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Higher Topological Complexity for Fibrations

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Abstract. We introduce the higher topological complexity (TC_n) of a fibration in two ways: the higher homotopic distance and the Schwarz genus. Then we have some results on this notion related to TC, TC_n or cat of a topological space or a fibration. We also show that TC_n of a fibration is a fiber homotopy equivalence.

1. Introduction

The works on determining the topological complexity have been developing since Farber first computed the number of the topological complexity (TC) for a topological space [4] (see [5] for a comprehensive analysis of the investigation of this concept). Note that such a space must be path-connected. Rudyak [10] has a number of higher topological complexity TC_n for a topological space for the first time (similarly the space is path-connected). It is a reasoned generalization on existing works of TC because of the fact that $\mathrm{TC}_2 = \mathrm{TC}$. After that Pavešić [9] gives a new point of view to the developing explorations by determining the topological complexity of a map. Such a map must be surjective and continuous or one can choose a surjective fibration. Murillo and Wu [8] present the homotopy invariant version of the topological complexity of maps. These studies have a common point: All of these investigations can be considered by using the concept of the Schwarz genus (secat) for a fibration, introduced by Schwarz [12]. One can use a fibrational substitute, as well. On the other hand, there is one more option, which is the homotopic distance [6]. It can be used for the computations of TC or TC_n of a topological space or a map. The higher homotopic distance [2] is a general case of the homotopic distance and especially useful for the studies of TC_n of a topological space. Since LS-category (cat) for a topological space or a map is the base point for the research of TC, cat is obtained by using both the Schwarz genus and the homotopic distance [3, 6].

These studies lead us to solve an open problem and introduce a new concept the higher topological complexity of a surjective fibration f between any path-connected topological spaces, denoted by $\mathrm{TC}_n(f)$. For $n = 2$, we show that $\mathrm{TC}_2(f) = \mathrm{TC}(f)$. In addition, we have that $\mathrm{TC}_n(f) = \mathrm{TC}_n(Y)$ if f is $1_Y : Y \to Y$. Therefore, $\mathrm{TC}_n(f)$ is a general version of respective notions $\mathrm{TC}(Y)$, $\mathrm{TC}_n(Y)$ and $\mathrm{TC}(f)$, where $f : X \to X'$ is a surjective fibration and Y is path-connected. We define $\mathrm{TC}_n(f)$ by using the two options the Schwarz genus and the higher homotopic distance.

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In this study, we first give explicit definitions and significant results of TC, TC_n and their related invariants such as $D(f, g)$ and cat. Then we express $TC(f)$ as the homotopic distance. With this expression, we state some results on $TC(f)$. So, we introduce the definition of $TC_n(f)$ for a surjective fibration f. Later, we state some properties on $TC_n(f)$. In addition, we prove that $TC_n(f)$ is invariant on fiber homotopy equivalence fibrations. We also research the relationship between $TC_n(f)$ and other related notions such as $TC_n(Y)$ and cat(Y). We choose some special fibrations as examples for computing their higher topological complexities. In Section 4, we define the higher topological complexity of a surjective fibration by using secat.

2. Preliminaries

We begin with recalling some basic notions and results on the homotopic distance, the topological complexities, and the Lusternik-Schnirelmann category.

Definition 2.1. ([6]) Given two continuous functions $f, f^{'} : X \to X^{'}$. Then the *homotopic distance* for f and $f^{'}$, denoted by $D(f, f^{'})$, is the smallest integer $l > 0$ satisfying that there is a covering

$$X = V_1 \cup \cdots \cup V_l$$

and the restrictions of f and $f^{'}$ on V_j, $j = 1, \cdots, l$, are homotopic, i.e., $f_{|V_j} \simeq f^{'}_{|V_j}$.

Definition 2.2. ([2, 6]) Let $f_i : X \to X^{'}$ be any continuous map for $i = 1, \cdots, k$. Then the *higher homotopic distance*, denoted by $D(f_1, \cdots, f_k)$, is the smallest positive integer l for which

$$X = U_1 \cup \cdots \cup U_l$$

is a covering and the condition

$$f_1^j|_{U_j} \simeq \cdots \simeq f_k^j|_{U_j}$$

satisfies for each $j = 1, \cdots, l$. If such a covering does not exist, then $D(f_1, \cdots, f_k)$ equals ∞.

Note that, in the definition of (higher) homotopic distance, the covering of X has $k + 1$ open sets $U_0, U_1, \cdots U_k$, but we consider that U_1 is the first open set in this covering for consistency of Farber's topological complexity definition (see [4]).

Proposition 2.3. ([6]) *Each of the following conditions on the notion homotopic distance is satisfied:*
i) *If f and g are homotopic ($f \simeq g$), and $f^{'}$ and $g^{'}$ are homotopic ($f^{'} \simeq g^{'}$), then*

$$D(f, f^{'}) = D(g, g^{'}).$$

ii) *For any functions $f, f^{'} : X \to X^{'}$, and $k : Z \to X$, we have*

$$D(f \circ k, f^{'} \circ k) \leq D(f, f^{'}).$$

iii) *Let $f, f^{'} : X \to X^{'}$ be any functions. If $k : X^{'} \to Z$ is a function such that there is a function $k^{'} : Z \to X^{'}$ satisfying $k^{'} \circ k \simeq 1_{X^{'}}$, then*

$$D(k \circ f, k \circ f^{'}) = D(f, f^{'}).$$

iv) *Let $f, f^{'} : X \to X^{'}$ be any functions. If $k : Z \to X$ is a function such that there is a function $k^{'} : X \to Z$ satisfying $k \circ k^{'} \simeq 1_X$, then*

$$D(f \circ k, f^{'} \circ k) = D(f, f^{'}).$$

v) *Let X be normal. If $f, f^{'} : X \to X^{'}$, and $g, g^{'} : X^{'} \to Y$ are any functions, then*

$$D(g \circ f, g^{'} \circ f^{'}) \leq D(f, f^{'}) + D(g, g^{'}).$$

Proposition 2.4. ([2]) *Each of the following conditions on the notion of higher homotopic distance is satisfied:*
i) *If $f_1, f_2, \cdots, f_l, \cdots, f_k : X \to X'$ are any functions for $1 < l < k$, then*

$$D(f_1, \cdots, f_l) \leq D(f_1, \cdots, f_k).$$

ii) *If $f_1, \cdots, f_k : X \to X'$ and $g_1, \cdots, g_k : X' \to Y$ are functions such that $g_i \simeq g_{i+1}$ for every $i = 1, \cdots, k-1$, then*

$$D(g_1 \circ f_1, \cdots, g_k \circ f_k) \leq D(f_1, \cdots, f_k).$$

iii) *Given two path-connected spaces X and X'. If $f_1, \cdots, f_k : X \to X'$ are any functions, then*

$$D(f_1, \cdots, f_k) \leq cat(X).$$

iv) *If $f_1, \cdots, f_k : X \to X'$ are any functions, then*

$$D(f_1, \cdots, f_k) \leq TC_k(X').$$

v) *Let $X \times \overline{X'}$ be normal. If $f_1, \cdots, f_k : X \to X'$ and $g_1, \cdots, g_k : \overline{X} \to \overline{X'}$ are any functions, then*

$$D(f_1 \times g_1, \cdots, f_k \times g_k) \leq D(f_1, \cdots, f_k) + D(g_1, \cdots, g_k).$$

If $f : X \to Z$ is a fibration, then the *Schwarz genus* [12] of it, denoted by *secat*(f), is the smallest integer $l > 0$ for which

$$Z = V_1 \cup \cdots \cup V_l$$

is a covering and the condition

$$f \circ s_i = 1_{V_i}$$

satisfies for each V_i, $i = 1, \cdots, l$. If there is no such a covering of Z, then we point out that *secat*$(f) = \infty$. We again note that we use l open sets in the covering of Y instead of $l+1$ for consistency of our results with Definition 2.2. Just as homotopic distance, the notion Schwarz genus is an important common point in the exploration of motion planning problems because each of TC, TC_n, and cat is motivated by the Schwarz genus of some special fibrations.

Definition 2.5. ([4]) For any path-connected Y, the *topological complexity* of Y, denoted by $TC(Y)$, is *secat*(π), where $\pi : Y^I \to Y \times Y$, $\pi(\beta) = (\beta(0), \beta(1))$, is a path fibration.

Definition 2.6. ([10]) For any path-connected Y, if $\beta = (\beta_1, \cdots, \beta_n) : I \to Y^n$ is a multipath in Y such that the starting point of each β_i, $i = 1, \cdots, n$, is the same. Then the *higher topological complexity* of Y, denoted by $TC_n(Y)$, is *secat*(e_n), where $e_n : (Y^n)^I \to Y^n$, $\beta \mapsto e_n(\beta) = (\beta_1(1), \cdots, \beta_n(1))$, is a fibration.

Denote J_n by the wedge of n copies of $[0,1]$ such that the initial point 0_i is the same point for each $i = 1, \cdots, n$. Then the function space Y^{J_n} includes all multipaths in a path-connected space Y. Therefore, $TC_n(Y)$ is also defined as

$$secat(e_n : Y^{J_n} \to Y_n),$$

$e_n(\beta) = (\beta_1(1), \cdots, \beta_n(1))$ for a multipath $\beta = (\beta_1, \cdots, \beta_n) \in Y^{J_n}$. $TC_n(Y) = 1$ when $n = 1$. For $n = 2$, the Definition 2.5 coincides with Definition 2.6. Moreover, $TC_n(Y) \leq TC_r(Y)$ if $n + 1 \leq r$.

Definition 2.7. ([9]) Let $g : Y \to Y'$ be a surjective fibration. Then the *topological complexity of* g, denoted by $TC(g)$, is *secat*(π_g), where $\pi_g : Y^I \to Y \times Y'$ with $\pi_g(\beta) = (\beta(0), g \circ \beta(1))$ is a fibration.

For a fibration g, $TC(g)$ is also motivated by the homotopic distance [7]. We observe the same thing with an alternative method in Theorem 3.3, as well.

Proposition 2.8. ([9]) *The following hold:*

i) *Let $g : Y \to Y'$ be a surjective and continuous function. If $k : Z \to Y$ is a function such that there is a function $k' : Y \to Z$ satisfying $k \circ k' \simeq 1_Y$, then*

$$TC(g \circ k) \geq TC(g).$$

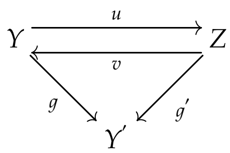
ii) *Let $g : Y \to Y'$ be a surjective and continuous function. If $k : Z \to Y$ is a function such that there is a function $k' : Y \to Z$ satisfying $k' \circ k \simeq 1_Z$, then*

$$TC(g \circ k) \leq TC(g).$$

iii) *If $g : Y \to Y'$ is a surjective and continuous map, and $k : Z \to Y$ is a fibration, then*

$$TC(g \circ k) \leq TC(g).$$

Recall that two functions $g : Y \to Y'$ and $g' : Z \to Y'$ are *fiber homotopy equivalent* (FHE-equivalent) if the diagram



is commutative with $u \circ v \simeq 1_Z$ and $v \circ u \simeq 1_Y$.

Theorem 2.9. ([9]) *Let $g : Y \to Y'$ and $g' : Z \to Y'$ be maps. Given any fibrewise maps $u : Y \to Z$ and $v : Z \to Y$ with $u \circ v \simeq 1_Z$ and $v \circ u \simeq 1_Y$, we have*

$$TC(g) = TC(g').$$

Denote P_0Y as the space of entire paths in a topological space Y such that the starting points of the paths are the same point. Then the *LS-category* [3] of Y, denoted by $cat(Y)$, is $secat(\pi_Y)$, where $\pi_Y : P_0Y \to Y$, $\pi_Y(\beta) = \beta(1)$, is a fibration. If $g : Y \to Y'$ is a function, then the *LS-category* [3] of g, denoted by $cat(g)$, is the smallest integer $m > 0$ for which

$$Y = W_1 \cup \cdots \cup W_m$$

is a covering such that $g|_{W_i}$ is nullhomotopic for each $i = 1, \cdots, m$. In a similar manner, for consistency, we use m open sets for the cover of X instead of $m + 1$ open sets in the definition of $cat(g)$.

Using the higher homotopic distance; alternative expressions of $TC(Y)$, $TC_n(Y)$, $TC(g)$, $cat(Y)$, and $cat(g)$ are given by the next theorem.

Theorem 2.10. ([2, 6, 7]) *Each of the following holds:*

i) *Let Y be path-connected. Then*

$$TC(Y) = D(p_1, p_2),$$

where $p_1, p_2 : Y \times Y \to Y$ are functions with $p_1(y, y') = y$ and $p_2(y, y') = y'$, respectively.

ii) *Let Y be path-connected. Then*

$$TC_n(Y) = D(p_1, \cdots, p_n),$$

where $p_i : Y^n \to Y$ is the projection for each $j \in \{1, \cdots, n\}$.

iii) *If $g : Y \to Y'$ is a surjective fibration, then*

$$TC(g) = D(g \circ \pi_1, \pi_2),$$

where $\pi_1 : Y \times Y' \to Y$ and $\pi_2 : Y \times Y' \to Y$ are projection maps.

iv) *For any space Y,*

$$cat(Y) = D(j_1, j_2),$$

where $j_1 : Y \to Y \times Y$ is defined with $j_1(y) = (y, y_0)$ and $j_2 : Y \to Y \times Y$ is defined with $j_2(y) = (y_0, y)$ for the point $y_0 \in Y$.

v) *If $g : Y \to Y'$ is a function such that Y' is path-connected, then*

$$cat(g) = D(g, *),$$

where $*$ is a constant map.

3. The higher topological complexity using the homotopic distance

Let $f : X \to Y$ be a fibration, $p_1 : Y \times Y \to Y$ be a projection defined by $p_1(y, y') = y$, and $\pi_2 : X \times Y \to Y$ be another projection defined by $\pi_2(x, y) = y$. In Theorem 2.8 of [6], if we take $X \times Y$, $p_1 \circ (f \times 1_Y)$, and π_2 instead of X, f and g, respectively, then we find that

$$\begin{array}{ccc} P & \longrightarrow & PY \\ {\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X \times Y & \xrightarrow[h]{} & Y \times Y \end{array}$$

is commutative, where h is the map $(p_1 \circ (f \times 1_Y), \pi_2) : X \times Y \to Y \times Y$. This proves that

$$D(p_1 \circ (f \times 1_Y), \pi_2) = secat(q_1). \tag{1}$$

On the other hand, in Theorem 4.9 of [9], if we take $p_1 \circ (f \times 1_Y)$ instead of the fibration f, then we observe that

$$\begin{array}{ccc} X \sqcap PY & \longrightarrow & PY \\ {\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X \times Y & \xrightarrow[k]{} & Y \times Y \end{array}$$

is also commutative, where k is $(p_1 \circ (f \times 1_Y)) \times 1_Y : X \times Y \to Y \times Y$. This gives

$$TC(p_1 \circ (f \times 1_Y)) = secat(q_2). \tag{2}$$

Theorem 3.1. *For a fibration $f : X \to Y$, if $p_1 : Y \times Y \to Y$ and $\pi_2 : X \times Y \to Y$ are projections with $p_1(y, y') = y$ and $\pi_2(x, y) = y$, respectively, then*

$$D(p_1 \circ (f \times 1_Y), \pi_2) = TC(p_1 \circ (f \times 1_Y)).$$

Proof. By (1) and (2), it is enough to show that q_1 and q_2 are fibre homotopy equivalent fibrations. Define the maps $u : P \to X \sqcap PY$ and $v : X \sqcap PY \to P$ with $u(x, y, \alpha) = (x, \alpha)$ and $v(x, \alpha) = (x, \alpha(1), \alpha)$, respectively. Since $P \subset (X \times Y) \times PY$ is given by $\{((x, y), \alpha) \ : \ \alpha(0) = p_1 \circ (f \times 1_Y)(x, y), \alpha(1) = \pi_2(x, y)\}$, we have that $\alpha(0) = f(x)$ and $\alpha(1) = y$. It follows that

$$u \circ v = 1_{X \sqcap PY} \quad \text{and} \quad v \circ u = 1_P.$$

Thus, q_1 and q_2 are fibre homotopy equivalent to each other. □

Corollary 3.2. *For a fibration $f : X \to Y$, if $p_1 : Y \times Y \to Y$ and $\pi_2 : X \times Y \to Y$ are projections with $p_1(y, y') = y$ and $\pi_2(x, y) = y$, respectively, then*

$$D(p_1 \circ (f \times 1_Y), \pi_2) \leq TC(Y).$$

Moreover, if Y is contractible, then we conclude that $D(p_1 \circ (f \times 1_Y), \pi_2) = 1$.

Proof. By Theorem 3.1, we obtain

$$D(p_1 \circ (f \times 1_Y), \pi_2) = TC(p_1 \circ (f \times 1_Y)).$$

Since f is a fibration, so is $f \times 1_Y$. Using Proposition 2.8 (iii), and Example 4.10 in [9], we get

$$TC(p_1 \circ (f \times 1_Y)) \leq TC(p_1) = TC(Y).$$

Finally, the contractibility of Y implies that $D(p_1 \circ (f \times 1_Y), \pi_2) = 1$. □

Theorem 3.3. *For a fibration $f : X \to Y$, if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are projections with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, respectively, we have*

$$D(f \circ \pi_1, \pi_2) = TC(f).$$

Proof. Given a fibration $f : X \to Y$, we rewrite the continuous function $p_1 \circ (f \times 1_Y)$ as $f \circ \pi_1$. By Proposition 2.3 (i), we have

$$D(p_1 \circ (f \times 1_Y), \pi_2) = D(f \circ \pi_1, \pi_2).$$

From Theorem 2.8, we find $D(f \circ \pi_1, \pi_2) = TC(f \circ \pi_1)$. Define a map $u : X \to X \times Y$ with $u(x) = (x, f(x))$. Therefore, we get $\pi_1 \circ u \simeq 1_X$. By Proposition 2.8 (i), we obtain

$$D(f \circ \pi_1, \pi_2) = TC(f \circ \pi_1) \geq TC(f).$$

On the other hand, Proposition 2.8 (iii) gives us that

$$D(f \circ \pi_1, \pi_2) = TC(f \circ \pi_1) \leq TC(f).$$

As a consequence, $D(f \circ \pi_1, \pi_2) = TC(f)$. □

Theorem 3.3 confirms the following result [9]:

Corollary 3.4. *The topological complexity of a space coincides with the topological complexity of the identity function on it.*

Proposition 3.5. *If $f : X \to X'$ is a fibration, then we have*

$$D(f, f) \leq TC(f).$$

Proof. Consider a fibration $f : X \to X'$ and a map $h : X \to X \times X'$ defined as $h(x) = (x, f(x))$. By Proposition 2.3 (ii), we obtain

$$TC(f) = D(f \circ \pi_1, \pi_2) \geq D((f \circ \pi_1) \circ h, \pi_2 \circ h) = D(f, f),$$

which completes the proof. □

Proposition 3.5 also confirms the well-known fact that TC of a fibration $f : X \to X'$ is a nonnegative integer when considering $D(f, f) = 1$.

Proposition 3.6. *For any fibrations $f, f' : X \to X'$, we have*

$$D(f, f') \leq TC(f) \cdot TC(f').$$

Proof. Let $TC(f) = k$ and $TC(f') = l$. Then $\{U_i\}_{i=1}^{k}$ and $\{V_j\}_{j=1}^{l}$ are corresponding coverings of $X \times X'$. In addition, we get

$$(f \circ \pi_1)\big|_{U_i} \simeq \pi_2\big|_{U_i} \quad \text{and} \quad (f' \circ \pi_1)\big|_{V_j} \simeq \pi_2\big|_{V_j}.$$

Let $W_{i,j} = U_i \cap V_j$. This covers $X \times X'$ and we derive

$$f \circ \pi_1\big|_{W_{i,j}} = (f \circ \pi_1)\big|_{W_{i,j}} \simeq \pi_2\big|_{W_{i,j}} \simeq (f' \circ \pi_1)\big|_{W_{i,j}} = f' \circ \pi_1\big|_{W_{i,j}}.$$

This means that $D(f \circ \pi_1\big|_{W_{i,j}}, f' \circ \pi_1\big|_{W_{i,j}}) \leq k \cdot l$. Recalling that $h : X \to X \times X'$, $h(x) = (x, f(x))$, is a right homotopy inverse of π_1, by Proposition 2.3 (iii), we conclude that

$$D(f, f') = D(f \circ \pi_1\big|_{W_{i,j}}, f' \circ \pi_1\big|_{W_{i,j}}) \leq k \cdot l.$$

This gives the desired result. □

Assume that $f = (f_1, \cdots, f_n) : X \to Y^n$ is a function. If f is a fibration, then $f_i : X \to Y$ is a fibration for all $i = 1, \cdots, n$. Indeed, for the projection map $p_i : Y^n \to Y$ onto the $ith-$factor for each i, we have $p_i \circ f = f_i$. Since projections are fibrations, f_i is a fibration for all i.

Definition 3.7. Let $f = (f_1, f_2, \cdots, f_n) : X \to Y^n$ be a surjective fibration for $n > 1$. Let $p_i : X^n \to X$ be the projection map onto the $ith-$factor for $1 \leq i \leq n$. Then the $n-$dimensional higher topological complexity of f is

$$TC_n(f) = D(f \circ p_1, f \circ p_2, \cdots, f \circ p_n).$$

In Definition 3.7, we assume that $TC_1(f)$ is always equal to 1. If f is identity, then $TC_n(f) = TC_n(X)$. In addition, the following result proves that $TC_2(f)$ coincides with the notion $TC(f)$ in Theorem 3.3.

Proposition 3.8. *$TC_2(f)$ in Definition 3.7 coincides with $TC(f)$ in Theorem* 3.3.

Proof. Let $f = (f_1, f_2) : X \to Y \times Y$ and $f^{'} : X \to Y$ be any continuous functions. For each projection $p_i : X^n \to X$, $i = 1, 2$, and the functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, respectively, we shall prove that $D(f \circ p_1, f \circ p_2) = D(f^{'} \circ \pi_1, \pi_2)$. We take

$$\alpha = 1_Y : Y \to Y$$

and

$$\beta = 1_X \times f^{'} : X \times X \to X \times Y$$

and consider the following two diagrams:

$$\begin{array}{ccc} X \times Y & \xrightarrow{f^{'} \circ \pi_1} & Y \\ {\scriptstyle 1_X \times f^{'}} \uparrow & & \downarrow {\scriptstyle 1_Y} \\ X \times X & \xrightarrow[f \circ p_1]{} & Y, \end{array}$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle 1_X \times f^{'}} \uparrow & & \downarrow {\scriptstyle 1_y} \\ X \times X & \xrightarrow[f \circ p_2]{} & Y. \end{array}$$

There is an element $x^{'} \in X$ with $f^{'}(x^{'}) = y$ because $f^{'}$ is surjective. Define a new continuous function $\gamma : X \times Y \to X \times X$ with $\gamma(x, y) = (x, x^{'})$. Then, we observe that $\gamma \circ \beta$ and $\beta \circ \gamma$ are identity maps, namely that, γ is a homotopy inverse of the function β. By Proposition 3.14 in [6], we conclude that $D(f \circ p_1, f \circ p_2)$ equals $D(f^{'} \circ \pi_1, \pi_2)$. $\square$

Proposition 3.9. *For a fibration $f : X \to Y^n$, we have $TC_n(f) \leq TC_{n+1}(f)$.*

Proof. Let $TC_{n+1}(f) = k$. Then $D(f \circ p_1, \cdots, f \circ p_{n+1}) = k$ for a fibration $f : X \to Y^n$. By Proposition 2.4 (i), we obtain $D(f \circ p_1, \cdots, f \circ p_n) \leq k$. It follows that $TC_n(f) \leq k$. $\square$

Proposition 3.10. *If $f = (f_1, \cdots, f_n) : X \to Y^n$ and $f^{'} = (f_1^{'}, \cdots, f_n^{'}) : \overline{X} \to \overline{Y}^n$ are two fibrations with a normal space $X \times \overline{X}$, then*

$$TC_n(f \times f^{'}) \leq TC_n(f) + TC_n(f^{'}).$$

Proof. Assume that $TC_n(f) = k$ and $TC_n(f') = l$. Then $D(f \circ p_1, \cdots, f \circ p_n) = k$ and $D(f' \circ \overline{p}_1, \cdots, f' \circ \overline{p}_n) = l$ for projections maps $p_i : X^n \to X$ and $\overline{p}_i : \overline{X}^n \to \overline{X}$ with each $i \in \{1, \cdots, n\}$. By Proposition 2.4 (v), we have

$$\begin{aligned} D(f \circ p_1, \cdots, f \circ p_n) + D(f' \circ \overline{p}_1, \cdots, f' \circ \overline{p}_n) &\geq D((f \circ p_1) \times (f' \circ \overline{p}_1), \cdots, (f \circ p_n) \times (f' \circ \overline{p}_n)) \\ &= D((f \times f') \circ (p_1 \times \overline{p}_1), \cdots, (f \times f') \circ (p_n \times \overline{p}_n)) \\ &= D((f \times f') \circ p'_1, \cdots, (f \times f') \circ p'_n), \end{aligned}$$

where $p'_i : X^n \times \overline{X}^n \to X \times \overline{X}$ is $ith-$projection map. This shows that $k + l \geq TC_n(f \times f')$. □

Proposition 3.11. *Let $X \times X$ be normal. For a fibration $f : X \to X'$, we have $TC(f) \leq TC(X) + 1$.*

Proof. By Proposition 2.3 (v), we get

$$TC(f) = D(f \circ p_1, f \circ p_2) \leq D(p_1, p_2) + D(f, f) = TC(X) + 1,$$

where $X \times X$ is normal. □

Proposition 3.12. *Given two fibrations $f, f' : X \to Y^n$ ($f \simeq f'$) homotopic to each other, we have $TC_n(f) = TC_n(f')$.*

Proof. $f \simeq f'$ implies that $f \circ p_i \simeq f' \circ p_i$ for $i = 1, \cdots, n$. Therefore, we find

$$D(f \circ p_1, \cdots, f \circ p_n) = D(f' \circ p_1, \cdots, f' \circ p_n).$$

Thus, we conclude that $TC_n(f) = TC_n(f')$. □

Note that Proposition 3.14 of [6] can be easily generalized as follows:

Proposition 3.13. *Given any homotopy equivalences $\alpha : Y \simeq \overline{Y}$ and $\beta : \overline{X} \simeq X$, assume that $\alpha \circ f_i \circ \beta \simeq f'_i$ for every $i = 1, \cdots, n$, where $f_i : X \to Y$ and $f'_i : \overline{X} \to \overline{Y}$, i.e., the following diagram commutes for each i:*

$$\begin{array}{ccc} X & \xrightarrow{f_i} & Y \\ \beta \uparrow & & \downarrow \alpha \\ \overline{X} & \xrightarrow[f'_i]{} & \overline{Y}. \end{array}$$

Then $D(f_1, \cdots, f_n) = D(f'_1, \cdots, f'_n)$.

Corollary 3.14. *$TC_n(f)$ is a fiber homotopy equivalent invariant.*

Proof. Let $f = (f_1, \cdots, f_n) : X \to Y^n$ and $f' = (f'_1, \cdots, f'_n) : \overline{X} \to Y^n$ be any fibrations for which there are two functions $u : X \to \overline{X}$ and $v : \overline{X} \to X$. These satisfy the conditions $u \circ v \simeq 1_{\overline{X}}$ and $v \circ u \simeq 1_X$. For each $i \in \{1, \cdots, n\}$, define $\alpha = 1_{Y^n}$ and $\beta = d_n \circ v \circ \overline{p}_i$, where $d_n : X \to X^n$ denotes the diagonal function and $\overline{p}_i : \overline{X}^n \to \overline{X}$ is the projection. It follows that α and β are two homotopy equivalences. Indeed, $\overline{d_n} \circ u \circ p_i$ is the homotopy inverse of β, where $\overline{d_n} : \overline{X} \to \overline{X}^n$ denotes the diagonal function and $p_i : X^n \to X$ is the projection. Then the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{f \circ p_i} & Y^n \\ \beta \uparrow & & \downarrow \alpha \\ \overline{X}^n & \xrightarrow[f' \circ \overline{p}_i]{} & Y^n \end{array}$$

commutes for each i. By Proposition 3.13, we get

$$D(f \circ p_1, \cdots, f \circ p_n) = D(f' \circ \overline{p}_1, \cdots, f' \circ \overline{p}_n).$$

This proves that $TC_n(f) = TC_n(f')$ for two fiber homotopy equivalent fibrations f and f'. □

Proposition 3.15. *$TC_n(f) \leq TC_n(X)$ for a fibration $f = (f_1, \cdots, f_n) : X \to Y^n$.*

Proof. The proof follows directly from Proposition 2.4 (ii). □

Proposition 3.16. *Let $n > 1$. Then $TC(f) \leq TC_n(X)$ for a fibration $f : X \to Y$.*

Proof. By Proposition 2.4 (i) and (ii), we obtain

$$TC(f) = D(f \circ p_1, f \circ p_2) \leq D(p_1, p_2) \leq D(p_1, \cdots, p_n) = TC_n(X)$$

for a fibration f. □

We state a confirmation for the following well-known result [1]:

Theorem 3.17. *Let X be a path-connected space. Then we have*

$$cat(X^{n-1}) \leq TC_n(X) \leq cat(X^n).$$

Proof. Let $i_k : X \to X^{n-1}$ be the inclusion map such the kth−component of $i_k(x)$ is $x^{'} \in X^{n-1}$, and the other components are always x for $1 \leq k \leq n-1$. Then we find

$$cat(X^{n-1}) = D(1_{X^{n-1}}, x^{'}) = D(p_2 \circ i_1, p_1 \circ i_1) \leq D(p_1, p_2).$$

Proposition 2.4 (i) says that

$$D(p_1, p_2) \leq D(p_1, \cdots, p_n) = TC_n(X),$$

which proves the first inequality. On the other hand, consider the projection maps $p_1, \cdots, p_n : X^n \to X$ on the path-connected space X. By Proposition 2.4 (iii), we have

$$TC_n(X) = D(p_1, \cdots, p_n) \leq cat(X^n),$$

which completes the proof. □

Corollary 3.18. *If X or Y is contractible, then $TC_n(f) = 1$ for a fibration $f = (f_1, \cdots, f_n) : X \to Y^n$.*

Proof. Let X be contractible. Then $cat(X^n) = 1$. From Theorem 3.17, we obtain $TC_n(X) = 1$. By Proposition 3.15, we conclude that $TC_n(f) = 1$. Let Y be contractible. Hence, Y^n is contractible. In a similar manner, we get $TC_n(Y^n) = 1$. By Proposition 2.4 (iv), we find

$$TC_n(f) = D(f \circ p_1, \cdots, f \circ p_n) \leq TC_n(Y^n) = 1.$$

As a result, $TC_n(f) = 1$ when Y is contractible. □

Example 3.19. Let $f_i : X \to \{y_0\}$ be a fibration for each $i \in \{1, \cdots, n\}$, where y_0 is any point of Y. Then, by Corollary 3.18, we get $TC_n(f) = 1$ for a fibration $f = (f_1, \cdots, f_n) : X \to \{y_0\}$.

Corollary 3.20. *Let $n > 2$. For a fibration $f : X \to Y$ with path-connected X and Y, we have*

$$TC(f) \leq cat(X^{n-1}) \leq TC_n(X) \leq cat(X^n).$$

Proof. By Proposition 3.16 and Theorem 3.16, it is enough to prove that $TC(f) \leq cat(X^{n-1})$ for $n > 2$. Proposition 2.4 (iii) gives us that

$$TC(f) = D(f^{'} \circ p_1, f^{'} \circ p_2) \leq cat(X^2) \leq cat(X^{n-1})$$

for a fibration $f^{'} = (f_1, f_2) : X \to Y^2$. □

Example 3.21. Let X be any space such that $f = (p_1, \cdots, p_n) : X^n \to X^n$ is a fibration whose components are projections $p_i : X^n \to X$ for $i \in \{1, \cdots, n\}$. Then, by using Proposition 3.15, and Corollary 3.14 of [2], we get

$$TC_n(f) \leq TC_n(X^n) \leq TC_n(X) + TC_n(X) + \cdots + TC_n(X) = n \cdot TC_n(X).$$

4. The higher topological complexity using Schwarz genus

Besides the higher homotopic distance, we also interpret $TC_n(f)$ by using the well-known concept Schwarz genus of a fibration.

Lemma 4.1. *For a surjective fibration $f = (f_1, \cdots, f_n) : X \to Y^n$, the induced map*

$$e_n^f : X^{J_n} \to Y^n$$

defined by $e_n^f(\alpha) = (f_1(\alpha_1(1)), \cdots, f_n(\alpha_n(1)))$ is a fibration.

Proof. Let $e_n^f = (f_1 \times f_2 \times \cdots \times f_n) \circ e_n$. Since f is a fibration, for each i, f_i is a fibration. Then $f_1 \times f_2 \times \cdots \times f_n$ is a fibration. The composition of two fibrations is again a fibration. This completes the proof. □

Definition 4.2. For a fibration $f = (f_1, \cdots, f_n) : X \to Y^n$, $TC_n(f)$ is defined as $secat(e_n^f : X^{J_n} \to Y^n)$.

Remark 4.3. The first observation is $\mathrm{TC}_1(f) = 1$. Also, we obtain the equality $\mathrm{TC}_n(f) = \mathrm{TC}_n(Y)$ when f is the identity $Y \to Y$. The next proof [10] confirms Proposition 3.9:

Proof of Proposition 3.9: Let $TC_{n+1}(f) = k$. Then $\{U_1, \cdots, U_k\}$ is an open cover of Y^{n+1} and $s_i : U_i \to X^{J_{n+1}}$ is a section of e_{n+1}^f for all $i = 1, \cdots, k$. Let

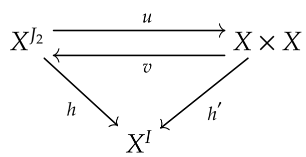
$$V_i = \{(y_1, \cdots, y_n) \ : \ (y_1, \cdots, y_n, a) \in Y^{n+1}\} \subset U_i$$

for $a \in Y$. Define two maps $h : T_{n+1}(Y) \to T_n(Y)$, $h(\alpha_1, \cdots, \alpha_{n+1}) = (\alpha_1, \cdots, \alpha_n)$ and $k : Y^n \to Y^{n+1}$, $k(y_1, \cdots, y_n) = (y_1, \cdots, y_n, a)$, where $T_n(Y)$ is a set consists of ordered set of n paths in Y. It follows that $\{V_1, \cdots, V_k\}$ is an open cover of Y^n and $t_i = h \circ s_i \circ k$ is a section of e_n^f on V_i for each i. Thus, $TC_n(f) \leq k$.

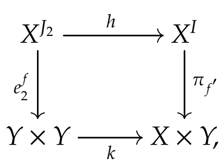
Lemma 4.4. *Given any fibrations $f' : X \to Y$ and $f = (f_1, f_2) : X \to Y \times Y$, we have*

$$secat(\pi_{f'} : X^I \to X \times Y) = secat(e_2^f : X^{J_2} \to Y \times Y).$$

Proof. Let α be a path in X^{J_2} such that $\alpha(0) = x_0$, $\alpha_1(1) = x_1$ and $\alpha_1(1) = x_2$. Define $h : X^{J_2} \to X^I$, $h(\alpha) = \beta$, where β is a path from x_1 to x_2. $h' : X \times X \to X^I$, $h(x, x')$ denotes a path, its starting point is x and its final point is x'. Therefore, h is a fiber homotopy equivalence. Indeed, consider two maps $u : X^{J_2} \to X \times X$, $u(\alpha) = (\alpha_1(1), \alpha_2(1))$ and $v : X \times X \to X^{J_2}$, $v(x, x') = \beta$, where β is a path from any point of X to two points $\beta_1(1) = x$, $\beta_2(1) = x'$. Then



commutes such that $u \circ v$ and $v \circ u$ are homotopic to respective identity maps $1_{X\times X}$ and $1_{X^{J_2}}$. Recall that f_1, f_2 and f' are surjective maps. Using this fact, we define a map $k : Y \times Y \to X \times Y$, $k(y_1, y_2) = (x_1, f'(x_2))$, where $y_1 = f_1(x_1)$ and $y_2 = f_2(x_2)$. We shall show that k is one of the equivalences of homotopic functions. Consider the map $k' : X \times Y \to Y \times Y$, $k'(x, y) = (f_1(x), f_2(x'))$, where $y = f'(x')$. Then $k \circ k'$ and $k' \circ k$ are respective identity maps $1_{X\times Y}$ and $1_{Y\times Y}$. Finally, by Theorem 6.4 of [11], the following diagram gives the desired result:



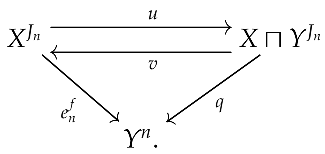
i.e., $secat(\pi_{f'} : X^I \to X \times Y) = secat(e_2^f : X^{J_2} \to Y \times Y)$. □

By Lemma 4.4, we have the quick result:

Corollary 4.5. *$TC_2(f)$ in Definition 4.2 coincides with $TC(f)$ in Definition 2.7.*

Theorem 4.6. *Given a fibration $f : X \to Y^n$, $e_n^f : X^{J_n} \to Y^n$ is a fiber homotopy equivalent to the function $q : X \sqcap Y^{J_n} \to Y^n$ defined by $q(x, \alpha) = (\alpha_1(1), \cdots, \alpha_n(1))$.*

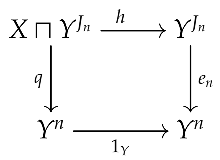
Proof. Let $\pi_2 : X \sqcap Y^{J_n} \to Y^{J_n}$. Since $q = e_n \circ \pi_2$, q is a fibration. Define a function $u : X^{J_n} \to X \sqcap Y^{J_n}$ by $u(\alpha) = (\alpha(0), (f_1 \times \cdots \times f_n) \circ \alpha)$, where α is a path with endpoints $x_1, \cdots, x_n$, i.e., $\alpha_1(1) = x_1, \cdots, \alpha_n(1) = x_n$, and $v : X \sqcap Y^{J_n} \to X^{J_n}$ by $v(x, \beta) = \alpha$, where β is a path from x to $(f_1(x_1), \cdots, f_n(x_n))$. Then $u \circ v$ and $v \circ u$ are homotopic to identity maps $1_{X \sqcap Y^{J_n}}$ and $1_{X^{J_n}}$, respectively. Moreover, the following diagram commutes:



This completes the proof. □

Corollary 4.7. *The function $q : X \sqcap Y^{J_n} \to Y^n$ is a pullback fibration of the map $e_n : X^{J_n} \to X^n$. Moreover, $TC_n(f) = secat(q)$.*

Proof. Let $h : X \sqcap Y^{J_n} \to Y^{J_n}$ be the projection. Then



commutes. Furthermore, by Theorem 4.6, we get

$$TC_n(f) = secat(e_n^f) = secat(q),$$

which is the desired result. □

5. Conclusion

The topological complexity is an essential homotopy invariant for the work of topological robotics. $\mathrm{TC}_n(X)$ and $\mathrm{TC}(f)$ improve this investigation, where f is a continuous and surjective map. The next step is to reveal the expression of $\mathrm{TC}_n(f)$. We give answer to this problem when f is a surjective fibration. If we consider the case that f does not have to be a fibration, in other saying, f is just a continuous and surjective map, then this is still an open problem. In this study, with the generalization of $\mathrm{TC}(f)$, we have one extra method to examine the problem of motion planning in topological robotics. We also contribute to the investigation of the relationship between $\mathrm{D}(f, g)$ and secat because another common point of these two notions is given by $\mathrm{TC}_n(f)$. One can have a certain fibration f to determine TC_n or cat of a space of a fibration by using the properties in Section 3 or Section 4. The reader is free to choose the way following on $\mathrm{D}(f_1, \cdots, f_n)$ for a multipath $f = (f_1, \cdots, f_n) : X \to Y^n$ or the way following on secat $e_n^f : X^{J_n} \to Y^n$ with $e_n^f = (f_1 \times f_2 \times \cdots \times f_n) \circ e_n$, where e_n is the path fibration in the definition of TC_n of a path-connected topological space X.

In digital images, there are many different computations on the notions TC, TC_n and cat rather than topological spaces. Analogously, the definition of $\mathrm{TC}_n(f)$ and the related results can be adapted to the digital images using the digital meaning of the (higher) homotopic distance. The task is to determine the similarities and differences on TC_n of a map between ordinary spaces and digital images.

One of the future researches on $\mathrm{TC}_n(f)$ is to state its symmetric version, in other stating, the higher symmetric topological complexity for a map (or a fibration). Each of directed and monodial versions of $\mathrm{TC}_n(f)$ is another topic for the reader.

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