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On Stević-Sharma Operator from $Q_K(p,q)$ Space to Zygmund-Type Space

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Abstract. The aim of this paper is to investigate the boundedness and compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces to Zygmund-type space and little Zygmund-type space. We also give the upper and lower estimations for the norm of $T_{\psi_1,\psi_2,\varphi}$.

1. Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, the weighted composition operator is defined by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

In particular, we can get the composition operator C_φ and multiplication operator M_ψ when $\psi \equiv 1$ and $\varphi(z) \equiv z$, respectively. For the theory of (weighted) composition operators on analytic function spaces, we refer to [2]. The differentiation operator D, which is defined by $(Df)(z) = f'(z)$, $f \in H(\mathbb{D})$, plays an important role in operator theory and dynamical system.

In [32, 33], Stević et al. introduced the following so-called Stević-Sharma operator:

$$(T_{\psi_1,\psi_2,\varphi}f)(z) = \psi_1(z)f(\varphi(z)) + \psi_2(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can easily get the general product-type operators:

$$\begin{aligned}
&M_\psi C_\varphi = T_{\psi,0,\varphi}, \quad C_\varphi M_\psi = T_{\psi\circ\varphi,0,\varphi}, \quad M_\psi D = T_{0,\psi,id}, \quad DM_\psi = T_{\psi',\psi,id}, \quad C_\varphi D = T_{0,1,\varphi},\\
&DC_\varphi = T_{0,\varphi',\varphi}, \quad M_\psi C_\varphi D = T_{0,\psi,\varphi}, \quad M_\psi DC_\varphi = T_{0,\psi\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},\\
&DM_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi}, \quad C_\varphi DM_\psi = T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi}, \quad DC_\varphi M_\psi = T_{\varphi'(\psi'\circ\varphi),\varphi'(\psi\circ\varphi),\varphi}.
\end{aligned}$$

Some of these operators had been investigated before introduction of Stević-Sharma operator for example in [6, 13, 14, 21, 27–29]. Recently, the research of $T_{\psi_1,\psi_2,\varphi}$ between analytic function spaces has aroused

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the interest of experts. Under some assumptions, Stević et al. [32, 33] characterized the boundedness, compactness and essential norm of $T_{\psi_1,\psi_2,\varphi}$ on the weighted Bergman space. Liu et al. [16–18, 38] studied the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from several specific analytic function spaces to the weighted-type space or Zygmund-type space. Wang et al. [34] considered the differences of two Stević-Sharma operators and investigated its boundedness, compactness and order boundedness between Banach spaces of analytic functions. Some more related results can be found (see, e.g.,[1, 3–5, 7] and the references therein).

A positive continuous function ϕ on [0,1) is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0, 1)$ such that (see[24])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta, 1), \ \lim_{r\to 1} \frac{\phi(r)}{(1-r)^s} = 0;$$
$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1), \ \lim_{r\to 1} \frac{\phi(r)}{(1-r)^t} = \infty.$$

Let $\mu : \mathbb{D} \to (0, +\infty)$ be a normal function satisfying $\mu(z) = \mu(|z|)$. The Bloch-type space, denoted by $\mathcal{B}^\mu$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

$\mathcal{B}^\mu$ is a Banach space under the above norm. Moreover, $\mathcal{B}^\mu$ induces the α-Bloch space $\mathcal{B}^\alpha$ when $\mu(z) = (1-|z|^2)^\alpha$, $\alpha > 0$. In particular, we get the classical Bloch space $\mathcal{B}$ if $\alpha = 1$.

An $f \in H(\mathbb{D})$ is said to belong to Zygmund-type space $\mathcal{Z}_\mu$ if

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Under the above norm, $\mathcal{Z}_\mu$ becomes a Banach space. The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ consists of those functions f in $\mathcal{Z}_\mu$ satisfying

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0,$$

and it can be shown that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$. Some results on Bloch-type space and Zygmund-type space and operators on them can be found, for instance, in [4, 7, 9–12, 14, 19, 22, 23, 26, 30, 31, 38–40].

Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function and $g(z, a)$ the Green function with logarithmic singularity at a, i.e., $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. For $p > 0$, $q > -2$, $Q_K(p, q)$ space consists of those $f \in H(\mathbb{D})$ such that (see [20, 35])

$$\|f\|^p_{Q_K(p,q)} = |f(0)| + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z, a))dA(z) < \infty,$$

where dA denotes the normalized Lebesgue area measure in $\mathbb{D}$. Under the norm $\|\cdot\|_{Q_K(p,q)}$, $Q_K(p, q)$ is a Banach space when $p \ge 1$. An $f \in H(\mathbb{D})$ is said to belong to $Q_{K,0}(p, q)$ space if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z, a))dA(z) = 0.$$

Throughout the paper we assume that (see [35])

$$\int_0^1 (1-r^2)^q K(-\log r)rdr < \infty,$$

since otherwise $Q_K(p, q)$ consists only of constant functions. Recently, many researchers have studied various concrete operators from or to $Q_K(p, q)$ space. For instance, Kotilainen [8] characterized the boundedness and compactness of composition operator between $\mathcal{B}^\alpha$ and $Q_K(p, q)$ spaces. The boundedness and compactness of an integral-type operator from $Q_K(p,q)$ space to Bloch-type space and Zygmund-type space were studied by Pan [22] and Ren [23], respectively. Some more related results can be found (see, e.g.,[9, 15, 36, 37] and the references therein).

Inspired by the above results, this paper is devoted to investigating the boundedness and compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces to Zygmund-type space and little Zygmund-type space.

Throughout the paper we use the letter C to denote a positive constant whose value may change at each occurrence. The notation abbreviation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y means that there is a positive constant C such that $X \leq CY$. Moreover, if both $X \lesssim Y$ and $Y \lesssim X$ hold, then one says that $X \approx Y$.

2. Auxiliary results

In this section, we state several auxiliary results which will be used in the proofs of the main results.

Lemma 2.1. [25] *Let $f \in \mathcal{B}^\alpha$, $0 < \alpha < \infty$. Then*

$$|f(z)| \lesssim \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1, \\ \|f\|_{\mathcal{B}} \ln \frac{e}{1-|z|^2}, & \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha}, & \alpha > 1. \end{cases}$$

The following lemma is well-known (see [40]).

Lemma 2.2. *Suppose $\alpha > 0$, $n \in \mathbb{N}$ and $f \in \mathcal{B}^\alpha$. Then*

$$\|f\|_{\mathcal{B}^\alpha} \approx |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} |f^{(n)}(z)|.$$

Lemma 2.3. [35] *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. For $f \in Q_K(p,q)$, we have $f \in \mathcal{B}^{\frac{q+2}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \leq \|f\|_{Q_K(p,q)}.$$

Lemma 2.4. [39] *Fix $0 < \alpha < 1$ and let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in B^α which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Then we have*

$$\lim_{k\to\infty} \sup_{z\in\mathbb{D}} |f_k(z)| = 0.$$

By a standard arguments in [2, Proposition 3.11], which is omitted here, we can get the following lemma.

Lemma 2.5. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Then the operator $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and for each sequence $\{f_k\}_{k\in\mathbb{N}}$ which is bounded in $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$.*

The lemma below can be obtained by the same method as [19, Lemma 1].

Lemma 2.6. *A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to1} \sup_{f\in K} \mu(z)|f''(z)| = 0.$$

3. Main results

In this section, our main results are stated and proved. For simplicity of notation, we set

$$\begin{aligned}
\widetilde{A_0}(z) &:= \mu(z)|\psi_1''(z)|,\\
\widetilde{A_1}(z) &:= \mu(z)|2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)|,\\
\widetilde{A_2}(z) &:= \mu(z)|\psi_1(z)\varphi'(z)^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)|,\\
\widetilde{A_3}(z) &:= \mu(z)|\psi_2(z)\varphi'(z)^2|,\\
E_0 &:= |\psi_1(0)| + |\psi_1'(0)|,\\
E_1 &:= |\psi_2(0)| + |\psi_2'(0)| + |\psi_1(0)\varphi'(0)|,\\
E_2 &:= |\psi_2(0)\varphi'(0)|,
\end{aligned}$$

and

$$\begin{aligned}
M_j &:= \sup_{z\in\mathbb{D}} \frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}},\\
N_j &:= \lim_{|\varphi(z)|\to 1} \frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}},\\
R_j &:= \lim_{|z|\to 1} \frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}},
\end{aligned}$$

where $j = 0, 1, 2, 3$.

Theorem 3.1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ such that $q+2 \geq p$ and K be a nonnegative nondecreasing function on $[0, \infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\Big(\log\frac{1}{1-r}\Big)^{\chi_{-1}(q)} r\,dr < \infty, \tag{1}$$

where $\chi_O(x)$ denotes the characteristic function of the set O. Then the following statements are true.

(i) *If $q + 2 > p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_\mu$ is bounded if and only if $M_0, M_1, M_2, M_3 < \infty$. Moreover, the following asymptotic relations hold:*

$$\begin{aligned}
M_0 + M_1 + M_2 + M_3 &\lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q)\ (or\ Q_{K,0}(p,q))\to\mathcal{Z}_\mu}\\
&\lesssim M_0 + M_1 + M_2 + M_3 + \sum_{j=0}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^{\frac{q+2}{p}-1+j}}.
\end{aligned} \tag{2}$$

(ii) *If $q + 2 = p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_\mu$ is bounded if and only if $M_1, M_2, M_3 < \infty$ and*

$$M_4 := \sup_{z\in\mathbb{D}} \widetilde{A_0}(z) \ln\frac{e}{1-|\varphi(z)|^2} < \infty.$$

Moreover, the following asymptotic relations hold:

$$\begin{aligned}
M_1 + M_2 + M_3 + M_4 &\lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q)\ (or\ Q_{K,0}(p,q))\to\mathcal{Z}_\mu}\\
&\lesssim M_1 + M_2 + M_3 + M_4 + E_0 \ln\frac{e}{1-|\varphi(0)|^2} + \sum_{j=1}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^j}.
\end{aligned} \tag{3}$$

(iii) *If* $q+2<p$, *then* $T_{\psi_1,\psi_2,\varphi}:Q_K(p,q)$ *(or* $Q_{K,0}(p,q)$*)* $\to\mathcal{Z}_\mu$ *is bounded if and only if* $\psi_1\in\mathcal{Z}_\mu$ *and* $M_1,M_2,M_3<\infty$.

Moreover, the following asymptotic relations hold:

$$L_0+M_1+M_2+M_3\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q)\ (or\ Q_{K,0}(p,q))\to\mathcal{Z}_\mu}$$
$$\lesssim L_0+M_1+M_2+M_3+E_0+\sum_{j=1}^{2}\frac{E_j}{(1-|\varphi(0)|^2)^j},\tag{4}$$

where $L_0:=\sup_{z\in\mathbb{D}}\widetilde{A_0}(z)$.

Proof. (i) Suppose that $q+2>p$ and $T_{\psi_1,\psi_2,\varphi}:Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to\mathcal{Z}_\mu$ is bounded. Note that if $T_{\psi_1,\psi_2,\varphi}:Q_K(p,q)\to\mathcal{Z}_\mu$ is bounded, then $T_{\psi_1,\psi_2,\varphi}:Q_{K,0}(p,q)\to\mathcal{Z}_\mu$ is bounded, and

$$\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}\le\|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q)\to\mathcal{Z}_\mu}.\tag{5}$$

Taking the function $f(z)=1\in Q_{K,0}(p,q)$, we get

$$L_0:=\sup_{z\in\mathbb{D}}\widetilde{A_0}(z)\le\|T_{\psi_1,\psi_2,\varphi}1\|_{\mathcal{Z}_\mu}\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}<\infty.\tag{6}$$

Likewise, using the function $f(z)=z\in Q_{K,0}(p,q)$ we obtain

$$\sup_{z\in\mathbb{D}}\mu(z)|\psi_1''(z)\varphi(z)+2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|\le\|T_{\psi_1,\psi_2,\varphi}z\|_{\mathcal{Z}_\mu}\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}<\infty,$$

which along with (6), the triangle inequality and the fact that $|\varphi(z)|<1$ implies that

$$L_1:=\sup_{z\in\mathbb{D}}\widetilde{A_1}(z)\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}<\infty.\tag{7}$$

Taking the functions $f(z)=\frac{z^2}{2}$ and $f(z)=\frac{z^3}{6}\in Q_{K,0}(p,q)$, in the same manner we have

$$L_2:=\sup_{z\in\mathbb{D}}\widetilde{A_2}(z)\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}<\infty,\tag{8}$$

and

$$L_3:=\sup_{z\in\mathbb{D}}\widetilde{A_3}(z)\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}<\infty.\tag{9}$$

For $w\in\mathbb{D}$, set

$$f_{0,w}(z)=-\frac{\frac{q+2}{p}+4}{\frac{q+2}{p}+1}\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}}+\frac{\frac{3q+6}{p}+8}{\frac{q+2}{p}+2}\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}}$$
$$-\frac{\frac{3q+6}{p}+8}{\frac{q+2}{p}+3}\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}}+\frac{(1-|\varphi(w)|^2)^4}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+3}}.$$

Using the condition (1), we have $f_{0,w}\in Q_{K,0}(p,q)$ (see [8]). By a direct calculation, we obtain

$$f_{0,w}'(\varphi(w))=f_{0,w}''(\varphi(w))=f_{0,w}'''(\varphi(w))=0,$$

and

$$f_{0,w}(\varphi(w))=-\frac{\frac{4q+8}{p}+10}{(\frac{q+2}{p}+1)(\frac{q+2}{p}+2)(\frac{q+2}{p}+3)}\frac{1}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}-1}},$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}$ implies that

$$\begin{aligned}\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}f_{0,w}\|_{\mathcal{Z}_\mu}\\ &\geq \mu(w)\Big|(T_{\psi_1,\psi_2,\varphi}f_{0,w})''(w)\Big|\\ &\gtrsim \frac{\widetilde{A_0}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}-1}}.\end{aligned} \tag{10}$$

Thus

$$M_0 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} < \infty. \tag{11}$$

For $w \in \mathbb{D}$, take the function

$$\begin{aligned}f_{1,w}(z) =& -\frac{\frac{q+2}{p}+4}{\frac{q+2}{p}+2}\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} + \frac{(\frac{q+2}{p}+4)(\frac{3q+6}{p}+7)}{(\frac{q+2}{p}+2)(\frac{q+2}{p}+3)}\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}}\\ &-\frac{\frac{3q+6}{p}+11}{\frac{q+2}{p}+3}\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}} + \frac{(1-|\varphi(w)|^2)^4}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+3}},\end{aligned}$$

then $f_{1,w} \in Q_{K,0}(p,q)$ by using the condition (1). Moreover, we have

$$f_{1,w}(\varphi(w)) = f''_{1,w}(\varphi(w)) = f'''_{1,w}(\varphi(w)) = 0,$$

and

$$f'_{1,w}(\varphi(w)) = -\frac{\frac{q+2}{p}+5}{\frac{q+2}{p}+3}\frac{\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}},$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}$ implies that

$$\begin{aligned}\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}f_{1,w}\|_{\mathcal{Z}_\mu}\\ &\geq \mu(w)\Big|(T_{\psi_1,\psi_2,\varphi}f_{1,w})''(w)\Big|\\ &\gtrsim \frac{\widetilde{A_1}(w)|\varphi(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}}.\end{aligned} \tag{12}$$

From (7) and (12), we have

$$\begin{aligned}&\sup_{w\in\mathbb{D}}\frac{\widetilde{A_1}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}}\\ \leq& \sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{\widetilde{A_1}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}} + \sup_{\frac{1}{2}<|\varphi(w)|<1}\frac{\widetilde{A_1}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}}\\ \leq&\Big(\frac{4}{3}\Big)^{\frac{q+2}{p}}\sup_{|\varphi(w)|\leq\frac{1}{2}}\widetilde{A_1}(w) + 2\sup_{\frac{1}{2}<|\varphi(w)|<1}\frac{\widetilde{A_1}(w)|\varphi(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}}}\\ \lesssim&\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}.\end{aligned}$$

It follows that

$$M_1 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} < \infty. \tag{13}$$

For $w \in \mathbb{D}$, consider the function

$$f_{2,w}(z) = -\frac{\frac{q+2}{p}+4}{\frac{q+2}{p}+3}\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} + \frac{\frac{3q+6}{p}+11}{\frac{q+2}{p}+3}\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} - \frac{\frac{3q+6}{p}+10}{\frac{q+2}{p}+3}\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}} + \frac{(1-|\varphi(w)|^2)^4}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+3}}.$$

Then $f_{2,w} \in Q_{K,0}(p,q)$ and

$$f_{2,w}(\varphi(w)) = f'_{2,w}(\varphi(w)) = f'''_{2,w}(\varphi(w)) = 0, \quad f''_{2,w}(\varphi(w)) = -\frac{2}{\frac{q+2}{p}+3}\frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}}.$$

Since $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathcal{Z}_\mu$ is bounded, we have

$$\begin{aligned}\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}f_{2,w}\|_{\mathcal{Z}_\mu}\\ &\geq \mu(w)\left|(T_{\psi_1,\psi_2,\varphi}f_{2,w})''(w)\right|\\ &\gtrsim \frac{\widetilde{A_2}(w)|\varphi(w)|^2}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}}.\end{aligned} \tag{14}$$

From (8) and (14) it follows that

$$\begin{aligned}&\sup_{w\in\mathbb{D}}\frac{\widetilde{A_2}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}}\\ \leq&\sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{\widetilde{A_2}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}} + \sup_{\frac{1}{2}<|\varphi(w)|<1}\frac{\widetilde{A_2}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}}\\ \leq&\Big(\frac{4}{3}\Big)^{\frac{q+2}{p}+1}\sup_{|\varphi(w)|\leq\frac{1}{2}}\widetilde{A_2}(w) + 4\sup_{\frac{1}{2}<|\varphi(w)|<1}\frac{\widetilde{A_2}(w)|\varphi(w)|^2}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+1}}\\ \lesssim&\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}.\end{aligned}$$

Consequently,

$$M_2 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} < \infty. \tag{15}$$

Set

$$f_{3,w}(z) = -\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} + 3\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} - 3\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}} + \frac{(1-|\varphi(w)|^2)^4}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+3}},$$

where $w \in \mathbb{D}$. Then $f_{3,w} \in Q_{K,0}(p,q)$ and

$$f_{3,w}(\varphi(w)) = f'_{3,w}(\varphi(w)) = f''_{3,w}(\varphi(w)) = 0, \quad f'''_{3,w}(\varphi(w)) = 6\frac{\overline{\varphi(w)}^3}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2}}.$$

Since $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathcal{Z}_\mu$ is bounded, we have

$$\begin{aligned}\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}f_{3,w}\|_{\mathcal{Z}_\mu}\\ &\geq \mu(w)\left|(T_{\psi_1,\psi_2,\varphi}f_{3,w})''(w)\right|\\ &\gtrsim \frac{\widetilde{A_3}(w)|\varphi(w)|^3}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2}}.\end{aligned} \tag{16}$$

From (9) and (16), we obtain

$$\begin{aligned}
&\sup_{w\in\mathbb{D}} \frac{\widetilde{A_3}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2}}\\
\le&\sup_{|\varphi(w)|\le\frac12} \frac{\widetilde{A_3}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2}} + \sup_{\frac12<|\varphi(w)|<1} \frac{\widetilde{A_3}(w)}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2}}\\
\le&\Big(\frac43\Big)^{\frac{q+2}{p}+2} \sup_{|\varphi(w)|\le\frac12} \widetilde{A_3}(w) + 8 \sup_{\frac12<|\varphi(w)|<1} \frac{\widetilde{A_3}(w)|\varphi(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+2n}}\\
\lesssim&\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}.
\end{aligned}$$

It follows that

$$M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} < \infty. \tag{17}$$

Combining (11), (13), (15) with (17) we see that

$$M_0 + M_1 + M_2 + M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}. \tag{18}$$

Conversely, assume that $M_0, M_1, M_2, M_3 < \infty$. By using Lemmas 2.1, 2.2 and 2.3, for each $f \in Q_K(p,q)$, we have

$$\begin{aligned}
&\mu(z)|(T^n_{\psi_1,\psi_2,\varphi}f)''(z)|\\
\le&\widetilde{A_0}(z)|f(\varphi(z))| + \widetilde{A_1}(z)|f'(\varphi(z))| + \widetilde{A_2}(z)|f''(\varphi(z))| + \widetilde{A_3}(z)|f'''(\varphi(z))|\\
\lesssim&\frac{\widetilde{A_0}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} + \frac{\widetilde{A_1}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\\
&+ \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} + \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+2}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\\
\le&(M_0 + M_1 + M_2 + M_3)\|f\|_{Q_K(p,q)}.
\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}
&|(T_{\psi_1,\psi_2,\varphi}f)(0)| + |(T_{\psi_1,\psi_2,\varphi}f)'(0)|\\
\le&E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))|\\
\lesssim&\sum_{j=0}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^{\frac{q+2}{p}-1+j}}\|f\|_{Q_K(p,q)}.
\end{aligned} \tag{20}$$

In view of (19) and (20), we conclude that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and

$$\|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q)\ (\text{or }Q_{K,0}(p,q))\to\mathcal{Z}_\mu} \lesssim M_0 + M_1 + M_2 + M_3 + \sum_{j=0}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^{\frac{q+2}{p}-1+j}}. \tag{21}$$

From (5), (18) and (21) we deduce that (2) holds.

(ii) Suppose that $q + 2 = p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded. From the proof of (i), we see that (13), (15) and (17) also hold in this case. That is, $M_1, M_2, M_3 < \infty$. Take the function

$$f_{4,w}(z) = \ln \frac{e}{1-\overline{\varphi(w)}z},$$

where $w \in \mathbb{D}$. Then $f_{4,w} \in Q_{K,0}(p,q)$ (see [8]) and it is easy to calculate that

$$f_{4,w}(\varphi(w)) = \ln \frac{e}{1-|\varphi(w)|^2}, \quad f'_{4,w}(\varphi(w)) = \frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2},$$
$$f''_{4,w}(\varphi(w)) = \frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^2}, \quad f'''_{4,w}(\varphi(w)) = \frac{2\overline{\varphi(w)}^3}{(1-|\varphi(w)|^2)^3},$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}$ and the triangle inequality implies that

$$\begin{aligned}\|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} \gtrsim & \|T_{\psi_1,\psi_2,\varphi} f_{4,w}\|_{\mathcal{Z}_\mu} \\ \geq & \mu(w)\left|(T_{\psi_1,\psi_2,\varphi} f_{4,w})''(w)\right| \\ \geq & \widetilde{A_0}(w) \ln \frac{e}{1-|\varphi(w)|^2} - \frac{\widetilde{A_1}(w)|\varphi(w)|}{1-|\varphi(w)|^2} \\ & - \frac{\widetilde{A_2}(w)|\varphi(w)|^2}{(1-|\varphi(w)|^2)^2} - \frac{2\widetilde{A_3}(w)|\varphi(w)|^3}{(1-|\varphi(w)|^2)^3}.\end{aligned}$$

From (13), (15), (17) and the fact that $|\varphi(w)| < 1$ it follows that

$$M_4 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu} < \infty.$$

Hence we have

$$M_1 + M_2 + M_3 + M_4 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}. \tag{22}$$

Conversely, assume that $M_1, M_2, M_3, M_4 < \infty$. For each $f \in Q_K(p,q)$, by using Lemmas 2.1, 2.2 and 2.3 we obtain

$$\begin{aligned}&\mu(z)|(T^n_{\psi_1,\psi_2,\varphi} f)''(z)| \\ \leq & \widetilde{A_0}(z)|f(\varphi(z))| + \widetilde{A_1}(z)|f'(\varphi(z))| + \widetilde{A_2}(z)|f''(\varphi(z))| + \widetilde{A_3}(z)|f'''(\varphi(z))| \\ \lesssim & \widetilde{A_0}(z)\|f\|_{\mathcal{B}} \ln \frac{e}{1-|\varphi(z)|^2} + \frac{\widetilde{A_1}(z)}{1-|\varphi(z)|^2}\|f\|_{\mathcal{B}} + \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)^2}\|f\|_{\mathcal{B}} + \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^3}\|f\|_{\mathcal{B}} \\ \leq & (M_1 + M_2 + M_3 + M_4)\|f\|_{Q_K(p,q)}.\end{aligned} \tag{23}$$

Furthermore,

$$\begin{aligned}&|(T_{\psi_1,\psi_2,\varphi} f)(0)| + |(T_{\psi_1,\psi_2,\varphi} f)'(0)| \\ \leq & E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| \\ \lesssim & \Big(E_0 \ln \frac{e}{1-|\varphi(0)|^2} + \sum_{j=1}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^j}\Big)\|f\|_{Q_K(p,q)}.\end{aligned} \tag{24}$$

From (23) and (24) we see that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and

$$\|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q) \text{ (or } Q_{K,0}(p,q))\to\mathcal{Z}_\mu} \lesssim M_1 + M_2 + M_3 + M_4 + E_0 \ln \frac{e}{1-|\varphi(0)|^2} + \sum_{j=1}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^j},$$

which along with (5) and (22) yields (3).

(iii) Suppose that $q + 2 < p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)(or\ Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded. From the proof of (i), we see that (6), (13), (15) and (17) also hold in this case. That is, $\psi_1 \in \mathcal{Z}_\mu$ and $M_1, M_2, M_3 < \infty$. We also have

$$L_0 + M_1 + M_2 + M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_\mu}. \tag{25}$$

On the contrary, assume that $\psi_1 \in \mathcal{Z}_\mu$ and $M_1, M_2, M_3 < \infty$. By using Lemmas 2.1, 2.2 and 2.3, for each $f \in Q_K(p,q)$, we have

$$\begin{aligned}
&\mu(z)|(T^n_{\psi_1,\psi_2,\varphi} f)''(z)| \\
\leq& \widetilde{A_0}(z)|f(\varphi(z))| + \widetilde{A_1}(z)|f'(\varphi(z))| + \widetilde{A_2}(z)|f''(\varphi(z))| + \widetilde{A_3}(z)|f'''(\varphi(z))| \\
\lesssim& \widetilde{A_0}(z)\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} + \frac{\widetilde{A_1}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} + \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} + \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+2}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \\
\leq& (L_0 + M_1 + M_2 + M_3)\|f\|_{Q_K(p,q)}.
\end{aligned} \tag{26}$$

Moreover,

$$\begin{aligned}
&|(T_{\psi_1,\psi_2,\varphi} f)(0)| + |(T_{\psi_1,\psi_2,\varphi} f)'(0)| \\
\leq& E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| \\
\lesssim& \Big(E_0 + \sum_{j=1}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^j}\Big)\|f\|_{Q_K(p,q)}.
\end{aligned} \tag{27}$$

From (26) and (27) we deduce that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded and

$$\|T_{\psi_1,\psi_2,\varphi}\|_{Q_K(p,q) \text{ (or } Q_{K,0}(p,q))\to\mathcal{Z}_\mu} \lesssim L_0 + M_1 + M_2 + M_3 + E_0 + \sum_{j=1}^{2} \frac{E_j}{(1-|\varphi(0)|^2)^j}. \tag{28}$$

Combining (5), (25) with (28) we can assert that (4) holds. □

Theorem 3.2. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ such that $q+2 \geq p$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (1) holds. Then the following statements are true.*

(i) If $q+2 > p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded and $N_0 = N_1 = N_2 = N_3 = 0$.

(ii) If $q+2 = p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded, $N_1 = N_2 = N_3 = 0$ and

$$N_4 := \lim_{|\varphi(z)|\to 1} \widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} = 0.$$

(iii) If $q+2 < p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded and $N_1 = N_2 = N_3 = 0$.

Proof. (i) Suppose that $q+2 > p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is compact. It is evident that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_\mu$ is bounded. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$f_{j,k}(z) = f_{j,z_k}(z), \quad j = 0,1,2,3,$$

where f_{j,z_k} is defined in the proof of Theorem 3.1. Moreover, we have $\{f_{j,k}\}_{k\in\mathbb{N},j=0,1,2,3}$ are norm bounded sequences in $Q_{K,0}(p,q)$, and it is easily seen that $f_{j,k}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By Lemma 2.5, we have

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_{j,k}\|_{\mathcal{Z}_\mu} = 0, \quad j = 0,1,2,3. \tag{29}$$

On the other hand, from (10), (12), (14) and (16) it follows that

$$\frac{\widetilde{A_j}(z_k)|\varphi(z_k)|^j}{(1-|\varphi(z_k)|^2)^{\frac{q+2}{p}-1+j}} \lesssim \|T_{\psi_1,\psi_2,\varphi} f_{j,k}\|_{\mathcal{Z}_\mu} \quad j = 0,1,2,3. \tag{30}$$

Letting $k \to \infty$ in (30) and employing (29), we can see that $N_0 = N_1 = N_2 = N_3 = 0$.

Conversely, assume that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and $N_0 = N_1 = N_2 = N_3 = 0$. Then for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\widetilde{A}_j(z_k)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} < \epsilon, \quad j = 0,1,2,3, \tag{31}$$

whenever $\delta < |\varphi(z)| < 1$. Moreover, by Theorem 3.1 we have L_0, L_1, L_2, L_3, which are defined in (6)–(9), are finite.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) such that $\sup_{k\in\mathbb{N}} \|f_k\|_{Q_K(p,q)} \lesssim 1$ and $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k \to \infty$. Applying (31), Lemmas 2.1, 2.2 and 2.3 we obtain

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \\
=&|(T_{\psi_1,\psi_2,\varphi} f_k)(0)| + |(T_{\psi_1,\psi_2,\varphi} f_k')(0)| + \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_k)''(z)| \\
\le& E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + \sum_{j=0}^{3}\Big(\sup_{|\varphi(z)|\le\delta} \widetilde{A}_j(z)|f_k^{(j)}(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \widetilde{A}_j(z)|f_k^{(j)}(\varphi(z))|\Big) \\
\lesssim& E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + \sum_{j=0}^{3} L_j \sup_{|\varphi(z)|\le\delta} |f_k^{(j)}(\varphi(z))| + \sum_{j=0}^{3} \sup_{\delta<|\varphi(z)|<1} \frac{\widetilde{A}_j(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} \\
\le& E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + \sum_{j=0}^{3} L_j \sup_{|w|\le\delta} |f_k^{(j)}(w)| + 4\epsilon.
\end{aligned} \tag{32}$$

Since $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k \to \infty$, we conclude that f_k', f_k'' and f_k''' also do by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w : |w| \le \delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k \to \infty$ in (32) yields

$$\limsup_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \le 4\epsilon.$$

From the arbitrariness of ϵ it follows that $\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} = 0$, and so by Lemma 2.5, $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact.

(ii) Suppose that $q + 2 = p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact, then it is bounded obviously. From the proof of (i), we can see that $N_1 = N_2 = N_3 = 0$. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ satisfying $|\varphi(z_k)| \to 1$ as $k \to \infty$, set

$$f_{4,k}(z) = \Big(\ln \frac{e}{1-\overline{\varphi(z_k)}z}\Big)^2 \Big(\ln \frac{e}{1-|\varphi(z_k)|^2}\Big)^{-1},$$

then $\{f_{4,k}\}_{k\in\mathbb{N}}$ is a bounded sequences in $Q_{K,0}(p,q)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By Lemma 2.5, we have

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_{4,k}\|_{\mathcal{Z}_\mu} = 0. \tag{33}$$

Furthermore,

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_{4,k}\|_{\mathcal{Z}_\mu} \\
\ge& \mu(z_k)\big|(T_{\psi_1,\psi_2,\varphi} f_{4,k})''(z_k)\big| \\
\ge& \widetilde{A_0}(z_k) \ln \frac{e}{1-|\varphi(z_k)|^2} - \frac{2\widetilde{A_1}(z_k)|\varphi(z_k)|}{1-|\varphi(z_k)|^2} \\
&- \Big(\frac{2}{\ln \frac{e}{1-|\varphi(z_k)|^2}} + 2\Big)\frac{\widetilde{A_2}(z_k)|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^2} - \Big(\frac{6}{\ln \frac{e}{1-|\varphi(z_k)|^2}} + 2\Big)\frac{\widetilde{A_3}(z_k)|\varphi(z_k)|^3}{(1-|\varphi(z_k)|^2)^3}.
\end{aligned} \tag{34}$$

Letting $k \to \infty$ in (34) and employing (33), the fact that $N_1 = N_2 = N_3 = 0$, we get $N_4 = 0$.

Conversely, assume that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and $N_1 = N_2 = N_3 = N_4 = 0$. Then for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} < \epsilon, \tag{35}$$

and

$$\frac{\widetilde{A}_j(z_k)}{(1-|\varphi(z)|^2)^j} < \epsilon, \quad j = 1,2,3, \tag{36}$$

whenever $\delta < |\varphi(z)| < 1$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) such that $\sup_{k\in\mathbb{N}} \|f_k\|_{Q_K(p,q)} \lesssim 1$ and $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k \to \infty$. Applying (35), (36), Lemmas 2.1, 2.2 and 2.3, we obtain

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \\
\lesssim &E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + \sum_{j=0}^{3} L_j \sup_{|\varphi(z)|\le\delta} |f_k^{(j)}(\varphi(z))| \\
&+ \sup_{\delta<|\varphi(z)|<1} \widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} + \sum_{j=1}^{3} \sup_{\delta<|\varphi(z)|<1} \frac{\widetilde{A}_j(z)}{(1-|\varphi(z)|^2)^j} \\
\le &E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + \sum_{j=0}^{3} L_j \sup_{|w|\le\delta} |f_k^{(j)}(w)| + 4\epsilon.
\end{aligned}$$

Analysis similar to (i) shows that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact.

(iii) Suppose that $q+2 < p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact, then it is bounded. Moreover, from (i) it follows that $N_1 = N_2 = N_3 = 0$.

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is bounded and $N_1 = N_2 = N_3 = 0$. Then for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\widetilde{A}_j(z_k)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} < \epsilon, \quad j = 1,2,3, \tag{37}$$

whenever $\delta < |\varphi(z)| < 1$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) such that $\sup_{k\in\mathbb{N}} \|f_k\|_{Q_K(p,q)} \lesssim 1$ and $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k \to \infty$. Applying (37), Lemmas 2.1, 2.2 and 2.3 we get

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathcal{Z}_\mu} \\
\lesssim &E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + L_0 \sup_{z\in\mathbb{D}} |f_k(\varphi(z))| \\
&+ \sum_{j=1}^{3} L_j \sup_{|\varphi(z)|\le\delta} |f_k^{(j)}(\varphi(z))| + \sum_{j=1}^{3} \sup_{\delta<|\varphi(z)|<1} \frac{\widetilde{A}_j(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} \\
\le &E_0|f(\varphi(0))| + E_1|f'(\varphi(0))| + E_2|f''(\varphi(0))| + L_0 \sup_{w\in\mathbb{D}} |f_k(w)| + \sum_{j=1}^{3} L_j \sup_{|w|\le\delta} |f_k^{(j)}(w)| + 3\epsilon.
\end{aligned} \tag{38}$$

Note that $0 < \frac{q+2}{p} < 1$, applying Lemma 2.4 yields

$$\lim_{k\to\infty} \sup_{w\in\mathbb{D}} |f_w(z)| = 0.$$

Letting $k \to \infty$ in (38) and by the same arguments as before, we can deduce that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_\mu$ is compact. □

When the target space is $\mathcal{Z}_{\mu,0}$, we have the following results.

Theorem 3.3. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}_0$, $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Then $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu,0}$ is bounded if and only if $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu}$ is bounded and*

$$\lim_{|z|\to 1} \widetilde{A_j}(z) = 0, \tag{39}$$

where $j = 0, 1, 2, 3$.

Proof. Assume that $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu,0}$ is bounded, then it is evident that $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu}$ is bounded and for every $f \in Q_{K,0}(p, q)$, we have $T_{\psi_1,\psi_2,\varphi} f \in \mathcal{Z}_{\mu,0}$. Taking $f(z) = 1 \in Q_{K,0}(p, q)$ yields

$$\lim_{|z|\to 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}1)'(z)| = \lim_{|z|\to 1} \widetilde{A_0}(z) = 0. \tag{40}$$

Instead of using the function $f(z) = z \in Q_{K,0}(p, q)$, we obtain

$$\lim_{|z|\to 1} \mu(z)|\psi_1''(z)\varphi(z) + 2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)| = 0,$$

which along with (40), the triangle inequality and the fact that $|\varphi(z)| < 1$, we deduce that (39) holds for $j = 1$. By using the functions $f(z) = \frac{z^2}{2}$ and $f(z) = \frac{z^3}{6} \in Q_{K,0}(p, q)$, in the same manner we can see that (39) holds for $j = 2, 3$.

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu}$ is bounded and (39) holds for $j = 0, 1, 2, 3$. Then for each polynomial $r(z)$, we have

$$\begin{aligned}\mu(z)|(T_{\psi_1,\psi_2,\varphi} r)''(z)| &\le \widetilde{A_0}(z)|r(\varphi(z))| + \widetilde{A_1}(z)|r'(\varphi(z))| + \widetilde{A_2}(z)|r''(\varphi(z))| + \widetilde{A_3}(z)|r'''(\varphi(z))| \\ &\lesssim \widetilde{A_0}(z) + \widetilde{A_1}(z) + \widetilde{A_2}(z) + \widetilde{A_3}(z).\end{aligned}$$

Letting $|z| \to 1$ in the above inequality and employing (39) gives

$$\lim_{|z|\to 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi} r)''(z)| = 0,$$

which says that $T_{\psi_1,\psi_2,\varphi} r \in \mathcal{Z}_{\mu,0}$. Since the set of all polynomials is dense in $Q_{K,0}(p, q)$ (see [8]), and hence for each $f \in Q_{K,0}(p, q)$, there is a sequence of polynomials $\{r_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|r_k - f\|_{Q_K(p,q)} = 0$, which along with the boundedness of $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu}$ implies that

$$\|T_{\psi_1,\psi_2,\varphi} r_k - T_{\psi_1,\psi_2,\varphi} f\|_{\mathcal{Z}_{\mu}} \le \|T_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathcal{Z}_{\mu}} \cdot \|r_k - f\|_{Q_K(p,q)} \to 0,$$

as $k \to \infty$. Since $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_{\mu}$, we have $T_{\psi_1,\psi_2,\varphi} f \in \mathcal{Z}_{\mu,0}$, and consequently $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p, q) \to \mathcal{Z}_{\mu,0}$ is bounded. □

Theorem 3.4. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ such that $q+2 \ge p$ and K be a nonnegative nondecreasing function on $[0, \infty)$ such that (1) holds. Then the following statements are true.*

(i) *If $q + 2 > p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_{\mu,0}$ is compact if and only if $R_0 = R_1 = R_2 = R_3 = 0$.*

(ii) *If $q + 2 = p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_{\mu,0}$ is compact if and only if $R_1 = R_2 = R_3 = 0$ and*

$$R_4 := \lim_{|z|\to 1} \widetilde{A_0}(z) \ln \frac{e}{1 - |\varphi(z)|^2} = 0.$$

(iii) *If $q + 2 < p$, then $T_{\psi_1,\psi_2,\varphi} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_{\mu,0}$ is compact if and only if $\psi_1 \in \mathcal{Z}_{\mu,0}$ and $R_1 = R_2 = R_3 = 0$.*

Proof. (i) Suppose that $q+2 > p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_{\mu,0}$ is compact. Then the compactness of $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_{\mu}$ easily follows. By using Theorem 3.2, for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that (31) holds whenever $\delta < |\varphi(z)| < 1$. Note that if $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$ is compact, then $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is compact. Moreover, the compactness of $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ implies that $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded. Then (39) follows from Theorem 3.3, and for any $\epsilon > 0$, there exists $\eta \in (0,1)$ such that

$$\widetilde{A_j}(z) \le \epsilon(1-\delta^2)^{\frac{q+2}{p}-1+j}, \quad j = 0,1,2,3, \tag{41}$$

whenever $\eta < |z| < 1$. From (31), when $\eta < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have

$$\frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} < \epsilon, \quad j = 0,1,2,3. \tag{42}$$

On the other hand, when $\eta < |z| < 1$ and $|\varphi(z)| \le \delta$, using (41) yields

$$\frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} \le \frac{\widetilde{A_j}(z)}{(1-\delta^2)^{\frac{q+2}{p}-1+j}} < \epsilon, \quad j = 0,1,2,3. \tag{43}$$

From (42) and (43) we deduce that $R_0 = R_1 = R_2 = R_3 = 0$.

Conversely, assume that $R_0 = R_1 = R_2 = R_3 = 0$. Let $f \in Q_K(p,q)$ (or $Q_{K,0}(p,q)$), analysis similar to (19) in the proof of Theorem 3.1 shows that

$$\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| \lesssim \sum_{j=0}^{3} \frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1+j}} \|f\|_{Q_K(p,q)}.$$

Taking the supremum in the above inequality over all $f \in Q_K(p,q)$ (or $Q_{K,0}(p,q)$) such that $\|f\|_{Q_K(p,q)} \le 1$ and letting $|z| \to 1$, we have

$$\lim_{|z|\to 1} \sup_{\|f\|_{Q_K(p,q)} \le 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| = 0.$$

Therefore, the operator $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_{\mu,0}$ is compact by Lemma 2.6.

(ii) Suppose that $q+2 = p$ and $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathcal{Z}_{\mu,0}$ is compact. By (i), we can see that $R_1 = R_2 = R_3 = 0$. From Theorem 3.2, for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that (36) holds whenever $\delta < |\varphi(z)| < 1$. Moreover, (39) follows from Theorem 3.3, and for any $\epsilon > 0$, there exists $\eta \in (0,1)$ such that

$$\widetilde{A_0}(z) \le \frac{\epsilon}{\ln \frac{e}{1-\delta^2}}, \tag{44}$$

whenever $\eta < |z| < 1$. From (36), when $\eta < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have

$$\widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} < \epsilon. \tag{45}$$

On the other hand, when $\eta < |z| < 1$ and $|\varphi(z)| \le \delta$, using (44) yields

$$\widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} \le \widetilde{A_0}(z) \ln \frac{e}{1-\delta^2} < \epsilon. \tag{46}$$

From (45) and (46) we conclude that $R_4 = 0$.

Conversely, assume that $R_1 = R_2 = R_3 = R_4 = 0$. Let $f \in Q_K(p,q)$ (or $Q_{K,0}(p,q)$), then we have

$$\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| \lesssim \left(\widetilde{A_0}(z) \ln \frac{e}{1-|\varphi(z)|^2} + \sum_{j=1}^{3} \frac{\widetilde{A_j}(z)}{(1-|\varphi(z)|^2)^j}\right) \|f\|_{Q_K(p,q)}.$$

Analysis similar to (i) shows that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_{\mu,0}$ is compact.

(iii) Suppose that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_{\mu,0}$ is compact. From (i) we see that $R_1 = R_2 = R_3 = 0$. Taking $f(z) = 1 \in Q_{K,0}(p,q)$, we have $T_{\psi_1,\psi_2,\varphi}1 = \psi_1 \in \mathcal{Z}_{\mu,0}$.

Conversely, assume that $R_1 = R_2 = R_3 = 0$ and $\psi_1 \in \mathcal{Z}_{\mu,0}$, i.e., $\lim_{|z|\to 1} \widetilde{A_0}(z) = 0$. Let $f \in Q_K(p,q)$ (or $Q_{K,0}(p,q)$), then we have

$$\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| \lesssim \Big(\widetilde{A_0}(z) + \sum_{j=1}^{3} \frac{\widetilde{A}_j(z)}{(1-|\varphi(z)|^2)^j}\Big)\|f\|_{Q_K(p,q)}.$$

Similar to (i) we deduce that $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q)$ (or $Q_{K,0}(p,q)) \to \mathcal{Z}_{\mu,0}$ is compact. □

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