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Multiple Curves on Punctured Orientable Surfaces

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Abstract. We describe each multiple curve on an orientable surface of genus-g, with n punctures and one boundary component by using this multiple curve's geometric intersection numbers with the embedded curves in this surface.

1. Introduction

Multiple curves, which are the disjoint unions of finitely many essential simple closed curves on an orientable surface modulo isotopy, play a central role in low dimensional topology and computational topology. Such curves are usually described combinatorially by using either Dehn-Thurston coordinates or train track coordinates [6, 7]. However, it has been observed that these methods are not computationally efficient enough for very complex problems. One of the approaches to describe the multiple curves on the standard n-times punctured disk ($n \geq 3$) is to use the geometric intersection numbers between the multiple curve and the embedded arcs in the disk [3, 8]. In [4], this approach is generalized for such curve systems on the orientable surface of genus-1 which has n ($n \geq 2$) punctures and one boundary component. The coordinate system [3] obtained for the punctured disk that uses this method was extensively preferred to solve various dynamical and combinatorial problems such as the word problem in the braid group [1, 2], calculate the topological entropy of a braid [5], and compute the geometric intersection number of two multiple curves on the disk [9] since it is a very effective way to coordinate a multiple curve on a finitely punctured disk due to the ease of computation. The aim of this work is to generalize the approach which describes each multiple curve by using the geometric intersection numbers with the embedded curves in the punctured orientable genus-1 surface which has one boundary to the orientable surface of genus-g ($g \geq 1$) which has n punctures and one boundary component. By using the formulas proposed in this work, it is thought that the previous works on curves on the disk can be expanded to this surface in the future.

Throughout the paper, $S_{n,g}$ shall denote an orientable genus-g ($g \geq 1$) surface with n ($n \geq 1$) punctures and one boundary component. We shall use this surface by pulling its boundary backward from the top and bottom as shown in Figure 1 for an easier understanding of the working process. In order to describe a given multiple curve on $S_{n,g}$, a system consisting of $3n+7g-5$ arcs and g simple closed curves on $S_{n,g}$ is used. Given a multiple curve $\mathcal{L}$, we shall introduce a vector in $\mathbb{Z}^{3n+8g-5}_{\geq 0} \setminus \{0\}$ by using the geometric intersection numbers with the curves in our system, and consider the linear combinations of these intersection numbers (see Section 2).

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2. Geometric intersection numbers with the customized curves embedded in $S_{n,g}$

In this section, we describe the multiple curves on $S_{n,g}$, whose geometric intersection numbers with the customized curves embedded in $S_{n,g}$ and whose directions are known. To do this, we use the model shown in Figure 1.

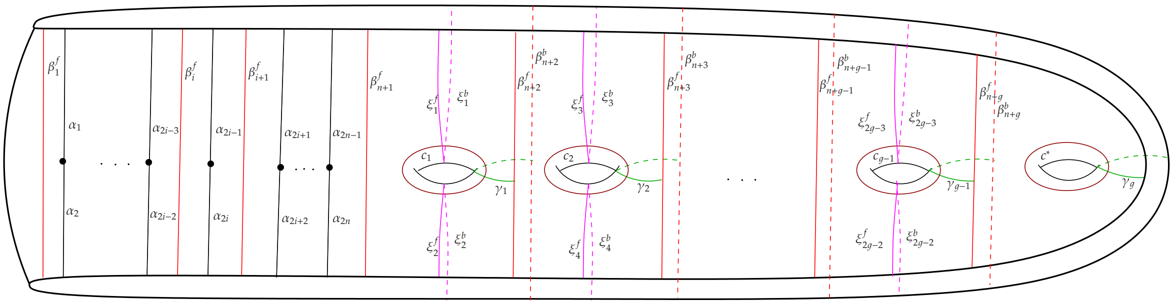

Figure 1: Curves on $S_{n,g}$

In this model, the endpoints of the arcs α_i $(1 \leq i \leq 2n)$, β_i^f $(1 \leq i \leq n+g)$, β_i^b $(n+2 \leq i \leq n+g)$, ξ_i^f $(1 \leq i \leq 2g-2)$, and ξ_i^b $(1 \leq i \leq 2g-2)$ are either on the boundary or on the puncture. While c_i $(1 \leq i \leq g-1)$ and c^* are the longitudes of each torus respectively, γ_i $(1 \leq i \leq g)$ is the arc whose endpoints are on β_i^f and β_i^b as depicted in Figure 1 and γ_g is the arc whose endpoints are on the boundary. Moreover, note that γ_i $(1 \leq i \leq g-1)$ and γ_g intersect c_i and c^*, respectively once transversally.

Definition 2.1. Let $\mathcal{L}_{n,g}$ be the set of the multiple curves on $S_{n,g}$ and $\mathcal{L} \in \mathcal{L}_{n,g}$. Moreover, let $i(\mathcal{L}, [\alpha])$ be the geometric intersection number between $\mathcal{L}$ and the isotopy class of α. The geometric intersection function $\psi : \mathcal{L}_{n,g} \to \mathbb{Z}_{\geq 0}^{3n+8g-5} \setminus \{0\}$ is defined by $\psi(\mathcal{L}) = (i(\mathcal{L}, [\alpha_1]), \ldots, i(\mathcal{L}, [\alpha_{2n}]); i(\mathcal{L}, [\beta_1^f]), \ldots, i(\mathcal{L}, [\beta_{n+g}^f]);$ $i(\mathcal{L}, [\beta_{n+2}^b]), \ldots, i(\mathcal{L}, [\beta_{n+g}^b]); i(\mathcal{L}, [\xi_1^f]), \ldots, i(\mathcal{L}, [\xi_{2g-2}^f]); i(\mathcal{L}, [\xi_1^b]), \ldots, i(\mathcal{L}, [\xi_{2g-2}^b]); i(\mathcal{L}, [\gamma_1]), \ldots, i(\mathcal{L}, [\gamma_g]);$ $i(\mathcal{L}, [c_1]), \ldots, i(\mathcal{L}, [c_{g-1}]); i(\mathcal{L}, [c^*]))$.

We simply write α for $i(\mathcal{L}, [\alpha])$. This shall also be the case for other coordinates. It will always be clear from the context whether we mean the coordinate curve or the geometric intersection number assigned on the coordinate curve. Throughout the paper, we work with the minimal representative (a multiple curve in the same isotopy class intersecting the customized curves embedded in $S_{n,g}$ minimally, that is, the sum of all the intersection numbers is minimal in the isotopy class) of $\mathcal{L}$ and denote it by L. Therefore, the map ψ is well defined on $\mathcal{L}_{n,g}$.

Example 2.2. Intersection numbers of the multiple curve L depicted in Figure 2 are given by $\psi(L) = (5, 2, 5, 2, 4, 3; 7, 5, 7, 1, 5, 5; 5, 3; 6, 3, 5, 2; 4, 1, 4, 1; 2, 2, 3; 2, 0; 3)$.

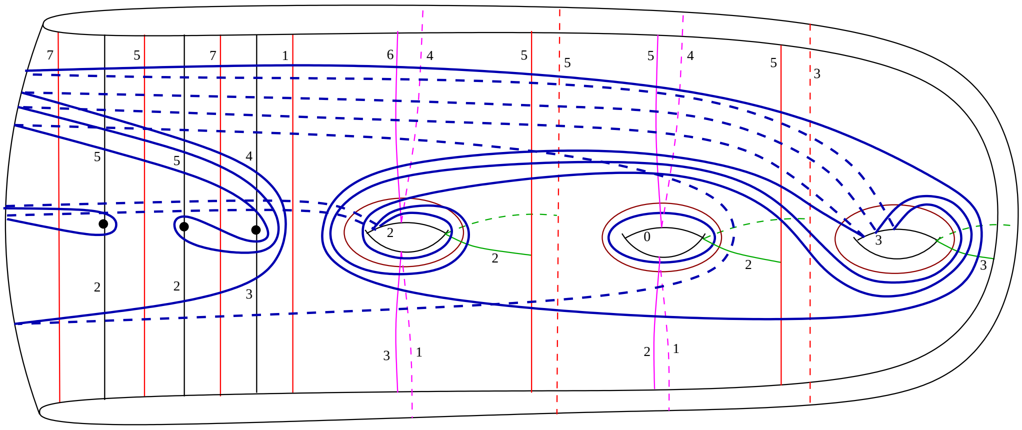

Figure 2: Intersection numbers with the curves embedded in $S_{3,3}$

2.1. Path components on $S_{n,g}$

In this subsection, we shall introduce the path components of a multiple curve L on $S_{n,g}$, and derive formulas for the number of these components.

Let U_i $(1 \le i \le n)$ be the region that is bounded by β_i^f and β_{i+1}^f, G_i $(1 \le i \le g-1)$ be the region bounded by β_{n+i}^f, β_{n+i}^b, β_{n+i+1}^f and β_{n+i+1}^b, and G^* be the region bounded by β_{n+g}^f, β_{n+g}^b and the boundary of $S_{n,g}$ $(\partial S_{n,g})$. Each component of $L \cap U_i$, $L \cap G_i$, and $L \cap G^*$ is called the *path component* of L in U_i, G_i, and G^*, respectively. Since L is minimal, there are four types of path components in the region U_i as on the disk [3] (see Figure 3). An *above component* has endpoints on β_i^f and β_{i+1}^f, and intersects α_{2i-1}. A *below component* has endpoints on β_i^f and β_{i+1}^f, and intersects α_{2i}. A *left loop component* has both endpoints on β_{i+1}^f, and intersects α_{2i-1} and α_{2i} (Figure 3a). A *right loop component* has both endpoints on β_i^f, and intersects α_{2i-1} and α_{2i} (Figure 3b). There are six types of path components in the region G^*. The first three of these components are *curve c^**, which is the longitude of the torus in G^* (Figure 4a); *visible genus component*, which has both endpoints on β_{n+g}^f and does not intersect the curve c^* (Figure 4b); *invisible genus component*, which has both endpoints on β_{n+g}^b and does not intersect the curve c^* (Figure 4c). The remaining three components called *twist* have endpoints on β_{n+g}^f and β_{n+g}^b, and intersect the curve c^* (see Figure 5). These components are nontwist, negative twist, and positive twist components. The *nontwist component* does not make any twist (see Figure 5a). The *negative twist component* makes clockwise twist (see Figure 5b). The *positive twist component* makes counterclockwise twist (see Figure 5c) [4]. There are 14 types of path components in each region G_i. These are *curve c_i*, which is the longitude of the torus in G_i (similar to Figure 4a); *visible left genus component*, which has both endpoints on β_{n+i+1}^f, and does not intersect the curve c_i (Figure 6a); *invisible left genus component*, which has both endpoints on β_{n+i+1}^b, and does not intersect the curve c_i (Figure 6a); *visible right genus component*, which has both endpoints on β_{n+i}^f, and does not intersect the curve c_i (Figure 6b); *invisible right genus component*, which has both endpoints on β_{n+i}^b, and does not intersect the curve c_i (Figure 6b); *upper diagonal component*, which has endpoints on β_{n+i}^b and β_{n+i+1}^f, and intersects the curve c_i and the arc ξ_{2i-1}^f (see Figure 6c); *lower diagonal component*, which has endpoints on β_{n+i}^b and β_{n+i+1}^f, and intersects the curve c_i and the arc ξ_{2i}^f (see Figure 6d); *visible above component*, which has endpoints on β_{n+i}^f and β_{n+i+1}^f, and intersects the arc ξ_{2i-1}^f (see Figure 6e); *invisible above component*, which has endpoints on β_{n+i}^b and β_{n+i+1}^b, and intersects the arc ξ_{2i-1}^b (see Figure 6e); *visible below component*, which has endpoints on β_{n+i}^f and β_{n+i+1}^f, and intersects the arc ξ_{2i}^f (see Figure 6f); *invisible below component*, which has endpoints on β_{n+i}^b and β_{n+i+1}^b, and intersects the arc ξ_{2i}^b (see Figure 6f); *negative twist component*, which has endpoints on β_{n+i}^b and β_{n+i}^f or β_{n+i}^b and β_{n+i+1}^f, and intersects the curve c_i, and makes clockwise twist (see Figures 7a and 7b); *positive twist component*, which has endpoints on β^b_{n+i} and β^f_{n+i} or β^b_{n+i} and β^f_{n+i+1}, and intersects the curve c_i, and makes counterclockwise twist (see Figures 7c and 7d); and *nontwist component* (see Figure 7e).

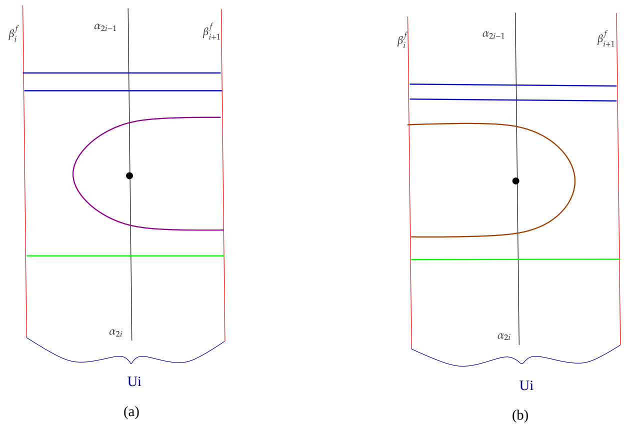

Figure 3: Above and below components, left and right loop components in the region U_i

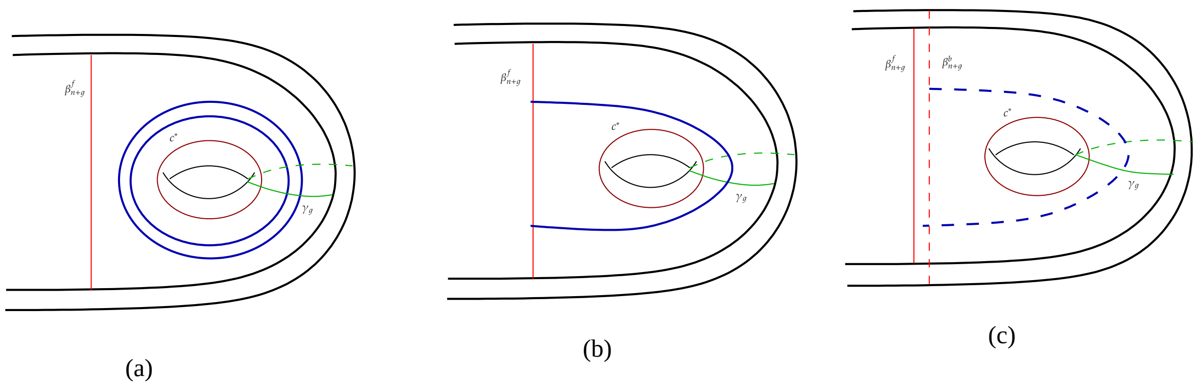

Figure 4: (a) c^* curves, (b) visible genus component, (c) invisible genus component in the region G^*

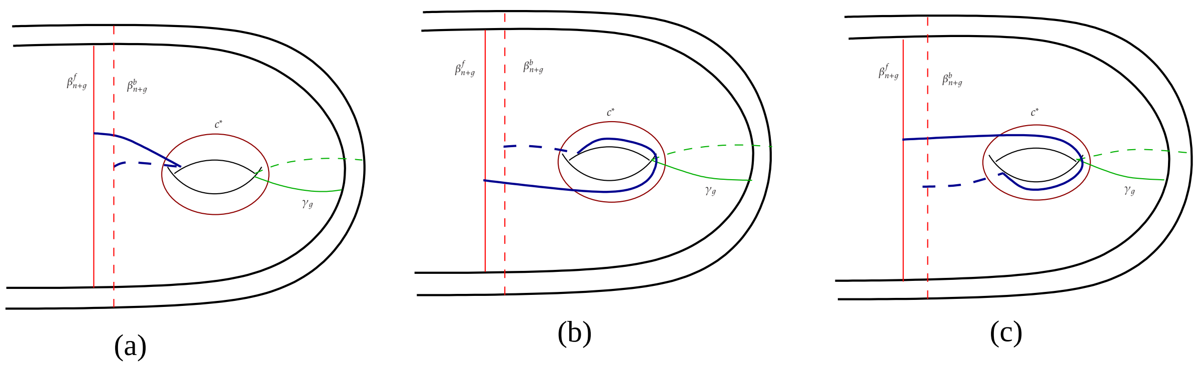

Figure 5: (a) Nontwist component, (b) negative twist component, (c) positive twist component.

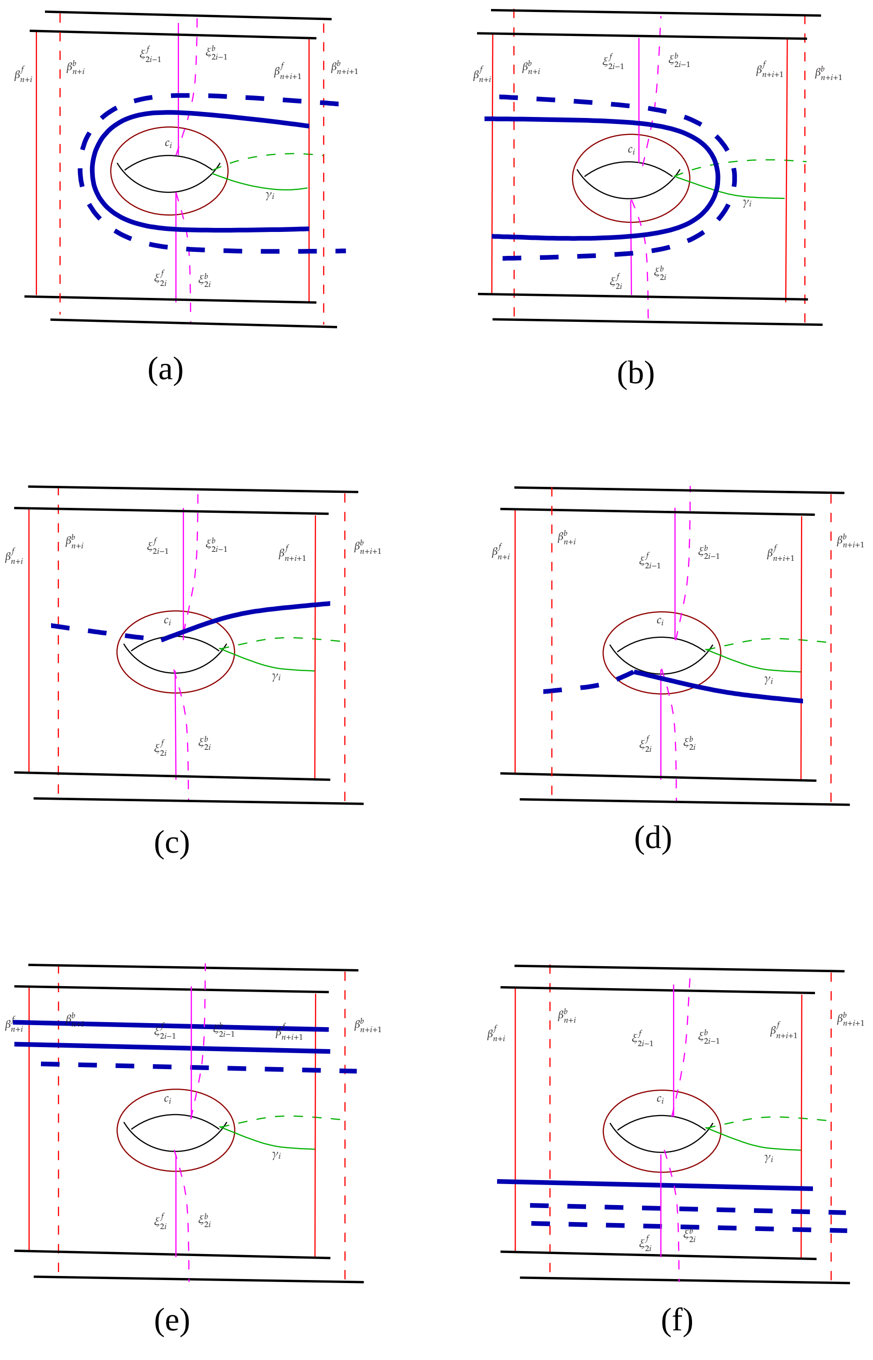

Figure 6: (a) Visible left and invisible left genus components, (b) visible right and invisible right genus components, (c) upper diagonal component, (d) lower diagonal component, (e) visible above and invisible above components, (f) visible below and invisible below components in the region G_i

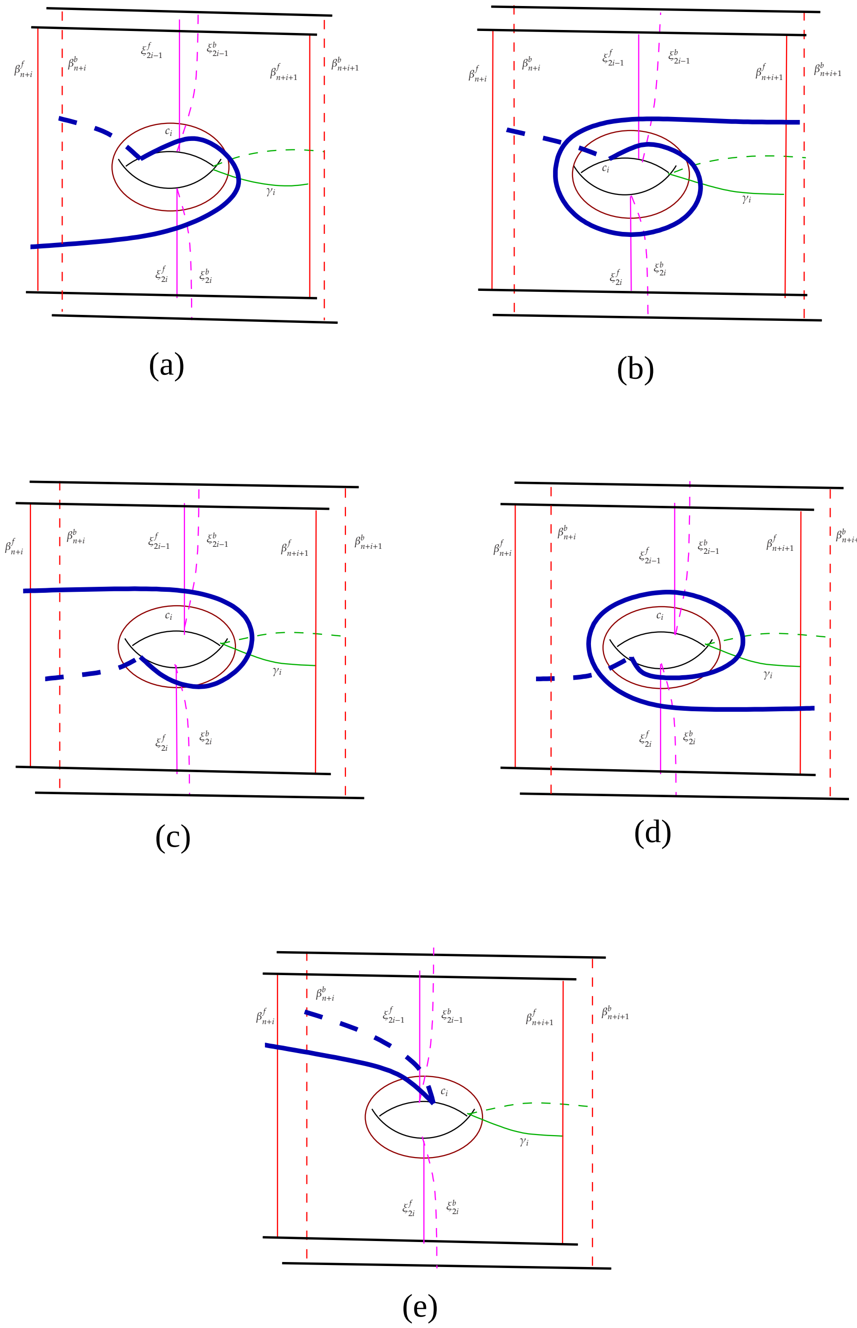

Figure 7: (a) and (b) Negative twist component; (c) and (d) positive twist component; (e) nontwist component in G_i.

Remark 2.3. For the ease of calculation, throughout the paper, we assume that each diagonal component (Figures 6c and 6d) and twist component (Figure 7) on $S_{n,g}$ intersect the arc ξ^f_{2i-1} instead of the arc ξ^b_{2i-1}, and intersect the arc ξ^f_{2i} instead of the arc ξ^b_{2i}. Moreover, we assume that the invisible (dashed) parts of these components are only on the invisible left side of $S_{n,g}$, as seen from the corresponding figures and that each G_i has either upper diagonal components or lower diagonal components.

Remark 2.4. Since a multiple curve $L \in \mathcal{L}_{n,g}$ consists of the simple closed curves that do not intersect each other, there cannot exist both the curve c_i and the twist or diagonal components at the same time in the region G_i, and there cannot exist both the curve c^* and the twist components at the same time in the region G^*.

Definition 2.5. Let d^u_{2i-1} and d^l_{2i} give the number of the upper and lower diagonal components in the region G_i for $1 \leq i \leq g-1$, respectively. Moreover, let c'_i denote the number of the twist components in G_i. Thus, throughout the paper, c_i shall be defined as the sum of these components. That is,

$$c_i = c'_i + d^u_{2i-1} + d^l_{2i}. \tag{1}$$

Note that since there cannot be any diagonal components in G^*, c_i is only equal to the number of the twist components in G^*, and in this case, we denote c_i with the number c^*.

Definition 2.6. A twist component's *twist number* is the signed number of the intersections with the arc γ_i $(1 \leq i \leq g)$.

Remark 2.7. Since a multiple curve on $S_{n,g}$ does not contain any self-intersections, the directions of the twists have to be the same. Moreover, in the regions G_i and G^*, the difference between the twist numbers of two different twist components cannot be greater than 1 ([4]).

If we denote the *smaller twist number* by t_i, and the *bigger twist number* by $t_i + 1$, then the *total twist number* T_i $(1 \leq i \leq g)$ in G_i $(1 \leq i \leq g-1)$ and G^* is the sum of the twist numbers of twist components (see Figure 7). Hence, if the difference between the twist numbers of any two twist components is 0, then $T_i = t_i(c_i - d^u_{2i-1} - d^l_{2i})$. On the other hand, if the difference between the twist numbers of any two twist components is 1, then $T_i = m_i(t_i + 1) + (c_i - d^u_{2i-1} - d^l_{2i} - m_i)t_i$, where $m_i \in \mathbb{Z}_{\geq 0}$ is the number of the twist components with the twist number $t_i + 1$, and $c_i - d^u_{2i-1} - d^l_{2i} - m_i$ is the number of the twist components with the twist number t_i.

Remark 2.8. Although T_i gives the total twist number in each region G_i and G^*, it cannot show the directions of twists by itself. Therefore, we first calculate the number of each T_i, and then we add a sign in front of T_i, denoting the negative direction by $-T_i$ and the positive direction by T_i. However, since only the total number of twists is required in the formulas throughout the paper, $|T_i|$ shall be used as the total number of twists in order not to cause any confusion.

Now, we calculate the number of each path component of L in the regions G_i for $1 \leq i \leq g-1$, and G^* for $i = g$.

Theorem 2.9. *Let L be given with the intersection numbers $(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*)$, and let the number of the visible genus components and the number of the invisible genus components in G_i be l^f_i and l^b_i, respectively. Further, let the number of the visible genus components and the number of the invisible genus components in G^* be l^f_g and l^b_g, respectively.*

Given the set $K = \{f, b\}$ and the function $\chi : \mathbb{Z}^{3n+8g-5}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, for $2 \leq i \leq g-1$ and each $x \in K$,

$$l^x_i = \chi^x_i(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*)$$

where

$$\chi_1^f(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*) = \max\{0, \frac{|\beta^f_{n+1} - \beta^f_{n+2}| - c_1}{2}\},$$

$$\chi_1^b(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*) = \max\{0, \frac{|\beta^f_{1} - \beta^b_{n+2}| - c_1}{2}\},$$

$$\chi_i^x(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*) = \max\{0, \frac{|\beta^x_{n+i} - \beta^x_{n+i+1}| - c_i}{2}\},$$

$$\chi_g^f(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*) = \frac{\beta^f_{n+g} - c^*}{2},$$

$$\chi_g^b(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*) = \frac{\beta^b_{n+g} - c^*}{2}.$$

Note that if $\beta^f_{n+i} < \beta^f_{n+i+1}$, then the visible genus component in G_i is left; if $\beta^f_{n+i} > \beta^f_{n+i+1}$, then the visible genus component in G_i is right. Similarly, if $\beta^b_{n+i} < \beta^b_{n+i+1}$, then the invisible genus component in G_i is left; if $\beta^b_{n+i} > \beta^b_{n+i+1}$, then the invisible genus component in G_i is right.

Proof. The absolute value of the difference between the intersection numbers on the arcs β^f_{n+i} and β^f_{n+i+1}, denoted as $|\beta^f_{n+i} - \beta^f_{n+i+1}|$, gives us the sum of the twist, diagonal and visible genus component numbers. If $\beta^f_{n+i} < \beta^f_{n+i+1}$, then the arc β^f_{n+i+1} intersects once with each twist component (Figures 7b and 7d) or diagonal component (Figures 6c and 6d), and twice with each visible left genus component (Figure 6a). Let the number of visible left genus components and the number of visible right genus components be denoted by l_i^L and l_i^R, respectively. Hence, $\beta^f_{n+i+1} - \beta^f_{n+i} = c_i' + d^u_{2i-1} + d^l_{2i} + 2l_i^L$. From Equation (1), $\beta^f_{n+i+1} - \beta^f_{n+i} = c_i + 2l_i^L$. Since a multiple curve consists of the simple closed curves that do not intersect each other, this curve system contains either the visible left genus components or the visible right genus components. Therefore, we can denote the number of both visible left genus components and visible right genus components as l_i^f. Thus, we can write

$$\beta^f_{n+i+1} - \beta^f_{n+i} = c_i + 2l_i^f. \tag{2}$$

If $\beta^f_{n+i} > \beta^f_{n+i+1}$, then the arc β^f_{n+i} intersects once with each twist component (Figures 7a, 7c, and 7e), and twice with each visible right genus component (Figure 6b). From Remark 2.3, there cannot be any diagonal component; otherwise, self-intersections occur in this curve system. Since $c_i = c_i' + d^u_{2i-1} + d^l_{2i}$, here $\beta^f_{n+i} - \beta^f_{n+i+1} = c_i + 2l_i^R$. That is,

$$\beta^f_{n+i} - \beta^f_{n+i+1} = c_i + 2l_i^f. \tag{3}$$

From Equations (2) and (3), we can write $|\beta^f_{n+i} - \beta^f_{n+i+1}| = c_i + 2l_i^f$. Therefore, $l_i^f = \frac{|\beta^f_{n+i} - \beta^f_{n+i+1}| - c_i}{2}$. When $c_i \geq |\beta^f_{n+i} - \beta^f_{n+i+1}|$, there cannot be any visible genus component in the multiple curve. Hence, $l_i^f = \max\{0, \frac{|\beta^f_{n+i} - \beta^f_{n+i+1}| - c_i}{2}\}$ is derived. Similarly, we can find l_1^f, l_1^b, and l_i^b. For the proofs of l_g^f and l_g^b, you can check [4]. □

In the following theorem, we calculate the total twist number of the twist components in each G_i and G^*:

Theorem 2.10. *Let L be given with the intersection numbers $(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*)$, and let the signed total twist number of the twist components in each G_i and G^* be denoted by T_i and T_g, respectively. For $2 \le i \le g-1$, we have*

$$|T_i| = \begin{cases} 0 & \text{if } c_i = 0, \\ \gamma_i - \max\{0, \frac{\max\{0,\beta^f_{n+i}-\beta^f_{n+i+1}\}-c_i}{2}\} - \max\{0, \frac{\max\{0,\beta^b_{n+i}-\beta^b_{n+i+1}\}-c_i}{2}\} & \text{if } c_i \neq 0; \end{cases} \tag{4}$$

for $i = 1$,

$$|T_1| = \begin{cases} 0 & \text{if } c_1 = 0, \\ \gamma_1 - \max\{0, \frac{\max\{0,\beta^f_{n+1}-\beta^f_{n+2}\}-c_1}{2}\} - \max\{0, \frac{\max\{0,\beta^f_{1}-\beta^b_{n+2}\}-c_1}{2}\} & \text{if } c_1 \neq 0; \end{cases} \tag{5}$$

for $i = g$,

$$|T_g| = \begin{cases} 0 & \text{if } c^* = 0, \\ \gamma_g - \frac{\beta^f_{n+g}-c^*}{2} - \frac{\beta^b_{n+g}-c^*}{2} & \text{if } c^* \neq 0. \end{cases} \tag{6}$$

The sign of the negative twist component is -1*, and the sign of the positive twist component is* 1.

Proof. Let us denote the total twist number of the twist components of L in each G_i by $|T_i|$. Observe that the curve γ_i intersects once with the curve c_i (Figure 4a), and intersects once with each visible right and invisible right genus components (Figure 6b). Moreover, γ_i intersects L by the total number of the twists of twist components (Figure 7). However, from Remark 2.4, there cannot exist the twists and the curve c_i in G_i at the same time. Therefore, when $c_i \neq 0$, we have

$$\gamma_i = l^f_i + l^b_i + |T_i|, \tag{7}$$

where l^f_i, l^b_i, and $|T_i|$ denote the number of the visible right genus, the number of invisible right genus components, and the total twist number of the twist components in G_i, respectively.

Since a multiple curve consists of the simple closed curves that do not intersect each other and from Definition 2.6, we can write

$$\gamma_i = \max\{0, \frac{\max\{0,\beta^f_{n+i}-\beta^f_{n+i+1}\}-c_i}{2}\} + \max\{0, \frac{\max\{0,\beta^b_{n+i}-\beta^b_{n+i+1}\}-c_i}{2}\} + |T_i|.$$

Hence, we have Equality (4) as follows:

$$|T_i| = \gamma_i - \max\{0, \frac{\max\{0,\beta^f_{n+i}-\beta^f_{n+i+1}\}-c_i}{2}\} - \max\{0, \frac{\max\{0,\beta^b_{n+i}-\beta^b_{n+i+1}\}-c_i}{2}\}.$$

Equalities (5) and (6) can be obtained similar to the calculations above and calculations in [4], respectively. □

Remark 2.11. If there exists one of the upper diagonal or the lower diagonal components in the region G_i, the equation $c_i = d^u_{2i-1} + d^l_{2i} + |T_i|$ is used so that the curves on the surface do not intersect each other. In this case, $|T_i|$ cannot be greater than c_i.

By using the following theorem, we calculate the number of the curves c_i and c^* (Figure 4a) in each region G_i $(1 \le i \le g-1)$ and G^*, respectively.

Theorem 2.12. *Let L be given with the intersection numbers $(\alpha;\beta^f;\beta^b;\xi^f;\xi^b;\gamma;c;c^*)$. We find the number of the curves c_i and c^* in L, denoting by $p(c_i)$ and $p(c^*)$, as follows:*

For $2 \le i \le g-1$,

$$p(c_i) = \begin{cases} \gamma_i - \max\{0, \frac{\max\{0,\beta^f_{n+i}-\beta^f_{n+i+1}\}}{2}\} - \max\{0, \frac{\max\{0,\beta^b_{n+i}-\beta^b_{n+i+1}\}}{2}\} & \textit{if } c_i = 0 \\ 0 & \textit{if } c_i \neq 0; \end{cases} \tag{8}$$

for $i = 1$,

$$p(c_1) = \begin{cases} \gamma_1 - \max\{0, \frac{\max\{0,\beta^f_{n+1}-\beta^f_{n+2}\}}{2}\} - \max\{0, \frac{\max\{0,\beta^b_{1}-\beta^b_{n+2}\}}{2}\} & \textit{if } c_1 = 0, \\ 0 & \textit{if } c_1 \neq 0; \end{cases} \tag{9}$$

for $i = g$,

$$p(c^*) = \begin{cases} \gamma_g - \frac{\beta^f_{n+g}}{2} - \frac{\beta^b_{n+g}}{2} & \textit{if } c^* = 0, \\ 0 & \textit{if } c^* \neq 0. \end{cases} \tag{10}$$

Proof. Whenever $c_i = 0$, we have $\gamma_i = l^f_i + l^b_i + p(c_i)$. Since a multiple curve consists of the simple closed curves that do not intersect each other and from Definition 2.6, we can write

$$\gamma_i = \max\{0, \frac{\max\{0, \beta^f_{n+i} - \beta^f_{n+i+1}\}}{2}\} + \max\{0, \frac{\max\{0, \beta^b_{n+i} - \beta^b_{n+i+1}\}}{2}\} + p(c_i).$$

Hence, $p(c_i) = \gamma_i - \max\{0, \frac{\max\{0,\beta^f_{n+i}-\beta^f_{n+i+1}\}}{2}\} - \max\{0, \frac{\max\{0,\beta^b_{n+i}-\beta^b_{n+i+1}\}}{2}\}$ is derived.

Equalities (9) and (10) can be obtained similar to the calculations above and calculations in [4], respectively. □

In the following proposition, we find the number of the upper diagonal components, d^u_{2i-1}, and the lower diagonal components, d^l_{2i}, in each region G_i $(1 \le i \le g-1)$.

Proposition 2.13. *Let L be given with the intersection numbers* $(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*)$, *and let the number of the upper diagonal components and the number of the lower diagonal components in* G_i *be* d^u_{2i-1} *and* d^l_{2i}, *respectively. Then for* $1 \le i \le g-1$,

$$d^u_{2i-1} = \max\{c_i - |T_i|, T_i c_i\} - \max\{0, T_i c_i\}, \tag{11}$$

and

$$d^l_{2i} = \max\{c_i - |T_i|, -T_i c_i\} - \max\{0, -T_i c_i\}. \tag{12}$$

Proof. First, we assume that there are upper diagonal components in the region G_i. When $T_i < 0$, from Remark 2.11, we see $d^u_{2i-1} + d^l_{2i} = c_i - |T_i|$. From Remark 2.3, G_i has no lower diagonal components. Therefore, we can write $d^u_{2i-1} = c_i - |T_i|$. When $T_i > 0$, it should be $d^u_{2i-1} = 0$ so that the curves do not intersect each other. Equation (11) provides these properties completely.

When there are lower diagonal components in G_i, we can find Equation (12) similar to the number of the upper diagonal components. □

The twist numbers of each twist component of a multiple curve whose intersection numbers are given are obtained by using Remark 2.7 and Theorem 2.10, which we can find these twist numbers with the following proposition. The proof of this proposition is similar to the proof in [4].

Proposition 2.14. *Let L be given with the intersection numbers* $(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*)$. *Let* $|T_i|$ $(1 \le i \le g-1)$ *and* $|T_g|$ *be the total twist numbers in each regions* G_i *and* G^*, *respectively. Moreover, let* m_i *and* m^* *be the number of the twist components, each with* $t_i + 1$ *and* $t_g + 1$ *twists, and let* $c_i - d^u_{2i-1} - d^l_{2i} - m_i$ *and* $c^* - m^*$ *be the number of the twist components, each with* t_i *and* t_g *twists in each* G_i *and* G^*, *respectively. In this case,*

$$m_i \equiv |T_i| \ (mod\ (c_i - d^u_{2i-1} - d^l_{2i})), \quad and \quad t_i = \frac{|T_i| - m_i}{c_i - d^u_{2i-1} - d^l_{2i}}; \tag{13}$$

and

$$m^* \equiv |T_g| \ (mod\ c^*), \quad and \quad t_g = \frac{|T_g| - m^*}{c^*}, \tag{14}$$

where $c_i - d^u_{2i-1} - d^l_{2i} \neq 0$, *and* $c^* \neq 0$.

In Lemma 2.15, we define some auxiliary components that shall be used to calculate the number of the visible above components denoted by u^{va}_{2i-1}, and the number of the visible below components denoted by u^{vb}_{2i} in the rest of the paper.

Lemma 2.15. *Let L be given with the intersection numbers* $(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*)$. *For* $1 \leq i \leq g-1$, *if* $\beta^f_{n+i} \leq \beta^f_{n+i+1}$, *then the number of the intersections of the twist components together with the total diagonals with the arc* β^f_{n+i+1}, *denoted by* n_i, *is as follows:*

$$n_i = \frac{\beta^f_{n+i+1} - \beta^f_{n+i} + c_i}{2} - \max\{0, \frac{|\beta^f_{n+i} - \beta^f_{n+i+1}| - c_i}{2}\}. \tag{15}$$

Hence, we can find the number of the intersections of the twist components together with the total diagonals with the arc β^f_{n+i} *as* $c_i - n_i$.

On the other hand, if $\beta^f_{n+i} \geq \beta^f_{n+i+1}$, *then the number of the intersections of the twist components together with the total diagonals with the arc* β^f_{n+i}, *denoted by* k_i, *is as follows:*

$$k_i = \frac{\beta^f_{n+i} - \beta^f_{n+i+1} + c_i}{2} - \max\{0, \frac{|\beta^f_{n+i} - \beta^f_{n+i+1}| - c_i}{2}\}. \tag{16}$$

Hence, we can find the number of the intersections of the twist components together with the total diagonals with the arc β^f_{n+i+1} *as* $c_i - k_i$.

Proof. When $\beta^f_{n+i} \leq \beta^f_{n+i+1}$, we can write the number of the intersections on the arcs β^f_{n+i+1} and β^f_{n+i} as follows:

$$\beta^f_{n+i+1} = n_i + 2l^f_i + u^{va}_{2i-1} + u^{vb}_{2i}, \tag{17}$$

$$\beta^f_{n+i} = c_i - n_i + u^{va}_{2i-1} + u^{vb}_{2i}. \tag{18}$$

From equations (17) and (18), we derive

$$n_i = \frac{\beta^f_{n+i+1} - \beta^f_{n+i} + c_i}{2} - l^f_i.$$

When $\beta^f_{n+i} \geq \beta^f_{n+i+1}$, we can find k_i similar to n_i. □

Remark 2.16. ([3]) In each region U_i, for $1 \leq i \leq n$, let the number of the loop components be denoted by $|b_i|$, where

$$b_i = \frac{\beta^f_i - \beta^f_{i+1}}{2}. \tag{19}$$

If $b_i < 0$, then the loop component is called *left*; if $b_i > 0$, then the loop component is called *right*.

Now, we can find the number of the above and below components in each U_i $(1 \leq i \leq n)$, and the number of the visible above, visible below, invisible above, and invisible below components in each G_i $(1 \leq i \leq g-1)$.

Theorem 2.17. *Let the number of the above and below components in each U_i, and the number of the visible above, visible below, invisible above and invisible below components in each G_i be denoted by u^a_{2i-1}, u^b_{2i}, u^{va}_{2i-1}, u^{vb}_{2i}, $u^{v'a}_{2i-1}$ and $u^{v'b}_{2i}$, respectively. Moreover, let $A = \{a, b, va, vb, v'a, v'b\}$. For the function $\lambda : \mathbb{Z}^{3n+8g-5}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$*

$$u^x_i = \lambda^x_i(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*),$$

where for $1 \leq i \leq g-1$ and each $x \in A$, if $|T_i| \neq 0$, when $\beta^f_{n+i} \leq \beta^f_{n+i+1}$,

$$\lambda^a_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i-1} - |b_i|, \text{ and } \lambda^b_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i} - |b_i|, \tag{20}$$

$$\lambda^{va}_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i-1} - |T_i| - \max\{n_i - d^l_{2i}, T_i\} + \max\{0, T_i\} - l^f_i, \tag{21}$$

$$\lambda^{vb}_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i} - |T_i| - \max\{n_i - d^u_{2i-1}, -T_i\} + \max\{0, -T_i\} - l^f_i; \tag{22}$$

when $\beta^f_{n+i} \geq \beta^f_{n+i+1}$,

$$\lambda^a_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i-1} - |b_i|, \text{ and } f^b_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i} - |b_i|,$$

$$\lambda^{va}_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i-1} - |T_i| - \max\{c_i - k_i - d^l_{2i}, T_i\} + \max\{0, T_i\} - l^f_i, \tag{23}$$

$$\lambda^{vb}_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i} - |T_i| - \max\{c_i - k_i - d^u_{2i-1}, -T_i\} + \max\{0, -T_i\} - l^f_i; \tag{24}$$

if $|T_i| = 0$,

$$\lambda^a_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i-1} - |b_i|, \text{ and } \lambda^b_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \alpha_{2i} - |b_i|,$$

$$\lambda^{va}_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i-1} - \max\{p(c_i), d^u_{2i-1}\} - l^f_i, \tag{25}$$

$$\lambda^{vb}_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^f_{2i} - \max\{p(c_i), d^l_{2i}\} - l^f_i. \tag{26}$$

Moreover,

$$\lambda^{v'a}_{2i-1}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^b_{2i-1} - l^b_i, \text{ and } \lambda^{v'b}_{2i}(\alpha; \beta^f; \beta^b; \xi^f; \xi^b; \gamma; c; c^*) = \xi^b_{2i} - l^b_i. \tag{27}$$

Proof. The proofs of Equations (20) are obvious since each above and below component intersects α_{2i-1} and α_{2i}, respectively (see Figure 3).

Let $|T_i| \neq 0$. When $\beta^f_{n+i} \leq \beta^f_{n+i+1}$, from Lemma 2.15, the number of the intersections of the twist components together with the total diagonal components with the arc β^f_{n+i+1} is n_i. When we subtract the number of the lower diagonal components (Figure 6d) from n_i, the arc ξ^f_{2i-1} intersects $n_i - d^l_{2i}$ times with the twist components. The arc ξ^f_{2i-1} also intersects l^f_i times with the visible genus components (Figures 6a and 6b), u^{va}_{2i-1} times with the visible above components (Figure 6e), and by the total number of twists, $|T_i|$. When $T_i > 0$, ξ^f_{2i-1} intersects by the total number of twists; whereas when $T_i < 0$, ξ^f_{2i-1} intersects by the total number of twists and $n_i - d^l_{2i}$. That is,

$$\xi^f_{2i-1} = |T_i| + \max\{n_i - d^l_{2i}, T_i\} - \max\{0, T_i\} + l^f_i + u^{va}_{2i-1}.$$

Hence, we get Equation (21) as follows:

$$u^{va}_{2i-1} = \xi^{f}_{2i-1} - |T_i| - \max\{n_i - d^{l}_{2i}, T_i\} + \max\{0, T_i\} - l^{f}_{i}.$$

Similarly, in addition to the number of the visible genus components and the visible below components, when $T_i > 0$, ξ^{f}_{2i} intersects by the total number of twists and $n_i - d^{u}_{2i-1}$; whereas when $T_i < 0$, ξ^{f}_{2i} intersects by the total number of twists. Hence,

$$\xi^{f}_{2i} = |T_i| + \max\{n_i - d^{u}_{2i-1}, -T_i\} - \max\{0, -T_i\} + l^{f}_{i} + u^{vb}_{2i}.$$

From here, we can write Equation (22) as follows:

$$u^{vb}_{2i} = \xi^{f}_{2i} - |T_i| - \max\{n_i - d^{u}_{2i-1}, -T_i\} + \max\{0, -T_i\} - l^{f}_{i}.$$

When $\beta^{f}_{n+i} \geq \beta^{f}_{n+i+1}$, we can derive Equations (23) and (24) with the similar derivations of Equations (21) and (22) using Lemma 2.15.

Let $|T_i| = 0$. In this case, in addition to the visible genus components and the visible above components, ξ^{f}_{2i-1} intersects either the curve c_i or the upper diagonal components (see Remark 2.4). That is,

$$\xi^{f}_{2i-1} = u^{va}_{2i-1} + \max\{p(c_i), d^{u}_{2i-1}\} + l^{f}_{i}.$$

Thus, we obtain Equation (25) as $u^{va}_{2i-1} = \xi^{f}_{2i-1} - \max\{p(c_i), d^{u}_{2i-1}\} - l^{f}_{i}$. u^{vb}_{2i} is also derived in the same manner.

From Remark 2.3, ξ^{b}_{2i-1} intersects only invisible genus components and invisible above components, and ξ^{b}_{2i} intersects only invisible genus components and invisible below components. Thus, we can write

$$u^{v'a}_{2i-1} = \xi^{b}_{2i-1} - l^{b}_{i}, \quad \text{and} \quad u^{v'b}_{2i} = \xi^{b}_{2i} - l^{b}_{i}.$$

□

Example 2.18. Let $(6,2,4,2,5,1;8,6,4,6,7,2;3,0;5,4,6,6;4,1,0,0;2,5,3;3,3;0)$ be the intersection numbers of a multiple curve $L \in \mathcal{L}_{3,3}$ with the corresponding arcs and the simple closed curves c_i and c^* in $S_{3,3}$. Moreover, $T_1 > 0$ and $T_2 < 0$. We shall show how we draw L from the given intersection numbers.

First, we find the number of each path component in each region G_i for $i = 1, 2$, and in G^*, respectively. From Theorem 2.9, we have $l^{f}_{1} = 0$, $l^{f}_{2} = 1$, $l^{f}_{3} = 1$, $l^{b}_{1} = 1$, $l^{b}_{2} = 0$, and $l^{b}_{3} = 0$. Namely, there is one right invisible genus component; however, there is not any visible genus component in the region G_1. In G_2, there is one right visible genus component and no invisible genus component. In G^*, there is one visible genus component and no invisible genus component.

According to Theorem 2.10, $|T_1| = 1$, $|T_2| = 4$, and since $c^* = 0$, $|T_3| = 0$. That is, the total twist number of the twist components in the region G_1 is 1. The total twist number of the twist components in G_2 is 4; however, there is not any twist in G^*. From Theorem 2.12, we observe that since $c_1 \neq 0$ and $c_2 \neq 0$, there are no curves c_1 and c_2 in the regions G_1 and G_2. Further, we have $p(c^*) = 2$. Therefore, there are two curves c^* in G^*.

We can find the number of the upper and lower diagonal components using Proposition 2.13 in each G_i, $i = 1, 2$. We know that $T_1 > 0$. Thus, $d^{u}_{1} = 0$ and $d^{l}_{2} = 2$.

While there are two lower diagonal components in the region G_1, there are no upper diagonal components. From Remark 2.11, since $|T_2|$ is greater than c_2, there are no lower and upper diagonal components in G_2.

We calculate the twist numbers of each twist component of L in each G_i and G^* by Proposition 2.14. In G_1, $m_1 = 0$, $t_1 = 1$, and $c_1 - d^{u}_{1} - d^{l}_{2} - m_1 = 1$.

Therefore, there is one twist component which has one twist; however, there is not any twist component with $t_1 + 1 = 1 + 1 = 2$ twists in G_1. In G_2, $m_2 = 1$, $t_2 = 1$, and $c_2 - d^{u}_{3} - d^{l}_{4} - m_2 = 2$.

Thus, there are two twist components, each with one twist and one twist component which has $t_2 + 1 = 1 + 1 = 2$ twists in G_2. Since $c^* = 0$, there is no twist in G^*.

According to Lemma 2.15, due to $\beta_4^f < \beta_5^f$, $n_1 = 2$. Hence, the number of the intersections of the twist components together with the total diagonals with the arc β_5^f in G_1 is 2. The number of the intersections of the twist components together with the total diagonals with the arc β_4^f in G_1 is $c_1 - n_1 = 3 - 2 = 1$. In G_2, due to $\beta_5^f > \beta_6^f$, $k_2 = 3$. Thus, the number of the intersections of the twist components together with the total diagonals with the arc β_5^f in G_2 is 3. The number of the intersections of the twist components together with the total diagonals with the arc β_6^f in G_2 is $c_2 - k_2 = 3 - 3 = 0$.

We find the loop components in each region U_i, $i = 1, 2, 3$ by Remark 2.16 as $b_1 = 1$, $b_2 = 1$, and $b_3 = -1$.

Namely, there is one right loop component in U_1, one right loop component in U_2, and one left loop component in U_3.

We calculate the number of the above and below components in each U_i $(1 \leq i \leq 3)$, and the number of the visible above, visible below, invisible above, and invisible below components in each G_i $(1 \leq i \leq 2)$ by using Theorem 2.17. Since $u_1^a = 5$, $u_2^b = 1$, $u_3^a = 3$, $u_4^b = 1$, $u_5^a = 4$, and $u_6^b = 0$, we have five above components and one below component in U_1, three above components and one below component in U_2, and four above components and zero below component in U_3.

Since $|T_1| \neq 0$ and $\beta_4^f < \beta_5^f$ in G_1; $u_1^{va} = 4$ and $u_2^{vb} = 1$. Moreover, $u_1^{v'a} = 3$ and $u_2^{v'b} = 0$. There are four visible above components, one visible below component, three invisible above components, and no invisible below component in G_1.

Since $|T_2| \neq 0$ and $\beta_5^f > \beta_6^f$ in G_2; $u_3^{va} = 1$ and $u_4^{vb} = 1$. Moreover, $u_3^{v'a} = 0$ and $u_4^{v'b} = 0$. There are one visible above component, one visible below component, zero invisible above component, and zero invisible below component in G_2. The calculated path components in each U_i, G_i, and G^* are connected in a unique way up to isotopy. Thus, the multiple curve L in Figure 8 is determined uniquely.

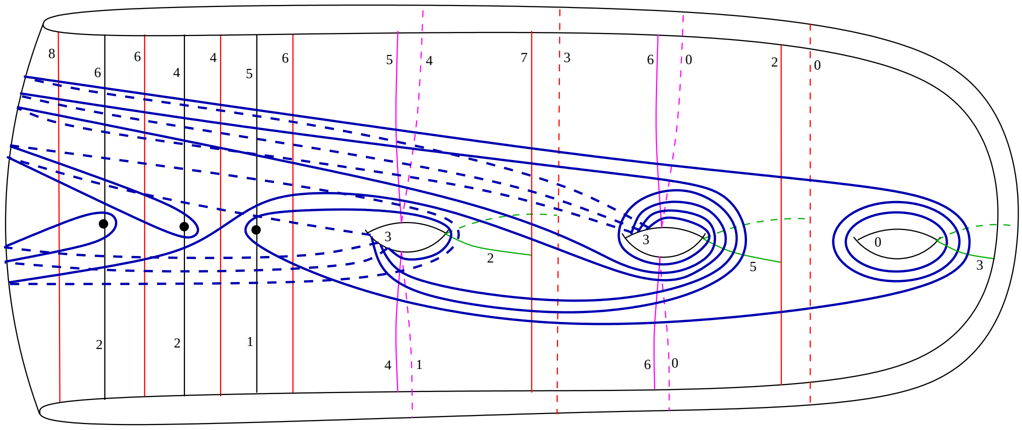

Figure 8: The multiple curve L with the intersection numbers (6, 2, 4, 2, 5, 1; 8, 6, 4, 6, 7, 2; 3, 0; 5, 4, 6, 6; 4, 1, 0, 0; 2, 5, 3; 3, 3; 0)

In the light of all this information, we have the following theorem.

Theorem 2.19. *The geometric intersection function* $\psi : \mathcal{L}_{n,g} \to \mathbb{Z}_{\geq 0}^{3n+8g-5} \setminus \{0\}$ *given by Definition 2.1 is injective.*

Proof. The numbers of above, below, right loop or left loop components in each region U_i (from Theorem 2.17 and Remark 2.16, respectively), the numbers of curves c^*, visible genus, invisible genus, twist components, the total twist of twist components and the number of twists of each twist component, the direction of these twists in the region G^* (see Theorem 2.12, Theorem 2.9, Definition 2.5, Theorem 2.10, Proposition 2.14, and Remark 2.8, respectively), the numbers of curves c, visible left genus, invisible left genus, visible right genus, invisible right genus, upper diagonal, lower diagonal, visible above, invisible above, visible below, invisible below, twist components, the total twist of twist components and the number of twists of each twist component, the direction of these twists in each region G_i (as obtained in Theorem 2.12, Theorem 2.9, Proposition 2.13, Theorem 2.17, Definition 2.5, Theorem 2.10, Proposition 2.14, and Remark 2.8, respectively) are calculated as given in this paper and as in Example 2.18, and the path components in the regions U_i, G_i, and G^* are combined uniquely up to isotopy by giving a direction to the twist components. Hence, ψ is injective. □

3. Conclusion

In this work, we have generalized the approach, which describes each multiple curve on the standard n-times punctured disk and punctured orientable genus-1 surface which has one boundary, to the orientable surface of genus-g which has n punctures and one boundary component by using the multiple curve's geometric intersection number with the embedded curves in this surface. It is thought that the previous works on curves on the disk, such as solving the word problem in the braid group [1, 2], calculating the topological entropy of an braid [5], and computing the geometric intersection number of two multiple curves on the disk [9], can be expanded to this surface in the future by using the formulas proposed in this work. It is the author's intention that this paper on multiple curves on the punctured orientable genus-g surface with one boundary will provide a useful reference for researchers who want to study this subject.

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