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Balanced-Euler Approximation Schemes for Stiff Systems of Stochastic Differential Equations

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Abstract. This paper aims to design new families of balanced-Euler approximation schemes for the solutions of stiff stochastic differential systems. To prove the mean-square convergence, we use some fundamental inequalities such as the global Lipschitz condition and linear growth bound. The mean-square stability properties of our new schemes are analyzed. Also, numerical examples illustrate the accuracy and efficiency of the proposed schemes.

1. Introduction

Consider the non-autonomous system of Itô stochastic differential equation (SDE)

$$\mathrm{d}Z_\tau = a(\tau, Z_\tau)\mathrm{d}\tau + \sum_{i=1}^{s} b_i(\tau, Z_\tau)\mathrm{d}\omega_\tau^i, \quad Z(\tau_0) = Z_0, \quad \tau \in [\tau_0, \mathcal{T}], \tag{1}$$

where $\{\omega_\tau^i : i = 1, \ldots, s\}$ are independent one-dimensional standard Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_\tau\}_{\tau\in[\tau_0,\mathcal{T}]})$ filtered probability space fulfilling the usual conditions. In SDE (1), the deterministic term $a : [\tau_0, \mathcal{T}] \times \mathbb{R}^d \to \mathbb{R}^d$ is called the drift coefficient, and the stochastic terms $b_i : [\tau_0, \mathcal{T}] \times \mathbb{R}^d \to \mathbb{R}^{d\times s}$, $i = 1, \ldots, s$ are called the diffusion coefficients.

In this paper, notation $\mathcal{Y}_r$ used to indicate the value of the approximation of the analytic solution Z_{τ_r}. Also, we define a mesh with uniform step Δ on the interval $[\tau_0, \mathcal{T}]$ and $\Delta = \frac{\mathcal{T}-\tau_0}{M}$, $\tau_r = \tau_0 + r\Delta$, $r = 0, 1, \ldots, M$, $M = 1, 2, \ldots$.

Definition 1.1. *[12, 17] A discrete-time approximation $\mathcal{Y}_r$ is said to be mean-square convergent with order $\varkappa > 0$ to the solution Z_τ of SDE* (1) *at time τ_r if there exist constants $K > 0$ and $\varepsilon_0 > 0$, such that*

$$\sqrt{\mathbb{E}\left[\left|Z_{\tau_r} - \mathcal{Y}_r\right|^2\right]} \leq K\Delta^{\varkappa},$$

for each $\Delta \in (0, \varepsilon_0)$. Especially, the constant K is independent of Δ.

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SDEs and RDEs(random differential equations) have successfully been used to model physical phenomena with uncertainty [6–10, 23, 37]. Because SDE (1) generally does not have an explicit solution, many effective numerical schemes provide for exploring the properties of this equation [3, 4, 11, 15, 17, 20, 27, 31, 34, 36, 39, 42], during the past few decades. The (semi-)implicit methods, are on the important family of these methods which face an increase in the computational cost when the SDE (1) is stiff [26, 28–30, 38]. Therefore, explicit schemes with extended stability regions could greatly help to the solution of stiff SDEs. See, for instance, [1, 18, 19, 24, 25, 40].

As we know, the stability theory of numerical solutions is one of the essential criteria for providing efficient and effective numerical methods in solving SDEs. The stability properties of numerical schemes for scalar SDEs, first, are investigated by Saito and Mitsui [32]. Many interesting articles have studied the stability behavior of numerical methods. See, for example, [2, 13, 14, 33, 35, 41].

The paper is organized as follows. In Section 2, we propose the new balanced-Euler schemes for solving SDE (1). The strong convergence order of the methods are obtained, in Section 3. In Section 4, we study the mean-square stability of the methods for scalar SDEs. Numerical problems are carried out in Section 5 and, we will conclude our work finally in Section 6.

2. New balanced schemes formulation

For solving SDE (1), Milstein et al. [22] consider a linear-implicit balanced method which defines an approximation process $\mathcal{Y}_r \approx Z_{\tau_r}$ with $\mathcal{Y}_0 = Z_0$ and

$$\mathcal{Y}_{r+1} = \mathcal{Y}_r + \Delta a(\tau_r, \mathcal{Y}_r) + \sum_{i=1}^{s} b_i(\tau_r, \mathcal{Y}_r)\Delta\omega_r^i + C_r(\mathcal{Y}_r - \mathcal{Y}_{r+1}), \tag{2}$$

with $d \times d$ matrix-valued functions

$$C_r(\tau_r, \mathcal{Y}_r) = c_0(\tau_r, \mathcal{Y}_r)\Delta + \sum_{i=1}^{s} c_i(\tau_r, \mathcal{Y}_r)|\Delta\omega_r^i|, \tag{3}$$

where $\Delta\omega_r^i = \omega_{r+1}^i - \omega_r^i$ and in general form $c_i(\tau_r, \mathcal{Y}_r)$, $i = 0, 1, 2, \ldots, s$ are $d \times d$ matrix-valued functions which called control functions.

Assumption 2.1. *([12]) For any real numbers $\varrho_0 \in [0, \overline{\varrho}]$, $\varrho_1, \varrho_2, \ldots, \varrho_s \geq 0$ and $\overline{\varrho} \geq \Delta$ for all step-sizes Δ considered and $(\tau, z) \in [0, \infty] \times \mathbb{R}^d$, the function*

$$\mathcal{M}(\tau, z) = \mathcal{I} + \varrho_0 c_0(\tau, z) + \sum_{i=1}^{s} \varrho_i c_i(\tau, z),$$

has an inverse and satisfies the inequality

$$\left|\mathcal{M}^{-1}(\tau, z)\right| \leq \widetilde{\mathcal{M}} < \infty,$$

where $\mathcal{I}$ is the unit $d \times d$-matrix, $\widetilde{\mathcal{M}}$ is a positive constant.

We constructed a new balanced Euler (BE-I) method with taking following control function

$$c_0^I(\tau_r, \mathcal{Y}_r) = -\alpha \sinh\left(\Delta J_a(\tau_r, \mathcal{Y}_r)\right) + \sum_{i=1}^{s} \beta_i \sin\left(\Delta J_{b_i}^2(\tau_r, \mathcal{Y}_r)\right), \quad \alpha, \beta_i \in [0, 1] \tag{4}$$

and BE-II method with control function

$$c_0^{II}(\tau_r, \mathcal{Y}_r) = \Delta \left| \theta J_a(\tau_r, \mathcal{Y}_r) - \sum_{i=1}^{s} \sigma_i J_{b_i}^2(\tau_r, \mathcal{Y}_r) \right|, \quad \theta, \sigma_i \in [0, 1]. \tag{5}$$

Here J_a and J_{b_i} are the Jacobian matrices of a and b_i, respectively.

3. Mean-square convergence analysis

In this section, we investigate the mean-square (MS) convergence of the BE-I (2) and (4), BE-II (2) and (5) schemes using the following lemma.

Lemma 3.1. *([23]) Assume for a one-step discrete-time approximation $\mathcal{Y}$ that the local mean error and MS error for all $\mathcal{R} = 1, 2, \dots,$ and $r = 0, 1, \dots, \mathcal{R} - 1$ satisfy the estimates*

$$\left|\mathbb{E}\left[(\mathcal{Y}_{r+1} - Z_{\tau_{r+1}})|\mathcal{F}_{\tau_r}\right]\right| \le \mathcal{K}(1 + |\mathcal{Y}_r|^2)^{1/2}\Delta^{\varkappa_1}, \tag{6}$$

and

$$\left|\mathbb{E}\left[\left|\mathcal{Y}_{r+1} - Z_{\tau_{r+1}}\right|^2 |\mathcal{F}_{\tau_r}\right]\right|^{1/2} \le \mathcal{K}(1 + |\mathcal{Y}_r|^2)^{1/2}\Delta^{\varkappa_2}, \tag{7}$$

when $\varkappa_2 \ge \frac{1}{2}$ and $\varkappa_1 \ge \varkappa_2 + \frac{1}{2}$. Then

$$\left|\mathbb{E}\left[\left|\mathcal{Y}_{n} - Z_{\tau_{n}}\right|^2 |\mathcal{F}_{r_0}\right]\right|^{1/2} \le \mathcal{K}(1 + |\mathcal{Y}_0|^2)^{1/2}\Delta^{\varkappa_2 - 1/2},$$

holds for each $n = 0, 1, \dots, \mathcal{R}$. Here $\mathcal{K}$ is independent of Δ but dependent on the length of the time interval $\mathcal{T} - \tau_0$.

Theorem 3.2. *Let $0 < \theta, \sigma_i, \alpha, \beta_i \le 1$, $i = 1, 2, \dots, s$, than under the linear growth bound,*

$$|a(h, f)|^2 \vee \sum_{i=1}^{s} \left|b_i(h, f)\right|^2 \le \widehat{K}\left(1 + |f|^2\right), \quad f \in \mathbb{R}^d. \tag{8}$$

The numerical solution produced by the methods (2)-(4) *and* (2) *and* (5) *converges to the exact solution of SDE* (1) *in the MS sense with strong order of convergence $\frac{1}{2}$.*

Proof. First, we will prove that the inequality (7) holds for our methods with $\varkappa_1 = 2$. For this purpose, we were applying Euler approximation step

$$\mathcal{Y}^{\text{EM}}_{r+1} = \mathcal{Y}^{\text{EM}}_{r} + \Delta a(\tau_r, \mathcal{Y}^{\text{EM}}_{r}) + \sum_{i=1}^{s} b_i(\tau_r, \mathcal{Y}^{\text{EM}}_{r})\Delta\omega^i_r. \tag{9}$$

The local mean and MS errors of method (9) are [27–30]

$$\left|\mathbb{E}\left[\left(\mathcal{Y}^{\text{EM}}_{r+1} - Z_{\tau_{r+1}}\right)\middle|\mathcal{F}_r\right]\right| = O(\Delta^2), \tag{10a}$$

$$\left|\mathbb{E}\left[\left|\mathcal{Y}^{\text{EM}}_{r+1} - Z_{\tau_{r+1}}\right|^2\middle|\mathcal{F}_r\right]\right|^{\frac{1}{2}} = O\left(\Delta\right), \tag{10b}$$

respectively. Then from (10a), we have

$$\begin{aligned}
\delta_1 =& \left|\mathbb{E}\left[(\mathcal{Y}_{r+1} - Z_{\tau_{r+1}})\middle|\mathcal{F}_r\right]\right| \\
\le& \left|\mathbb{E}\left[\left(\mathcal{Y}^{\text{EM}}_{r+1} - Z_{\tau_{r+1}}\right)\middle|\mathcal{F}_r\right]\right| + \left|\mathbb{E}\left[\left(\mathcal{Y}_{r+1} - \mathcal{Y}^{\text{EM}}_{r+1}\right)\middle|\mathcal{F}_r\right]\right| \\
\le& O(\Delta^2) + \delta_2.
\end{aligned} \tag{11}$$

Using the linear growth bounds (8) and $\mathbb{E}[\Delta\omega^i_r] = 0$, $i = 1, \dots, s$, one can derive that

$$\begin{aligned}
\delta_2 =& \left|\mathbb{E}\left[(\mathcal{Y}_{r+1} - \mathcal{Y}^{\text{EM}}_{r+1})\middle|\mathcal{F}_r\right]\right| \\
=& \left|\mathbb{E}\left[C^{I,II}_r\left(\mathcal{I} + C^{I,II}_r\right)^{-1}\left(\Delta a(\tau_r, \mathcal{Y}_r) + \sum_{i=1}^{s} b_i(\tau_r, \mathcal{Y}_r)\Delta\omega^i_r\right)\right]\right| \\
\le& \Delta\sqrt{\widehat{K}}\left|\mathbb{E}\left[C^{I,II}_r\left(\mathcal{I} + C^{I,II}_r\right)^{-1}\right]\right|\left(1 + |\mathcal{Y}_r|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By (3)-(5), Assumption 2.1 and the symmetry property of $\Delta\omega_r^i$, $i = 1, \ldots, s$ in the above relation, we have

$$\begin{aligned}
\delta_2 \leq &\Delta\widetilde{\mathcal{M}}\sqrt{\widehat{K}}\left|\mathbb{E}\left[C_r^{I,II}\right]\right|\left(1+|\mathcal{Y}_r|^2\right)^{\frac{1}{2}} \\
\leq &\Delta\widetilde{\mathcal{M}}\sqrt{\widehat{K}}\left|\mathbb{E}\left[c_0^{I,II}(\tau_r,\mathcal{Y}_r)\Delta+\sum_{i=1}^{s}c_i(\tau_r,\mathcal{Y}_r)|\Delta\omega_r^i|\right]\right|\left(1+|\mathcal{Y}_r|^2\right)^{\frac{1}{2}} \\
\leq &\Delta^{3/2}\left(\sqrt{\frac{2}{\pi}}+\Delta^{1/2}\right)\mathcal{L}\widetilde{\mathcal{M}}\sqrt{\widehat{K}}\left(1+|\mathcal{Y}_r|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{12}$$

where we have used the $\mathbb{E}[|\Delta\omega_r^i|] = \sqrt{\frac{2\Delta}{\pi}}$, $i = 1, \ldots, s$ and

$$\left|c_0^{I}(\tau_r,\mathcal{Y}_r)\right| \vee \left|c_0^{II}(\tau_r,\mathcal{Y}_r)\right| \vee \left|\sum_{i=1}^{s}c_i(\tau_r,\mathcal{Y}_r)\right| \leq \mathcal{L}. \tag{13}$$

Hence, from inequalities (11) and (12), we obtain $\varkappa_1 = 3/2$.

We prove that the inequality (7) with $\varkappa_2 = 1$ holds for the BE-I (2)-(4) and BE-II (2) and (5) methods. For this aim, using the Euler approximation step (9) and inequality $(p+q)^2 \leq 2(p^2+q^2)$, we divide (7) into two parts as follows

$$\begin{aligned}
\delta_3 = &\left|\mathbb{E}\left[\left|\mathcal{Y}_{r+1}-Z_{\tau_{r+1}}\right|^2\Big|\mathcal{F}_r\right]\right| \\
\leq &2\left|\mathbb{E}\left[\left|\mathcal{Y}_{r+1}^{\mathrm{EM}}-Z_{\tau_{r+1}}\right|^2\Big|\mathcal{F}_r\right]\right|+2\left|\mathbb{E}\left[\left|\mathcal{Y}_{r+1}-\mathcal{Y}_{r+1}^{\mathrm{EM}}\right|^2\Big|\mathcal{F}_r\right]\right| \\
\leq &\mathcal{O}(\Delta^2)+2\delta_4,
\end{aligned} \tag{14}$$

Using (3), (4), (5), Assumption 2.1, linear growth bounds (8) and (13), $\mathbb{E}[\Delta\omega_r^i] = 0$, $\mathbb{E}[(\Delta\omega_r^i)^2] = \Delta$, $\mathbb{E}[(\Delta\omega_r^i)^4] = 3\Delta^2$, $i = 1, \ldots, s$ and

$$(v_1+v_2+\ldots+v_u)^2 \leq u(v_1^2+v_2^2+\ldots+v_u^2),$$

we can write

$$\begin{aligned}
\delta_4 = &\mathbb{E}\left[\left|\mathcal{Y}_{r+1}-\mathcal{Y}_{r+1}^{\mathrm{EM}}\right|^2\Big|\mathcal{F}_r\right] \\
= &\mathbb{E}\left[\left|C_r^{I,II}\left(\mathcal{I}+C_r^{I,II}\right)^{-1}\left(\Delta a(\tau_r,\mathcal{Y}_r)+\sum_{i=1}^{s}b_i(\tau_r,\mathcal{Y}_r)\Delta\omega_r^i\right)\right|^2\right] \\
\leq &(1+s)\widetilde{\mathcal{M}}^2\mathbb{E}\left[\left|C_r^{I,II}\right|^2\left(\Delta^2|a(\tau_r,\mathcal{Y}_r)|^2+\sum_{i=1}^{s}|b_i(\tau_r,\mathcal{Y}_r)|^2(\Delta\omega_r^i)^2\right)\right] \\
\leq &(1+s)^2\widetilde{\mathcal{M}}^2\mathbb{E}\left[\left(\Delta^2|c_0^{I,II}(\tau_r,\mathcal{Y}_r)|^2+\sum_{i=1}^{s}|c_i(\tau_r,\mathcal{Y}_r)|^2(\Delta\omega_r^i)^2\right)\left(\Delta^2|a(\tau_r,\mathcal{Y}_r)|^2+\sum_{i=1}^{s}|b_i(\tau_r,\mathcal{Y}_r)|^2(\Delta\omega_r^i)^2\right)\right] \\
\leq &\Delta^2\,(1+\Delta)^2\,(1+s)^2(\mathcal{L}\widetilde{\mathcal{M}})^2\widehat{K}\left(1+|\mathcal{Y}_r|^2\right).
\end{aligned} \tag{15}$$

Thus the inequality (7) with $\varkappa_2 = 1$ holds for our methods.

So, in Theorem 3.2 we can choose $\varkappa_2 = 1$ and $\varkappa_1 = 2$ to establish convergence rate 1/2 of the methods BE-I (2) with (4) and BE-II (2) with (5). □

4. Mean-square stability properties

For MS stability analysis of BE-I and BE-II methods, we take a one-dimensional linear Itô test SDE with multi-dimensional noise terms

$$Z_\tau = \lambda Z_\tau \mathrm{d}\tau + \sum_{i=1}^{s} \mu_i Z_\tau \mathrm{d}\omega_\tau^i, \tag{16}$$

where the parameters $\lambda, \mu_i \in \mathbb{R}$. The zero solution of (16) is said to be MS stable if $\lim_{\tau\to\infty} \mathbb{E}|Z_\tau|^2 = 0$. It is well known [5, 21] that the zero solution of (16) is MS stable if and only if

$$\sum_{i=1}^{s} \mu_i^2 + 2\lambda < 0.$$

Saito and Mitsui [32] introduce the following definition of MS stability for a numerical scheme.

Definition 4.1. *The numerical method is said to be MS stable if*

$$\Xi\left(\lambda, \{\mu_i\}_{i=1}^s, \Delta\right) = \mathbb{E}\left[\Theta^2(\lambda, \{\mu_i\}_{i=1}^s, \Delta, \{\xi_r^i\}_{i=1}^s)\right] < 1, \quad \xi_r^i \sim \mathcal{N}(0,1), \quad i = 1, \dots, s,$$

where $\Xi(\lambda, \{\mu_i\}_{i=1}^s, \Delta)$ is called MS stability function, and the set $\mathfrak{S}_{MS} = \{(\lambda, \{\mu_i\}_{i=1}^s) \in \mathbb{R}^2 : \Xi(\lambda, \{\mu_i\}_{i=1}^s, \Delta) < 1\}$ is called the MS stability domain of the numerical method.

Applying our methods (4) and (5) to the linear test (16), we obtained

$$\mathcal{Y}_{r+1} = \Theta\left(\lambda, \{\mu_i\}_{i=1}^s, \Delta, \{\xi_r^i\}_{i=1}^s\right) \mathcal{Y}_r,$$

where

$$\Theta\left(\lambda, \{\mu_i\}_{i=1}^s, \Delta, \{\xi_r^i\}_{i=1}^s\right) = \begin{cases} 1 + \dfrac{\lambda\Delta + \sqrt{\Delta}\sum_{i=1}^{s}\mu_i\xi_r^i}{1-\alpha\sinh(\Delta\lambda) + \sum_{i=1}^{s}\beta_i \sin\left(\Delta\mu_i^2\right) + \sqrt{\Delta}\sum_{i=1}^{s} c_i\Delta\left|\xi_r^i\right|}, & \alpha, \beta_i \in [0,1], \\[2ex] 1 + \dfrac{\lambda\Delta + \sqrt{\Delta}\sum_{i=1}^{s}\mu_i\xi_r^i}{1+\Delta\left|\theta\lambda - \sum_{i=1}^{s}\sigma_i\mu_i^2\right| + \sqrt{\Delta}\sum_{i=1}^{s} c_i\Delta\left|\xi_r^i\right|}, & \theta, \sigma_i \in [0,1]. \end{cases}$$

Now, using Definition 4.1 and $c_i \equiv 0, i = 1, 2, \dots, s$ yields

$$\Xi\left(\lambda, \{\mu_i\}_{i=1}^s, \Delta\right) = \mathbb{E}\left[\left|\Theta\left(\lambda, \{\mu_i\}_{i=1}^s, \Delta, \{\xi_r^i\}_{i=1}^s\right)\right|^2\right]$$

$$= \begin{cases} \left(1 + \dfrac{\lambda\Delta}{1-\alpha\sinh(\Delta\lambda) + \sum_{i=1}^{s}\beta_i\sin\left(\Delta\mu_i^2\right)}\right)^2 + \dfrac{\Delta\sum_{i=1}^{s}\mu_i^2}{\left(1-\alpha\sinh(\Delta\lambda) + \sum_{i=1}^{s}\beta_i\sin\left(\Delta\mu_i^2\right)\right)^2}, & \alpha, \beta_i \in [0,1], \\[2ex] \left(1 + \dfrac{\lambda\Delta}{1+\Delta\left|\theta\lambda - \sum_{i=1}^{s}\sigma_i\mu_i^2\right|}\right)^2 + \dfrac{\Delta\sum_{i=1}^{s}\mu_i^2}{\left(1+\Delta\left|\theta\lambda - \sum_{i=1}^{s}\sigma_i\mu_i^2\right|\right)^2}, & \theta, \sigma_i \in [0,1], \end{cases}$$

where used $\mathbb{E}[\xi_r^i] = 0$ and $\mathbb{E}[(\xi_r^i)^2] = 1$.

Figures 1 and 3 display the MS stability regions of the BE-I method showing better stability properties of the test equation (16) when $\alpha, \beta_i \to 1$. Also, Figures 2 and 4 show a comparison between the MS stability regions of the BE-II and linear SDE (16). From these figures, it is evident that the MS stability regions achieved by the BE-II method covered MS stability region test equation (16) when $\theta, \sigma_i \to 1$.

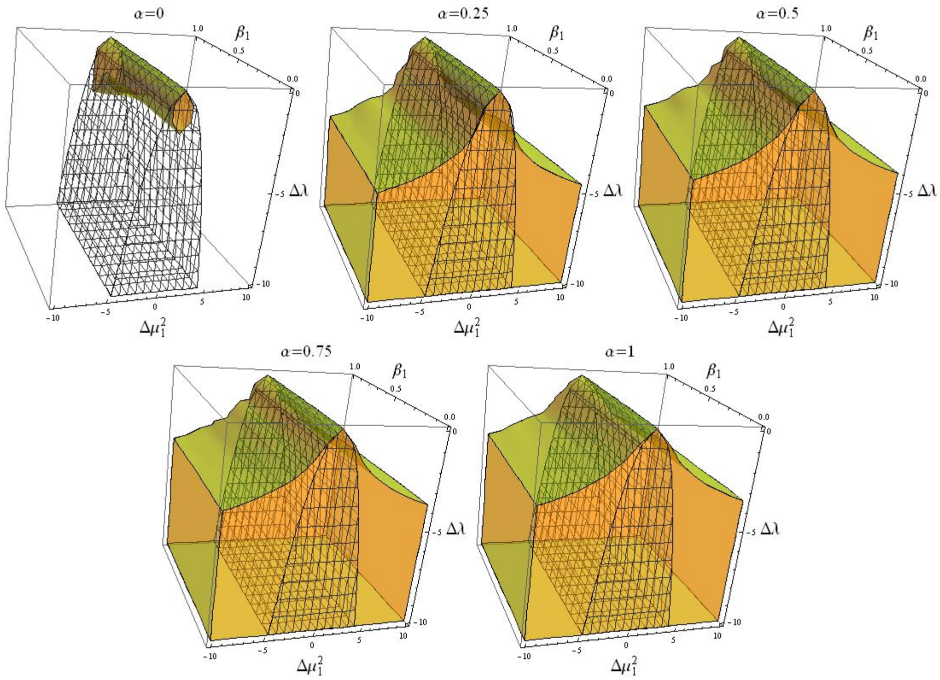


Figure 1: Real MS-stability areas of the linear SDE (16) (gridded) and BE-I scheme.

5. Numerical results

Some numerical experiments are considered to demonstrate the performance of BE-I and BE-II schemes. We compare our schemes with the Sine-Euler and Tanh-Euler methods [42]. All the simulations are performed using Matlab R2010a.

Example 5.1. *In the first example, we investigate the convergence and positivity properties of the Cox-Ingersoll-Ross (CIR) model*

$$\mathrm{d}Z_\tau = \kappa(\vartheta - Z_\tau)\mathrm{d}\tau + \eta\sqrt{Z_\tau}\mathrm{d}\omega_\tau. \tag{17}$$

It is proved that the CIR model (17) *is strictly positive when* $2\kappa\vartheta \geq \eta^2$ *[16]. In the CIR model* (17), *we set the initial value* $Z_0 = 0.1$ *and three cases of constants parameters*

(i) $\kappa = 0.3$, $\vartheta = 0.1$, $\eta = 0.1$,

(ii) $\kappa = 2$, $\vartheta = 0.75$, $\eta = 0.6$,

(iii) $\kappa = 100$, $\vartheta = 10$, $\eta = 10$.

Figures 5-7 presents 50 *paths of numerical solution of CIR model* (17) *for* $\mathcal{T} = 100$ *with step-size* $\Delta = 0.1$ *approximated by methods BE-I with* $\alpha = \beta_1 = 0.5$ *and BE-II with* $\theta = \sigma_1 = 0.5$. *In Figure 8, we show the strong convergence order of our methods applied to the CIR model for cases (i) and (ii) with* $Z_0 = 1$. *To estimate the MS errors (MSEs), we have*

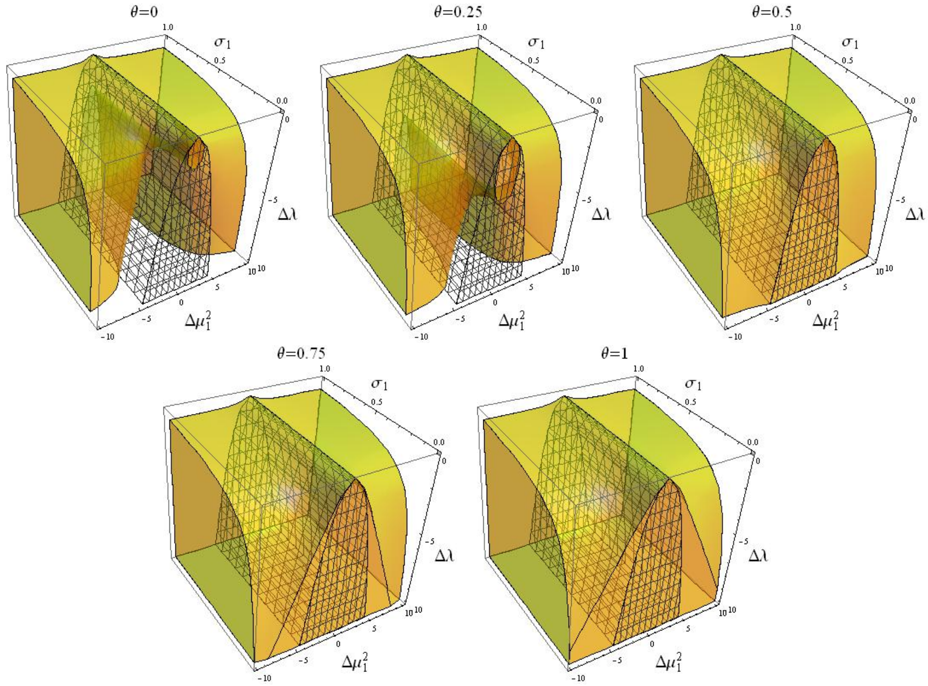


Figure 2: Real MS-stability areas of the linear SDE (16) (gridded) and BE-II scheme.

considered 5,000 different solution paths at the endpoint $\mathcal{T} = 1$ *with step-sizes* $\Delta = 2^{-k}$, $k = 4, \ldots, 11$ *and used a numerical solution with small step-size* $\Delta = 2^{-14}$ *as an exact solution. The numerical results confirm our theoretical analysis.*

Example 5.2. *The second example is the following two scalar test equations*

$$dZ_\tau = \lambda Z_\tau d\tau + \mu Z_\tau d\omega_\tau, \tag{18}$$

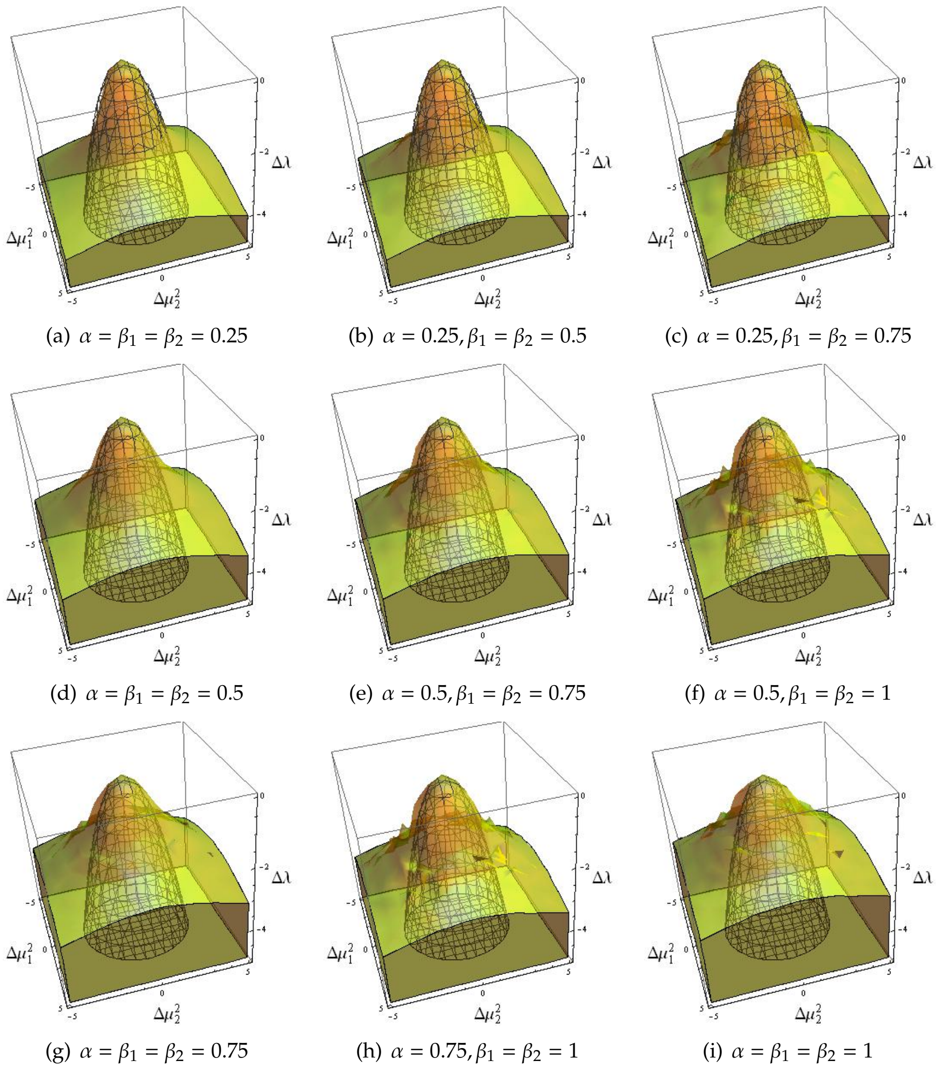
$$dZ_\tau = \lambda Z_\tau d\tau + \mu_1 Z_\tau d\omega_\tau^1 + \mu_2 Z_\tau d\omega_\tau^2. \tag{19}$$

For initial condition $Z_0 = 1$, *numerical MS stability* ($\mathbb{E}[Z_\tau^2]$) *of the methods is illustrated at* $\mathcal{T} = 25$. *Both equations* (18) *and* (19) *are simulated for* 50,000 *independent sample paths with step-sizes* $\Delta = 2^{-2}, 2^{-3}, 2^{-4}$. *The results are indicated in Figures 9-12. Figures 9 and 10 shows that BE-I method with* $\alpha = \beta_1 = 0.5$ *and BE-II method with* $\theta = \sigma_1 = 0.5$ *are MS stable for parameters* $\lambda = -10$, $\mu = \sqrt{19}$ *and* $\lambda = -50$, $\mu = \sqrt{99}$ *in equation* (18), *respectively. Also, from Figures 11 and 12, it can be deduced that the methods BE-I with* $\alpha = \beta_1 = \beta_2 = 0.5$ *and BE-II with* $\theta = \sigma_1 = \sigma_2 = 0.5$ *are mean-square stable for equation* (19) *with* $\mu_1 = \mu_2 = \frac{\sqrt{19}}{2}$ *and* $\mu_1 = \mu_2 = \frac{\sqrt{99}}{2}$, *respectively. From the figures it can be concluded that the proposed methods are better than other schemes, especially when the stiffness of the example 5.2 is added.*

Example 5.3. *In this example, we price a European call option with a current stock price of* $K = 82.96$, *a maturity of* $\mathcal{T} = 2$ *years, a risk-free interest rate* $\varpi = 0.75$, *an at-the-money volatilities* $\varsigma_1 = 0.75$, $\varsigma_2 = 0.5$. *For this set, the stock price process* Z_τ *is assumed to follow the dynamics*

$$dZ_\tau = \varpi Z_\tau d\tau + \varsigma_1 Z_\tau d\omega_\tau^1 + \varsigma_2 Z_\tau d\omega_\tau^2. \tag{20}$$

Figure 13 shows numerical results for the 5,000 *independent sample paths with step-size* $\Delta = 2^{-12}$ *and different initial values, ranging from* 65 *to* 100. *Computed option prices are shown for the exact value, BE-I method with* $\alpha = \beta_1 = \beta_2 = 0.5$ *and BE-II method with* $\theta = \sigma_1 = \sigma_2 = 0.5$. *The results reported in Figure 13 show that the agreement with actual value is good.*

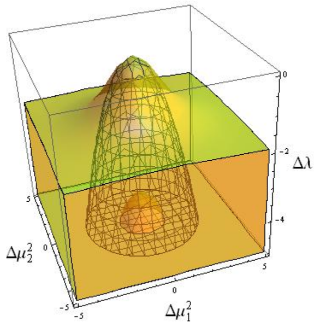


(a) $\alpha = \beta_1 = \beta_2 = 0.25$ (b) $\alpha = 0.25, \beta_1 = \beta_2 = 0.5$ (c) $\alpha = 0.25, \beta_1 = \beta_2 = 0.75$

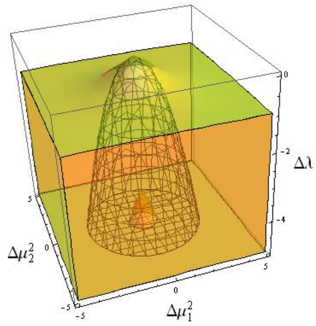
(d) $\alpha = \beta_1 = \beta_2 = 0.5$ (e) $\alpha = 0.5, \beta_1 = \beta_2 = 0.75$ (f) $\alpha = 0.5, \beta_1 = \beta_2 = 1$

(g) $\alpha = \beta_1 = \beta_2 = 0.75$ (h) $\alpha = 0.75, \beta_1 = \beta_2 = 1$ (i) $\alpha = \beta_1 = \beta_2 = 1$

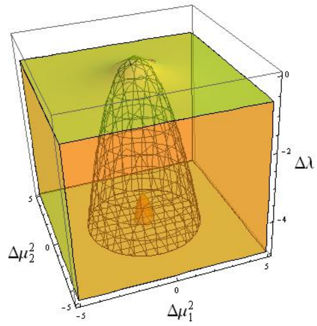
Figure 3: Real MS-stability areas of the linear SDE (16) (gridded) and BE-I scheme.



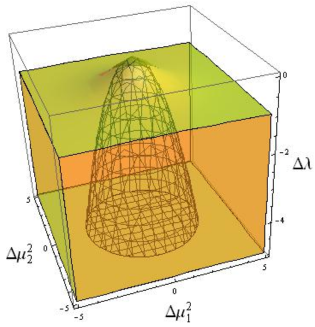
(a) $\theta = \sigma_1 = \sigma_2 = 0.25$



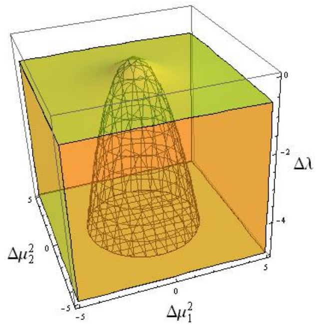
(b) $\theta = 0.25, \sigma_1 = \sigma_2 = 0.5$



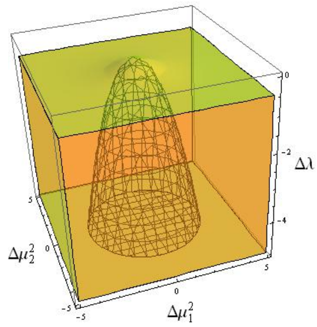
(c) $\theta = 0.25, \sigma_1 = \sigma_2 = 0.75$



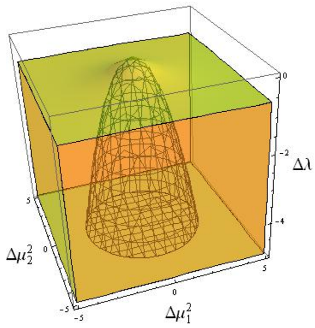
(d) $\theta = \sigma_1 = \sigma_2 = 0.5$



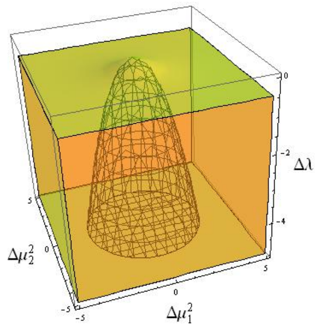
(e) $\theta = 0.5, \sigma_1 = \sigma_2 = 0.75$



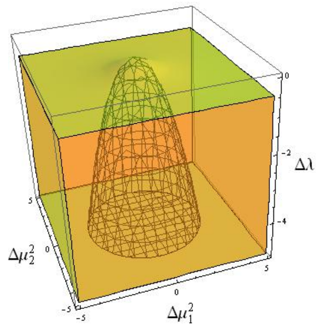
(f) $\theta = 0.5, \sigma_1 = \sigma_2 = 1$



(g) $\theta = \sigma_1 = \sigma_2 = 0.75$

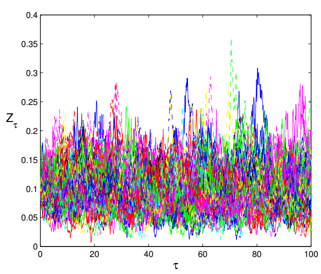


(h) $\theta = 0.75, \sigma_1 = \sigma_2 = 1$

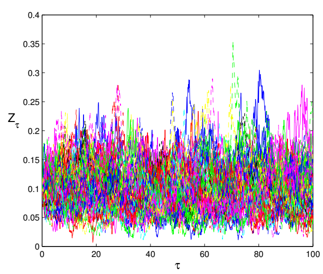


(i) $\theta = \sigma_1 = \sigma_2 = 1$

Figure 4: Real MS-stability areas of the linear SDE (16) (gridded) and BE-II scheme.

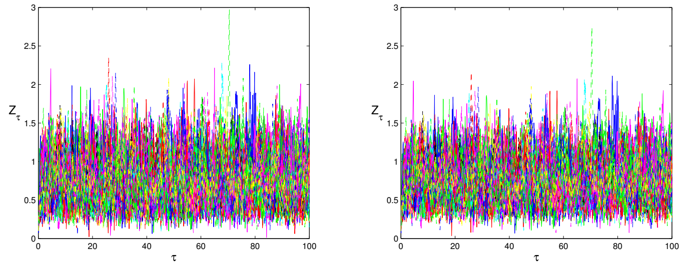


(a) BE-I method



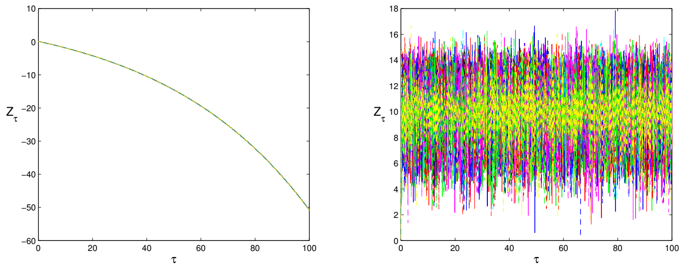
(b) BE-II method

Figure 5: Fifty different sample paths of the CIR model (17) with case (i) using BE-I and BE-II methods.



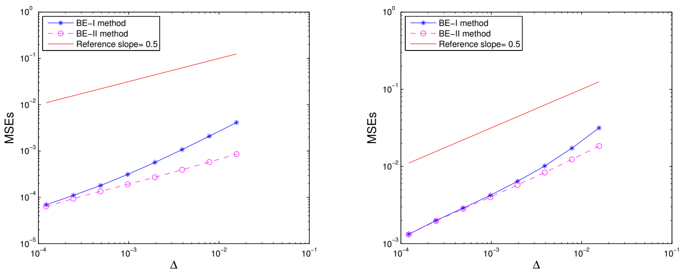
(a) BE-I method (b) BE-II method

Figure 6: Fifty different sample paths of the CIR model (17) with case (ii) using BE-I and BE-II methods.



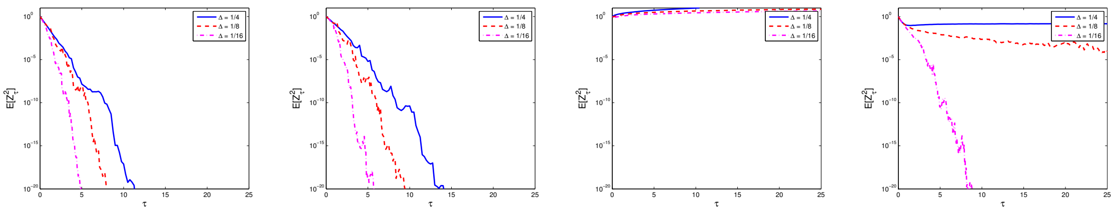
(a) BE-I method (b) BE-II method

Figure 7: Fifty different sample paths of the CIR model (17) with case (iii) using BE-I and BE-II methods.



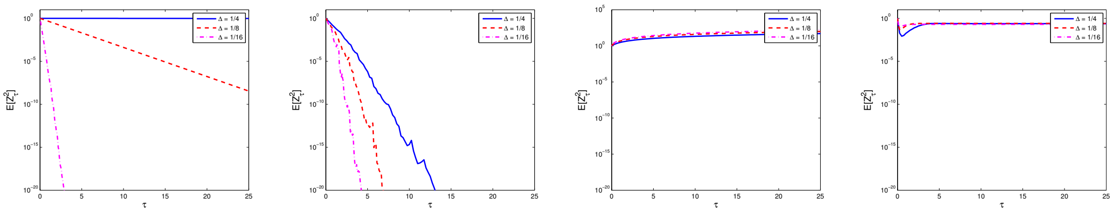
(a) case (i) (b) case (ii)

Figure 8: MSEs of the BE-I and BE-II methods applied to the CIR model (17).



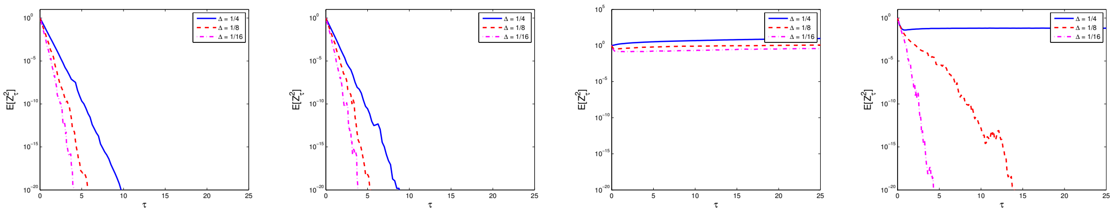
(a) BE-I method (b) BE-II method (c) Sine-Euler method [42] (d) Tanh-Euler method [42]

Figure 9: Numerical MS stability of the methods for equation (18) with $\lambda = -10$, $\mu = \sqrt{19}$.



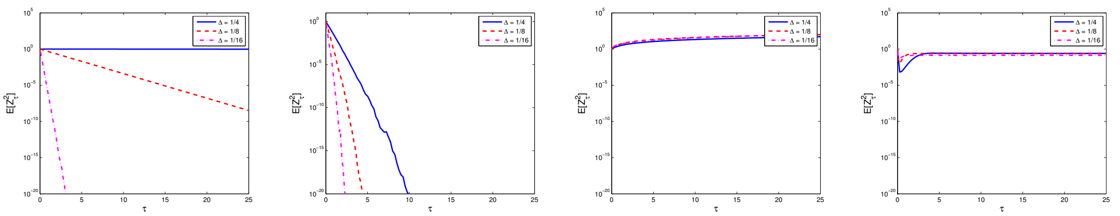
(a) BE-I method (b) BE-II method (c) Sine-Euler method [42] (d) Tanh-Euler method [42]

Figure 10: Numerical MS stability of the methods for equation (18) with $\lambda = -50$, $\mu = \sqrt{99}$.



(a) BE-I method (b) BE-II method (c) Sine-Euler method [42] (d) Tanh-Euler method [42]

Figure 11: Numerical MS stability of the methods for equation (19) with $\lambda = -10$, $\mu_1 = \mu_2 = \frac{\sqrt{19}}{2}$.



(a) BE-I method (b) BE-II method (c) Sine-Euler method [42] (d) Tanh-Euler method [42]

Figure 12: Numerical MS stability of the methods for equation (19) with $\lambda = -50$, $\mu_1 = \mu_2 = \frac{\sqrt{99}}{2}$.

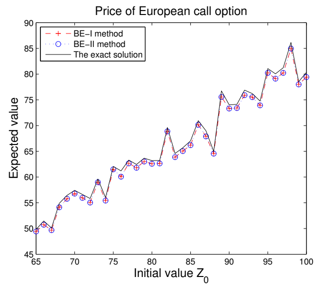
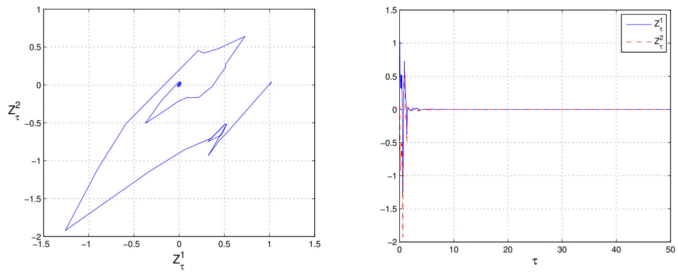
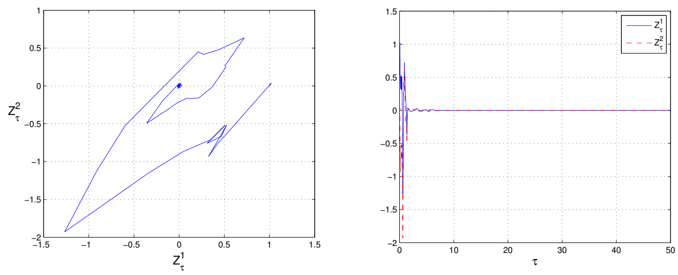


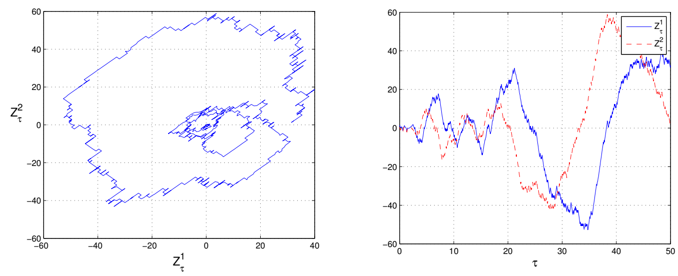
Figure 13: European call option prices.



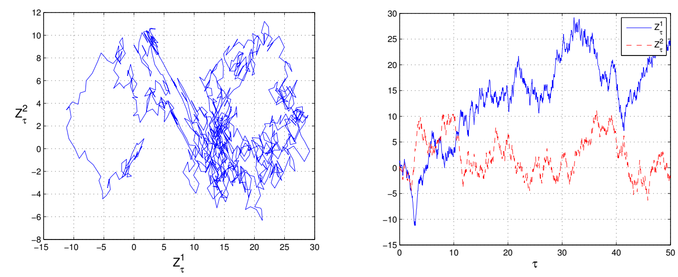
(a) EB-I method



(b) EB-II method



(c) Tanh-Euler method [42]



(d) Sine-Euler method [42]

Figure 14: Numerical simulation of stiff SDE (21).

Example 5.4. *Consider the following two-dimensional stiff stochastic system [22]*

$$dZ_\tau = v_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Z_\tau d\tau + \frac{v_2}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} Z_\tau d\omega_\tau^1 + \frac{v_3}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} Z_\tau d\omega_\tau^2. \tag{21}$$

The system (21) *was simulated with constant parameters* $v_1 = 5$, $v_2 = 4$ *and* $v_3 = \frac{1}{2}$ *on the interval* $\mathcal{T} = [0, 50]$ *with initial value at* $(Z_0^1, Z_0^2) = (1, 0)$ *and step-size* $\Delta = 0.05$. *The behavior of system* (21) *by Tanh-Euler [42], Sine-Euler [42], and proposed schemes is plotted in Figure 14. From Figure 14 it can be seen that only the BE-I method with* $\alpha = \beta_1 = \beta_2 = 1$ *and BE-II method with* $\theta = \sigma_1 = \sigma_2 = 1$ *the approximate trajectories stay close to the origin* $(0, 0)$, *which replicates the behavior of the exact solution.*

6. Conclusion

In this study, new balanced numerical schemes ware introduced for the SDE (1). Our primary strategy was to add new control functions to the balanced Euler-Maruyama method [22] and taking two numerical schemes BE-I and BE-II. We have succeeded in proving a strong form of a convergence of the methods by using some fundamental inequalities. Furthermore, we discussed the MS stability of our schemes. As we have shown, BE-I and BE-II methods are MS stable when $\alpha, \beta_i \to 1$ and $\theta, \sigma_i \to 1$, respectively. MS stability results, confirmed that the presented schemes are suitable in solving stiff SDEs. Numerical tests illustrate the effectiveness of the proposed schemes. Compared to Tanh-Euler and Sine-Euler methods [42], the proposed schemes are more capable of dealing with stiff SDEs.

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