero-divisor in $A_c(X, F)$ as well as in $C_c(X, F)$. So $q(A_c(X, F)) \leqq q_c(X, F)$. Now, let $\frac{f}{g} \in q_c(X, F)$ and notice that $\frac{f}{g} = \frac{f}{1+f^2+g^2} \diagup \frac{g}{1+f^2+g^2}$. Since both $\frac{f}{1+f^2+g^2}$ and $\frac{g}{1+f^2+g^2}$ belong to $C_c^*(X, F) \leqq A_c(X, F)$, and further the latter function is a non-zero-divisor, we get $\frac{f}{g} \in q(A_c(X, F))$. Hence, $q_c(X, F) \leqq q(A_c(X, F))$. Finally, the coincidence of the maximal rings of quotients follows from Lemma 1.2. □

Definition 2.8. A subset S of a space X is called C_cF*-embedded* (resp. C_c^*F*-embedded*) in X if every function in $C_c(S, F)$ (resp. $C_c^*(S, F)$) can be extended to a function in $C_c(X, F)$ (resp. $C_c^*(X, F)$).

Thus, in this terminology, $C_c\mathbb{R}$-embedded (resp. $C_c^*\mathbb{R}$-embedded) is precisely C_c-embedded (resp. C_c^*-embedded) introduced in [15].

Every C_cF-embedded subset of X is C_c^*F-embedded. To see this, let $S \subseteq X$ be C_cF-embedded and $f \in C_c^*(S, F)$. Take $m \in F$ such that $|f(x)| \leq m$ for each $x \in S$. Let $\bar{f}$ be the extension of f to X and let $f^* = (-m \vee \bar{f}) \wedge m$. Then $f^* \in C_c^*(X, F)$ and $f^*|_S = f$. But the converse is not true in general, see Example 3.7.

Proposition 2.9. *If $V \subseteq X$ is dense and C_cF-embedded in X, then $C_c(X, F) \cong C_c(V, F)$ as F-algebras.*

Theorem 2.10. *Let V be a dense open subset of X. Then $C_c(V, F)$ is a ring of quotients of $C_c(X, F)$. Moreover, $Q_c(X, F) \cong Q_c(V, F)$.*

Proof. By the above proposition, the density of V in X implies that $C_c(X, F)$ is isomorphic to a subring of $C_c(V, F)$ via the map: $f \mapsto f|_V$. According to Theorem 1.1, we must show if $0 \neq f \in C_c(V, F)$, then $f(f^{-1}C_c(X, F)) \neq 0$. So let $0 \neq f \in C_c(V, F)$. Take $v \in V$ such that $f(v) \neq 0$. Since X is zero-dimensional and $v \notin X \setminus V$, by [2, Theorem 2.10], there exists $g \in C_c(X, F)$ such that $g(v) \neq 0$ and $X \setminus V \subseteq \text{int}_X Z(g)$. Now, let us define the function h as follows:

$$h(x) = \begin{cases} f(x)g(x) & x \in V, \\ 0 & x \in X \setminus V. \end{cases}$$

We claim that $h \in C_c(X, F)$. To show this, it is enough to show that h is continuous on $X \setminus V = \overline{V} \cap \overline{X \setminus V} = \partial V$. Let $x \in X \setminus V$ and $(x_\lambda)_{\lambda \in \Lambda} \subseteq V$ be a net that converges to x. Then for some $\lambda_0 \in \Lambda$ and each $\lambda \geq \lambda_0$, we have $x_\lambda \in \text{int}_X Z(g)$ and therefore $h(x_\lambda) = 0$. So h is continuous, in fact, $h \in C_c(X, F)$. This yields h is an extension of fg to X. Hence, $g \in f^{-1}C_c(X, F)$ and $fg \neq 0$. As for the second part, we use Lemma 1.2. □

Corollary 2.11. *If V is a dense cofinite subset of X, then $C_c(V, F)$ is a ring of quotients of $C_c(X, F)$.*

Proposition 2.12. *Every dense open subset of X is C_cF-embedded (resp. C_c^*F-embedded) if and only if every open subset of X is C_cF-embedded (resp. C_c^*F-embedded).*

Proof. Let V be an open subspace of X and $f \in C_c(V, F)$ (resp. $f \in C_c^*(V, F)$). We must extend f to X. Notice that $V \cup (X \setminus \overline{V})$ is a dense open subset of X. Define $\overline{f}(x) = f(x)$ for each $x \in V$ and $\overline{f}(x) = 0$ for each $x \in X \setminus \overline{V}$. Since the subspace $V \cup (X \setminus \overline{V})$ is disconnected, $\bar{f}$ is continuous. By the hypothesis, $\bar{f}$ can be extended to X. The converse is obvious. □

Proposition 2.13. *Let $V \subseteq X$. Then V is open if and only if $V = \text{coz}(I)$, for some ideal I of $C_c(X, F)$.*

Proof. Clearly, $\emptyset = coz(0)$ and $X = coz(1)$. Suppose $V \neq \emptyset$ and put $I = \left\{f \in C_c(X, F) : coz(f) \subseteq V\right\}$. It is easy to verify that I is an ideal in $C_c(X, F)$ and $coz(I) = \bigcup_{f \in I} coz(f) \subseteq V$. To show equality, let $x \in V$. Then by Proposition 2.6 (or [2, Theorem 2.10]), there exists $f \in C_c(X, F)$ such that $f(x) = 1$ and $X \setminus V \subseteq \text{int}_X Z(f)$. So $x \in coz(f)$ and $f \in I$. Hence, $V \subseteq coz(I)$. □

The next example shows that not all open, even dense, sets are cozero-sets of elements of $C_c(X, F)$. Notice that by [2, Theorem 4.1], every zero-set $Z \in Z_c(X, F)$ is a G_δ-set.

Example 2.14. Let $X^* = X \cup \{x\}$ (where $x \notin X$) be the one-point compactification of an uncountable discrete space X and F, a totally ordered field. Clearly, X is a dense open set in X^*. We claim that X is not a cozero-set with respect to an element of $C_c(X^*, F)$. Otherwise, $\{x\}$ is a zero-set, hence a G_δ-set, i.e., $\{x\} = \bigcap_{n=1}^{\infty} V_n$, where each V_n is an open set in X^*. Since $X^* \setminus V_n$ is finite, we obtain $X = X^* \setminus \{x\} = \bigcup_{n=1}^{\infty}(X^* \setminus V_n)$ is countable, a contradiction. Consequently, we reach the claim. But if we let $I = \{f \in C_c(X^*, F) : \mathrm{coz}(f) \text{ is finite}\}$, then it easily follows that I is an ideal in $C_c(X^*, F)$, and $X = coz(I)$.

Lemma 2.15. *Let S be a subring of $C_c(X, F)$. Then for any ideal D of S, we have* $\mathrm{Hom}_S(D) \subseteq C_c(\mathrm{coz}(D), F)$.

Proof. The proof is more or less the same as the proof of [10, Lemma 2.5]. □

We remark that $Hom_S(D) = C_c(coz(D), F)$ if and only if for each $f \in C_c(coz(D), F)$ and each $g \in D$, fg has an extension to X. Also, in Lemma 2.15 the inclusion may be strict, see Example 3.11.

Proposition 2.16. *Let $f \in C_c(X, F)$ and $g \in C_c(\mathrm{coz}(f), F)$. Then $g \in \mathrm{Hom}(I)$; for some ideal I of $C_c(X, F)$.*

Proof. We first note that $1 + g^2$ is a unit and $(1 + g^2)^{-1} = \frac{1}{1+g^2} \leq 1$. Let $\bar{f} : X \to F$ be defined by

$$\bar{f}(x) = \begin{cases} \frac{f(x)}{1+g^2(x)} & x \in coz(f), \\ 0 & x \in Z(f). \end{cases}$$

Then $\bar{f} \in C_c(X, F)$. Set $I = (\bar{f})$. It is claimed that $g \in \mathrm{Hom}(I)$. Take $h\bar{f} \in I$, where $h \in C_c(X, F)$ and define $\bar{g} : I \to C_c(X, F)$ as follows:

$$\bar{g}(h\bar{f}) = \begin{cases} \frac{g(x)h(x)f(x)}{1+g^2(x)} & x \in coz(f), \\ 0 & x \in Z(f). \end{cases}$$

Since $\frac{g}{1+g^2}$ is bounded and $fh \in C_c(X, F)$, we obtain $\bar{g} \in C_c(X, F)$. This means that $gh\bar{f}$ can be continuously extended to X, i.e., $g \in \mathrm{Hom}(I)$. □

We call a subring of $C_c(X, F)$ *essential* if it intersects every nonzero ideal of $C_c(X, F)$ nontrivially. In the next result, we observe that an ideal of an essential subring of $C_c(X, F)$ is dense in that subring if and only if its cozero-set is dense in X.

Proposition 2.17. *Let S be an essential subring of $C_c(X, F)$. Then, an ideal D of S is dense in S if and only if* $\mathrm{coz}(D)$ *is dense in X.*

Proof. (⇒) Suppose D is dense in S and V is an open set in X such that $coz(D) \cap V = \emptyset$. We claim that $V = \emptyset$. By Proposition 2.13, there exists an ideal I of $C_c(X, F)$ such that $V = coz(I)$. Since $coz(D) \subseteq Z(I) = \bigcap_{f \in I} Z(f)$, we conclude that $f(coz(D)) = 0$ for every $f \in I$. Hence, $fd = 0$ for every $d \in D$. So $fD = 0$ for every $f \in I$. If $I \neq (0)$, then by the assumption, there must exist $0 \neq f \in I \cap S$, which is absurd since D is dense in S. Therefore, $I = (0)$ and so $V = \emptyset$.

(⇐) Suppose $f \in \mathrm{Ann}_S(D)$. Then for every $d \in D$, $fd = 0$ and therefore $coz(fd) = coz(f) \cap coz(d) = \emptyset$. Hence, $coz(D) \subseteq Z(f)$. Since $coz(D)$ is a dense subset of X, we have $Z(f) = X$ or $f = 0$. Thus $\mathrm{Ann}_S(D) = \{0\}$, i.e., D is dense in S. □

Corollary 2.18. *An ideal D in $C_c(X, F)$ is dense in $C_c(X, F)$ if and only if* $\mathrm{coz}(D)$ *is dense in X.*

Let S be a subring of $C_c(X, F)$ and $\mathcal{D}_0$ (resp. $\mathcal{D}$) be the family of all dense (resp. regular) ideals in S. Notice that $\mathcal{D}_0$ and $\mathcal{D}$ are closed under multiplication, i.e., if D_1 and D_2 are dense (resp. regular) ideals in S, then D_1D_2 is also a dense (resp. regular) ideal in S; and $(d) \in \mathcal{D}_0$ if and only if $(d) \in \mathcal{D}$. Furthermore, $d \in D$ gives $\mathrm{Hom}(D) \subseteq \mathrm{Hom}((d))$. Then $Q_c(S)$ (resp. $q_c(S)$), the maximal (resp. classical) ring of quotients of S has been described based on the S-modules $\mathrm{Hom}(D)$, where $D \in \mathcal{D}_0$ (resp. $D \in \mathcal{D}$), i.e.,

$$Q_c(S) = \varinjlim_{D \in \mathcal{D}_0} Hom(D), \text{ and } q_c(S) = \varinjlim_{D \in \mathcal{D}} Hom(D). \tag{1}$$

Observe that $Q_c(S)$ and $q_c(S)$ may be thought of as $\bigcup \mathrm{Hom}(D)$, where we identify $\varphi_1 \in \mathrm{Hom}(D_1)$ and $\varphi_2 \in \mathrm{Hom}(D_2)$ whenever φ_1 and φ_2 agree on D_1D_2 (see [10, 1.7], and also [19]). Therefore,

$$Q_c(S) = \bigcup \Big\{Hom(D) : D \in \mathcal{D}_0\Big\}, \text{ and } q_c(S) = \bigcup \Big\{Hom(D) : D \in \mathcal{D}\Big\}. \tag{2}$$

In [10, Theorem 2.6], $Q(X)$ (resp. $q(X)$) is determined in terms of the rings of continuous functions on dense open sets (resp. dense cozero-sets concerning to elements of $C(X)$) in X. Also, $Q_c(X)$ and $q_c(X)$, the maximal and the classical ring of quotients of $C_c(X)$ are characterized in [20, Theorem 2.12] and [6, Theorem 2.2] respectively. We are now ready to express the next theorem which generalizes these characterizations.

Theorem 2.19. *Let $Q_c(X,F)$ (resp. $q_c(X,F)$) be the maximal (resp. classical) ring of quotients of $C_c(X,F)$. Then*

(*i*) $Q_c(X,F) = \bigcup \Big\{C_c(V,F) : V \text{ is a dense open subset of } X\Big\}$.
(*ii*) $q_c(X,F) = \bigcup \Big\{C_c(\mathrm{coz}(f),F) : f \in C_c(X,F) \text{ and } \overline{\mathrm{coz}(f)} = X\Big\}$.

Proof. (*i*) Let $\mathbb{V}_0 = \Big\{V : V \text{ is a dense open subset of } X\Big\}$ and $Q_0 = \bigcup \Big\{C_c(V,F) : V \in \mathbb{V}_0\Big\}$. Note that $\mathbb{V}_0$ is a filter base, i.e., it is closed under finite intersection. An equivalence relation on Q_0 is obtained by defining $f \in C_c(V_1,F)$ and $g \in C_c(V_2,F)$ to be equivalent if and only if the restriction of f and g to $V_1 \cap V_2$ are equal. It is known that the above relation turns Q_0 into a commutative ring with identity. Now, we claim that $Q_c(X,F) = Q_0$. Combining (2) in the above discussion (where $S = C_c(X,F)$), Lemma 2.15 and Corollary 2.18, we get

$$Q_c(X,F) = \bigcup \Big\{Hom(D) : D \in \mathcal{D}_0\Big\} \leqq \bigcup \Big\{C_c(coz\,(D)) : D \in \mathcal{D}_0\Big\} \leqq Q_0.$$

On the other hand, we know from Theorem 2.10 that, if $V \subseteq X$ is dense and open, then $C_c(V,F)$ is a ring of quotients of $C_c(X,F)$, so Q_0 is too. Thus Q_0 is contained in $Q_c(X,F)$ and therefore $Q_0 = Q_c(X,F)$.
(*ii*) Recall that a function $f \in C_c(X,F)$ is regular if and only if $\overline{coz(f)} = X$. Also, an ideal D in $C_c(X,F)$ is regular if it contains a regular element. Let $f \in D$ be a regular element. Then for the regular principal ideal (f), we have $(f) \subseteq D$ and hence $\mathrm{Hom}(D) \subseteq \mathrm{Hom}((f)) \subseteq C_c(coz(f),F)$. So using relation (2), we get

$$\begin{aligned} q_c(X,F) &= \bigcup \Big\{\mathrm{Hom}(D) : D \in \mathcal{D}, \text{i.e, } D \text{ is a regular ideal in } C_c(X,F)\Big\} \\ &= \bigcup \Big\{\mathrm{Hom}((f)) : f \text{ is a regular element of } C_c(X,F)\Big\}. \end{aligned}$$

Now, let $\mathbb{V} = \Big\{coz(f) : f \in C_c(X,F) \text{ and } \overline{coz(f)} = X\Big\}$ and $Q = \bigcup \Big\{C_c(coz(f),F) : coz(f) \in \mathbb{V}\Big\}$. Note that $\mathbb{V}$ and Q have the same properties as $\mathbb{V}_0$ and Q_0 respectively, i.e., $\mathbb{V}$ is a filter base and Q is a commutative ring with identity. Applying Lemma 2.15, we obtain that $q_c(X,F) \leqq Q$. As for the reverse inclusion, suppose that $g \in Q$. So $g \in C_c(coz(f),F)$, where $coz(f) \in \mathbb{V}$. Since $\overline{coz(f)} = X$; f is regular. Now, according to Proposition 2.16, we consider $\bar{f}$ and set $D = (\bar{f})$. Since $\bar{f}$ is regular, D is also a regular ideal and further $g \in \mathrm{Hom}(D)$. Thus $g \in q_c(X,F)$, i.e., $Q \leqq q_c(X,F)$. □

Theorem 2.20. *let X be zero-dimensional and F be either uncountable or a countable subfield of $\mathbb{R}$. Then, a subset of X is a cozero-set (of a function lying in $C_c(X,F)$) if and only if it is a σ-clopen set in X.*

Proof. (⇒) Suppose F is uncountable and V is a cozero-set in X. Clearly, if $V=\emptyset$ or $V = X$, then V is a σ-clopen set. Now, let $V = \mathrm{coz}(f)$, where $0 \neq f \in C_c(X,F)$ is a non-unit. Since $\mathrm{coz}(f) = \mathrm{coz}(f^2)$, we can suppose that $f \geq 0$, i.e., $f(X) = \{0, a_1, a_2, \ldots, a_n, \ldots\}$, where $a_n > 0$ for all n. Since F is uncountable, the set $\{x \in F : x \geq 0\}$ is also uncountable, so there exist $0 < r_n, r_{n+1} \in F \setminus f(X)$ such that $a_n \in (r_n, r_{n+1})$. Now, $[r_n, r_{n+1}] \cap f(X) = (r_n, r_{n+1}) \cap f(X)$ is a clopen set in $f(X)$ and therefore $f^{-1}\Big((r_n, r_{n+1}) \cap f(X)\Big)$ is a clopen set in X. So

$$V = coz(f) = \bigcup_{n\in\mathbb{N}} f^{-1}\Big((r_n, r_{n+1}) \cap f(X)\Big) = \bigcup_{n\in\mathbb{N}} f^{-1}\Big((r_n, r_{n+1})\Big) = \bigcup_{n\in\mathbb{N}} f^{-1}\Big([r_n, r_{n+1}]\Big)$$

is a σ-clopen set in X.

Next, suppose F is a countable subfield of $\mathbb{R}$. So F contains $\mathbb{Q}$ as well as a countable subset of $\mathbb{R} \setminus \mathbb{Q}$. Suppose $F = \{a_n : n \in \mathbb{N}\}$ and let $r_n, r_{n+1} \in \mathbb{R} \setminus F$ such that $a_n \in (r_n, r_{n+1})$. Then $[r_n, r_{n+1}] \cap F = (r_n, r_{n+1}) \cap F$ is a clopen subset of F, and, $F = \bigcup_{n=1}^{\infty} \left([r_n, r_{n+1}] \cap F\right)$, i.e., F is a σ-clopen set. So $F \setminus \{0\}$ is a σ-clopen set (note, this is true for any countable subset of $\mathbb{R}$). Hence, for $f \in C(X,F) = C_c(X,F)$; $\operatorname{coz}(f) = f^{-1}(F \setminus \{0\})$ is a σ-clopen set in X.

($\Leftarrow$) Here it suffices that F is infinite and therefore contains $\mathbb{Q}$. Let V be a σ-clopen set in X. Then $V = \bigcup_{n=1}^{\infty} V_n$, where each V_n is a clopen set in X. Without loss of generality, we may assume that the sets V_n are disjoint. (To see this, it suffices to take $G_1 = V_1$ and $G_n = V_n \setminus \bigcup_{i=1}^{n-1} V_i$, for $n \geq 2$. So each G_n is a clopen set and $V = \bigcup_{n=1}^{\infty} G_n$.) Next, consider the function $f : X \to F$ as follows:

$$f(x) = \begin{cases} \frac{1}{n} & x \in V_n, \\ 0 & x \in X \setminus V. \end{cases}$$

Evidently, $f \in C_c(X,F)$ and $V = coz(f)$. $\square$

Example 2.21. Let $X = \mathbb{R} = F$ and $f(x) = x$. Then $f \in C(\mathbb{R}) = C(X,F)$ and F is uncountable. But $\operatorname{coz}(f) = \mathbb{R} \setminus \{0\}$ is not a σ-clopen set in X. So in the above theorem, the "countability of $f(X)$" is necessary.

Corollary 2.22. *let X be zero-dimensional and F be either uncountable or a countable subfield of $\mathbb{R}$. Then*

$$q_c(X,F) = \varinjlim_{D \in \mathcal{D}} \operatorname{Hom}(D) = \bigcup \left\{C_c(\operatorname{coz}(f),F) : f \in C_c(X,F) \text{ and } \overline{\operatorname{coz}(f)} = X\right\}$$
$$= \bigcup \left\{C_c(V,F) : V \text{ is a dense } \sigma\text{-clopen set in } X\right\}.$$

Proof. Using Theorems 2.19 and 2.20, we get the result. $\square$

In the next example, we observe that $q_c(X,F)$ is a proper subring of $Q_c(X,F)$.

Example 2.23. Let X^* be the space in Example 2.14. Remember that X is not a cozero-set in X^*. So the only dense cozero-set in X^* is itself. Hence, $q_c(X^*,F) = C_c(X^*,F)$. On the other hand, the only dense open sets in X^* are X and X^*. Thus $Q_c(X^*,F) = C_c(X,F) \cup C_c(X^*,F)$ and hence $Q_c(X^*,F) = C_c(X,F)$, by Theorem 2.10. To show that $q_c(X^*,F) \subsetneqq Q_c(X^*,F)$, it suffices to show that X is not C_cF-embedded in X^*. To see this, let Y be an infinite countable subset of X. Define a function $f : X \to F$ by $f(x) = 1$ for each $x \in Y$ and $f(x) = -1$ for each $x \in X \setminus Y$. Then $f \in C_c(X,F)$ while it cannot be extended to X^*. Hence, $C_c(X^*,F) \subsetneqq C_c(X,F)$.

Lemma 2.24. *Let S be a subring of $C_c(X,F)$. Then $Q_c(S)$ is a subring of a homomorphic image of a subring of $Q_c(X,F)$.*

Proof. Let us put

$$\mathcal{D}_0 = \left\{D : D \text{ is a dense ideal in } S\right\}, \text{ and } \mathcal{C}_1 = \left\{\operatorname{coz}(D) : D \in \mathcal{D}_0\right\}, \text{ and let}$$

$$\mathcal{C}_2 = \left\{V \subseteq X : V \text{ is dense open in } X \text{ and } V \supseteq \operatorname{coz}(D) \text{ for some } D \in \mathcal{D}_0\right\}.$$

Remind that $\mathcal{D}_0$ is closed under multiplication and $coz(D_1) \cap coz(D_2) = coz\,(D_1 \cap D_2)$. So $\mathcal{C}_1$ and $\mathcal{C}_2$ are filter base on X. Let

$$Q_1 = \bigcup \left\{C_c(\operatorname{coz}(D),F) : D \in \mathcal{D}_0\right\}, \text{ and } Q_2 = \bigcup \left\{C_c(V,F) : V \in \mathcal{C}_2\right\}.$$

Notice that Q_1 and Q_2 are commutative rings with identity, in fact, they are F-algebras. (An equivalence relation on Q_2 is obtained by defining $f \in C_c(V_1,F)$ and $g \in C_c(V_2,F)$ to be equivalent if and only if they agree on $V_1 \cap V_2$, and so on for Q_1.) Applying relation (2) and Lemma 2.15, we obtain that $Q_c(S) \leqq Q_1$.

Also, $Q_2 \leqq Q_c(X, F)$, by Theorem 2.19. Now, let $\psi : Q_2 \to Q_1$ be defined by $\psi(f) = f|_{\mathrm{coz}(D)}$. Clearly, ψ is an F-algebra homomorphism. Let $g \in C_c(\mathrm{coz}(D), F)$ and define $\bar{g}$ by

$$\bar{g}(x) = \begin{cases} g & x \in coz(D), \\ 0 & x \in X \setminus \overline{coz(D)}. \end{cases}$$

Since $\mathrm{coz}(D) \cup (X \setminus \overline{\mathrm{coz}(D)})$ is a dense open subset of X, we obtain $\bar{g} \in Q_2$ and $\psi(\bar{g}) = g$, i.e., ψ is onto. So $Q_c(S) \leqq Q_1 \xleftarrow{onto} \psi : Q_2 \leqq Q_c(X, F)$, and we are done. □

Corollary 2.25. *Let ψ be as defined in (the proof of) Lemma 2.24. Then, ψ is one-to-one if and only if every dense ideal in S has dense cozero-set in X.*

Proof. (⇒) Suppose D is a dense ideal in S such that $\overline{\mathrm{coz}(D)} \neq X$. Let $V = \mathrm{coz}(D) \cup (X \setminus \overline{\mathrm{coz}(D)})$. Define $f : V \to F$ by $f(x) = 0$ for each $x \in \mathrm{coz}(D)$ and $f(x) = 1$ for each $x \in X \setminus \overline{\mathrm{coz}(D)}$. Thus $f \in C_c(V, F)$ and hence $f \in Q_2$. Moreover, $\psi(f) = 0$ while $f \neq 0$, i.e., ψ is not one-to-one.

(⇐) Suppose $f \in Q_2$ and $\psi(f) = 0$. So $f \in C_c(V, F)$ for some $V \in \mathcal{C}_2$. Hence, $f|_{\mathrm{coz}(D)} = 0$, where D is a dense ideal D in S and $V \supseteq \mathrm{coz}(D)$. Now, the assumption, $\overline{\mathrm{coz}(D)} = X$, yields $f = 0$, and we are done. □

Theorem 2.26. *Let S be a subring of $C_c(X, F)$. Then $Q_c(S)$ is a subring of $Q_c(X, F)$ if and only if every dense ideal in S has dense cozero-set in X.*

Proof. (⇒) Let D be a dense ideal in S and take $f \in Hom(D)$. Then $f \in C_c(\mathrm{coz}(D), F)$, by Lemma 2.15. Furthermore, the assumption that $f \in Q_c(S) \leqq Q_c(X, F)$ gives f belongs to a ring of continuous functions on a dense open set in X. So $\mathrm{coz}(D)$ is dense.

(⇐) It follows from Corollary 2.25. □

The following is an immediate consequence of Proposition 2.17 and Theorem 2.26.

Corollary 2.27. *Let S be an essential subring of $C_c(X, F)$. Then $Q_c(S) \leqq Q_c(X, F)$.*

3. Equalities among various rings of quotients of $C_c(X, F)$

In this section, we deal with the coincidence of rings of quotients of the $C_c(X, F)$, in particular, we are interested in cases when some of these rings of quotients coincide with $C_c(X, F)$, itself. We show that the fixed ring of quotients and the cofinite ring of quotients of $C_c(X)$ coincide if and only if $\mathrm{Hom}(M_p^c) = C_c(X_p)$ for every p in X.

Proposition 3.1. *$Q_c(X, F) = C_c(X, F)$ if and only if every open set in X is C_cF-embedded.*

Proof. By Theorem 2.19, $Q_c(X, F) = C_c(X, F)$ if and only if every dense open subset of X is C_cF-embedded. Proposition 2.12 now yields the result. □

Notice that the "zero-dimensional condition" cannot be omitted from the above result. To see this, let $X = \mathbb{R}^2$, $F = \mathbb{R}$ and let $V = \mathbb{R}^2 \setminus \{(0, y) : y \in \mathbb{R}\}$. Remember that the relation $C_c(X, F) = F(= \mathbb{R})$ follows trivially from the connectedness of $X = \mathbb{R}^2$, and therefore $Q_c(X, F) = Q(F) = F$. Now, let $f : V \to \mathbb{R}$ be defined by $f(x, y) = 1$, if $x > 0$, and $f(x, y) = -1$, if $x < 0$. Then $f \in C_c(V, F)$, but it cannot be extended to X, i.e., V is not C_cF-embedded.

Using Proposition 3.1 and the fact that "C_cF-embedding gives C_c^*F-embedding", we get the following.

Corollary 3.2. *$Q_c(X, F) = C_c(X, F)$ implies that $Q_c^*(X, F) = C_c^*(X, F)$.*

The converse of the above corollary is not true in general, see Example 3.7.

Proposition 3.3. *For a zero-dimensional space X and a totally ordered field F, the following are equivalent.*

(i) $Q_c(X,F) = q_c(X,F)$.
(ii) *Each F-valued continuous function on a dense open subset of X agrees on the cozero-set* coz(h) *for some* $h \in C_c(X,F)$ *for which* coz(h) *is dense in X.*
(iii) $Q^*_c(X,F) = q^*_c(X,F)$.

Proof. (i) ⇔ (ii) It is evident.

(ii) ⇒ (iii) Let $f \in Q^*_c(X,F)$. Then for some dense open subset V of X, we have $f \in C^*_c(V,F)$. By the assumption, there is $h \in C_c(X,F)$ and $g \in C_c(coz(h),F)$ such that $\overline{coz(h)} = X$ and f agrees with g on $V \cap coz(h)$. So $f \in q^*_c(X,F)$. The reverse inclusion is clear.

(iii) ⇒ (i) If $f \in Q_c(X,F)$, then for some dense open subset V of X, $f \in C_c(V,F)$. Take $g = (1+f^2)^{-1} = \frac{1}{1+f^2}$. Hence, $g, fg \in C^*_c(V,F) \subseteqq Q^*_c(X,F) = q^*_c(X,F)$. So there exist $h_1 \in C_c(X,F)$ and $k_1 \in C^*_c(coz(h_1),F)$ such that $\overline{coz(h_1)} = X$ and fg agrees with k_1 on $V \cap coz(h_1)$. Also, we can take $h_2 \in C_c(X,F)$ and $k_2 \in C^*_c(coz(h_2),F)$ such that $\overline{coz(h_2)} = X$ and g agrees with k_2 on $V \cap coz(h_2)$. Notice that $g^{-1} \geq 1$ and agrees with k_2^{-1} on $V \cap coz(h_2)$, also, $\mathrm{coz}(h_1) \cap \mathrm{coz}(h_2) = \mathrm{coz}(h_1h_2)$ is dense. Therefore, $f = (fg)g^{-1} \equiv k_1k_2^{-1}$ on $V \cap \mathrm{coz}(h_1h_2)$. This yields $f \in q_c(X,F)$. The reverse inclusion is obvious. □

In the sequel, we will need the next proposition.

Proposition 3.4. *Let F be an infinite, totally ordered field and* $(a_\lambda)_{\lambda\in\Lambda} \subseteq F$ *be a net of nonzero elements. Then the following hold.*

(i) *If* $0 \neq a \in F$, *then* $a_\lambda \to a$ *if and only if* $a_\lambda^{-1} \to a^{-1}$.
(ii) $a_\lambda \to 0$ *if and only if the net* $(a_\lambda^{-1})_{\lambda\in\Lambda}$ *is unbounded.*

Proof. (i) Suppose $a > 0$ and $a_\lambda \to a$. Let (x_1, x_2) be an open set containing a^{-1} such that $x_1 > 0$. Then $a \in (x_2^{-1}, x_1^{-1})$. So for some $\lambda_0 \in \Lambda$ and each $\lambda \geq \lambda_0$; $a_\lambda \in (x_2^{-1}, x_1^{-1})$ and thus $a_\lambda^{-1} \in (x_1, x_2)$. This yields $a_\lambda^{-1} \to a^{-1}$. The converse is obvious, by the previous part.

(ii) It is obtained similarly. □

The CP_F-spaces are introduced and determined in [2, Section 4]. Now, with this terminology, we call a space X an *almost CP_F-space*, if $int_X Z \neq \emptyset$, for each nonzero zero-set $Z \in Z_c(X,F)$.

Recall that $f \in C_c(X,F)$ is a zero-divisor if and only if $\mathrm{int}_X Z(f) \neq \emptyset$ if and only if $\overline{\mathrm{coz}(f)} \neq X$.

Theorem 3.5. *For a zero-dimensional space X and a totally ordered field F, the following are equivalent*

(i) *Every non-unit element in* $C_c(X,F)$ *is a zero-divisor.*
(ii) *There is no proper dense cozero-set in X.*
(iii) $q_c(X,F) = C_c(X,F)$.
(iv) *X is almost CP_F-space.*

Proof. (i) ⇒ (ii) Let $f \in C_c(X,F)$ and $\overline{coz(f)} = X$. Then $int_X Z(f) = \emptyset$ which means f is a non-zero-divisor. By the hypothesis, f is a unit and thus $coz(f) = X$.

(ii) ⇒ (iii) Since the only dense cozero-set in X is, itself, the result holds.

(iii) ⇒ (iv) Let $Z = Z(f) \in Z_c(X,F)$ and $int_X Z = \emptyset$, i.e., $\overline{coz(f)} = X$. Notice that $f^{-1} \in C_c(coz(f),F)$ (Proposition 2.2). Therefore, by the assumption, f^{-1} has an extension to X, say $\bar{f}$. For $x \in X$, there is a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq coz(f)$ converging to x. Hence, $f(x_\lambda) \to f(x)$ and $\bar{f}(x_\lambda) \to \bar{f}(x)$, this means that $f^{-1}(x_\lambda) \to \bar{f}(x)$. We claim that $f(x) \neq 0$. Otherwise, by Proposition 3.4(ii), the net $(f^{-1}(x_\lambda))_{\lambda\in\Lambda}$ is unbounded which is absurd, since $\bar{f}(x) \in F$. Consequently, we reach the claim, i.e., $f(x) \neq 0$. This yields $coz(f) = X$, or $Z = \emptyset$.

(iv) ⇒ (i) Let $f \in C_c(X,F)$ be a non-unit. Then $Z(f) \neq \emptyset$. Now, by the assumption, $int_X Z(f) \neq \emptyset$, i.e., f is a zero-divisor. □

An immediate conclusion of the above theorem is the following.

Corollary 3.6. $q_c(X,F) = C_c(X,F)$ *implies that* $q^*_c(X,F) = C^*_c(X,F)$.

The converse of the above corollary is not true in general, see Example 3.7.

Recall that a topological space is called *extremally disconnected* if all open sets have open closures. Hence, an extremally disconnected space is zero-dimensional. By [13, 1H], a topological space is extremally disconnected if and only if every open set is C^*-embedded in it.

Example 3.7. *Let $X = \Sigma = \mathbb{N} \cup \{\sigma\}$ (where $\sigma \notin \mathbb{N}$) be the space in [13, 4M] and let $F = \mathbb{R}$. We recall that the open neighborhoods of σ in the space X are of the form: $G \cup \{\sigma\}$, where G is a member of a free ultrafilter on $\mathbb{N}$, and each point of $\mathbb{N}$ is isolated. If $f(n) = \frac{1}{n}$ and $f(\sigma) = 0$, then $f \in C_c(X, F) = C(X, F) = C(X)$. So $\{\sigma\}$ is a zero-set and thus $\mathbb{N}$ is a cozero-set in X. Notice that X and $\mathbb{N}$ are precisely the dense open sets as well as the dense cozero-sets in X. By Theorem 2.10, $C(X) \leqq C(\mathbb{N})$ and thus $Q(X) = q(X) = C(\mathbb{N})$ (note, $Q_c(X, F) = Q(X)$ and $q_c(X, F) = q(X)$). Hence, $Q^*(X) = q^*(X) = C^*(\mathbb{N})$. Since X is extremally disconnected, every open subspace is C^*-embedded, see [13, 1H]. This yields $C^*(X) = C^*(\mathbb{N})$. So $Q^*(X) = q^*(X) = C^*(X)$. Moreover, since $\mathbb{N}$ is not C-embedded in X (consider $f(n) = n$), we obtain $C(X) \lneqq C(\mathbb{N})$.*

In [2, Definition 2.7], an ideal I in $C_c(X, F)$ is called a *fixed ideal* if $Z(I) = \bigcap_{f \in I} Z(f) \neq \emptyset$. Also, by [2, Theorem 2.8], a fixed maximal ideal in $C_c(X, F)$ is in the form of $M^c_{p,F} = \left\{f \in C_c(X, F) : f(p) = 0\right\}$, where $p \in X$.

Proposition 3.8. *Let $p \in X$. Then $M^c_{p,F}$ is a dense ideal in $C_c(X, F)$ if and only if p is a non-isolated point.*

Proof. ($\Rightarrow$) Suppose p is an isolated point of X. Then $\{p\}$ and $X \setminus \{p\}$ are clopen sets. Define a map $g : X \to F$ by $g(p) = 1$ and $g(x) = 0$ for every $x \neq p$. So $g \in C_c(X, F)$ and $g.M^c_{p,F} = 0$. Since $g \neq 0$, it gives $M^c_{p,F}$ is not a dense ideal.

($\Leftarrow$) Let p be a non-isolated point of X and let $g \in C_c(X, F)$ such that $g.M^c_{p,F} = 0$. We claim that $g = 0$. Otherwise, $g(x) \neq 0$ for some $x \neq p$. By [2, Theorem 2.10], there exists $h \in C_c(X, F)$ such that $h(x) = 1$ and $p \in int_X Z(h)$. Now, $h \in M^c_{p,F}$ and $gh \neq 0$, a contradiction. □

If $F = \mathbb{R}$, then we let $C_c(X, F) = C_c(X)$ and $M^c_{p,F} = M^c_p$.

The fixed ring of quotients and the cofinite ring of quotients of $C(X)$ have been investigated in [19]. In the following, we follow these methods in determining the fixed ring of quotients and the cofinite ring of quotients of $C_c(X, \mathbb{R}) = C_c(X)$. Let $\mathfrak{F}_0$ be the family of all finite intersections of dense fixed maximal ideals of $C_c(X)$. Then $\mathfrak{F}_0$ is a filter base, i.e., it is closed under finite intersection. Let $\mathfrak{F}$ be the filter of ideals of $C_c(X)$ that is generated by $\mathfrak{F}_0$. Then $\mathfrak{F}_c(X) = \bigcup \left\{Hom(D') : D' \in \mathfrak{F}\right\}$, with the usual equivalence relation, is a ring of quotients of $C_c(X)$ because $\mathfrak{F} \subseteq \mathcal{D}_0$, the family of dense ideals of $C_c(X)$. Note that for each $D' \in \mathfrak{F}$ there is $D \in \mathfrak{F}_0$ such that $D' \supseteq D$, and, in fact, we have $\mathfrak{F}_c(X) = \bigcup \left\{Hom(D) : D \in \mathfrak{F}_0\right\}$. Hence, if $f \in \mathfrak{F}_c(X)$, then $f \in Hom(D)$ for some D that is a finite intersection of dense fixed maximal ideals of $C_c(X)$. By borrowing the terminology from [19], we call $\mathfrak{F}_c(X)$ the *fixed ring of quotients* of $C_c(X)$.

For a finite subset G of X, we let $X_G = X \setminus G$ and $M_G = \bigcap_{x \in G} M^c_x$, where $M^c_x = \left\{f \in C_c(X) : f(x) = 0\right\}$. If $G = \{p\}$, then we use X_p instead of X_G.

Lemma 3.9. *Let G be a finite subset of X and let $f \in C_c(X_G)$. If $f \in \mathfrak{F}_c(X)$, then $f \in Hom(M_G)$.*

Proof. Let G_1 be a finite set of non-isolated points of X such that $f \in \mathrm{Hom}(M_{G_1})$. If $G_1 \subseteq G$, then $M_G \subseteq M_{G_1}$, hence $\mathrm{Hom}(M_{G_1}) \subseteq \mathrm{Hom}(M_G)$, and we are done. Otherwise, for two disjoint finite (compact) sets $G_1 \setminus G$ and $G \setminus G_1$, there is $h \in C^*_c(X)$ such that $G_1 \setminus G \subseteq \mathrm{int}_X Z(h)$ and $G \setminus G_1 \subseteq \mathrm{int}_X Z(1 - h)$, by Proposition 2.6 (or [11, Proposition 4.3]). Let $g \in M_G$, then we must extend fg to a continuous function on X. Since $gh \in M_{G_1}$ and $f \in \mathrm{Hom}(M_{G_1})$, the function fgh has an extension to X. Now, if we define $(fg)(t) = (fgh)(t)$ for all $t \in G$, then fg is extended to X. So $f \in \mathrm{Hom}(M_G)$, and we are done. □

Recall that for each finite set G of isolated points of X, we have $\mathrm{Hom}(M_G) = C_c(X_G)$.

Theorem 3.10. *Let X be a zero-dimensional space. Then,* $\mathrm{Hom}(M_G) = C_c(X_G)$ *for every finite set G of non-isolated points of X if and only if* $\mathrm{Hom}(M^c_p) = C_c(X_p)$ *for every non-isolated point $p \in X$.*

Proof. (⇒) It is obvious.

(⇐) We provide the proof for the case that $G = \{p, q\}$. The general case is done in the same way. We first note that $M_G = M_p^c \cap M_q^c$ and it is dense by Proposition 3.8, moreover, $\mathrm{Hom}(M_G) \subseteq C_c(X_G)$, by Lemma 2.15. Now, take $f \in C_c(X_G)$ and $g \in M_G$. We must extend fg to a continuous function on X. Recall that $g = g^{\frac{1}{3}} g^{\frac{2}{3}} \in C_c(X)$ and $Z(g) = Z(g^{\frac{1}{3}}) = Z(g^{\frac{2}{3}})$. It is clear that $g^{\frac{1}{3}} \in M_p^c$ and $g^{\frac{2}{3}} \in M_q^c$ because a maximal ideal in $C_c(X)$ is a z_c-ideal (see [11]). Let $Y = X_p$ and $M_q'^c = \left\{h \in C_c(Y) : h(q) = 0\right\}$. Then $Y_q = X_G$ and $M_q'^c$ is a maximal ideal in $C_c(Y)$. Using the assumption, $\mathrm{Hom}(M_q'^c) = C_c(Y_q)$, we get $fg^{\frac{2}{3}} \in C_c(Y)$, since $g^{\frac{2}{3}} \in M_q'^c$ and $f \in C_c(Y_q)$. Again, applying the assumption, $\mathrm{Hom}(M_p^c) = C_c(Y)$, we obtain that $fg = (fg^{\frac{2}{3}})g^{\frac{1}{3}} \in C_c(X)$. Consequently, $f \in \mathrm{Hom}(M_G)$, and we are through. □

Let $\mathcal{F}_0$ be the set of all dense cofinite subsets of X. Then $\mathcal{F}_0$ is a filter base. Let $\mathcal{F}_c(X, F) = \varinjlim_{V \in \mathcal{F}_0} C_c(V, F)$. By Corollary 2.11, $C_c(V, F)$ is a ring of quotients of $C_c(X, F)$, and thus $\mathcal{F}_c(X, F)$ is too. We observe that $\mathcal{F}_c(X, F) = \bigcup \left\{C_c(V, F) : V \in \mathcal{F}_0\right\}$, where we identify $f_1 \in C_c(V_1, F)$ with $f_2 \in C_c(V_2, F)$ whenever f_1 and f_2 agree on $V_1 \cap V_2$. Now, let $\mathcal{F}$ be the filter of sets that is generated by $\mathcal{F}_0$. Then for $W \in \mathcal{F}$, there exists $V \in \mathcal{F}_0$ such that $V \subseteq W$. So W is a dense cofinite subset of X, and thus the chain $C_c(X, F) \leqq C_c(W, F) \leqq C_c(V, F)$ is a chain of rings of quotients. Moreover,

$$\bigcup \left\{C_c(W, F) : W \in \mathcal{F}\right\} \leqq \bigcup \left\{C_c(V, F) : V \in \mathcal{F}_0\right\}.$$

Note that in more general, we have

$$\mathcal{F}_c(X, F) = \varinjlim_{V \in \mathcal{F}_0} C_c(V, F) = \bigcup \left\{C_c(V, F) : V \in \mathcal{F}_0\right\} = \bigcup \left\{C_c(W, F) : W \in \mathcal{F}\right\}.$$

By borrowing the terminology from [19], we call $\mathcal{F}_c(X, F)$ the *cofinite ring of quotients* of $C_c(X, F)$.

In the case that $F = \mathbb{R}$, we let $\mathcal{F}_c(X, F) = \mathcal{F}_c(X)$. Applying Lemma 2.15, we obtain $\mathfrak{F}_c(X) \leqq \mathcal{F}_c(X)$. In the next example, we observe that $\mathfrak{F}_c(X) \lneqq \mathcal{F}_c(X)$.

Example 3.11. Let $X = \mathbb{Q} \times \mathbb{Q}$, $F = \mathbb{R}$, and $p = (a, b) \in X$ be fixed. Then $coz(M_p^c) = X_p$ is a dense cofinite subset of X. Also, $g(x, y) = (x - a)^2 + (y - b)^2 \in M_p^c$ and $f = \frac{1}{g^2} \in C(X_p) = C_c(X_p) \subseteq \mathcal{F}_c(X)$. We claim that $f \notin \mathfrak{F}_c(X)$. Otherwise, $f \in Hom(M_p^c)$, by Lemma 3.9, which is absurd because $fg = \frac{1}{g} \notin C(X)$. Now, we reach the claim, i.e., $\mathfrak{F}_c(X) \neq \mathcal{F}_c(X)$. Moreover, $\mathrm{Hom}(M_p^c) \subsetneq C(\mathrm{coz}(M_p^c)) = C(X_p)$.

Theorem 3.12. *Let X be zero-dimensional space. Then,* $\mathfrak{F}_c(X) = \mathcal{F}_c(X)$ *if and only if* $\mathrm{Hom}(M_p^c) = C_c(X_p)$ *for every* $p \in X$.

Proof. (⇒) Note first that if p is an isolated point, then the equation $\mathrm{Hom}(M_p^c) = C_c(X_p)$ is obtained quickly. Next, let $p \in X$ be non-isolated. Then, M_p^c is a dense ideal in $C_c(X)$ (Proposition 3.8) and further $\mathrm{Hom}(M_p^c) \subseteq C_c(X_p)$ (Lemma 2.15). Now, we take $f \in C_c(X_p)$. Since X_p is a dense cofinite subset of X; $f \in \mathcal{F}_c(X)$ and thus $f \in \mathfrak{F}_c(X)$, by the assumption. Using Lemma 3.9, we get $f \in \mathrm{Hom}(M_p^c)$. Therefore, $C_c(X_p) \subseteq \mathrm{Hom}(M_p^c)$.

(⇐) Let $f \in \mathcal{F}_c(X)$. Then for a finite set G of non-isolated points of X; $f \in C_c(X_G)$. Now, combining the assumption and Theorem 3.10 gives $f \in \mathrm{Hom}(M_G)$ which means that $f \in \mathfrak{F}_c(X)$. □

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