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Fixed Point Theorem for Question of Set-Valued Quasi-Contraction

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Abstract. In this work, we give a partial positive answer to the question concerning the set-valued quasi-contraction proposed by Amini-Harandi (Appl. Math. Lett. 24:1791–1794 2011). By a useful lemma, we prove a fixed point theorem for the set-valued quasi-contraction, which extends the range of contraction constant in result of Amini-Harandi from $\left[0, \frac{1}{2}\right)$ to $\left[0, \frac{1}{\sqrt[3]{3}}\right)$. Also, we give a new simple proof for the result of quasi-contraction type proposed by Haghi et al. (Appl. Math. Lett. 25:843–846 2012). Finally, a counterexample and a theorem concerning cyclic set-valued mapping are given, which improve some recent results.

1. Introduction

In 1974, Ćirić [9] introduced a class of well known contraction, called Ćirić type contraction, and established the corresponding fixed point theorem. Since then, many authors studied and extended Ćirić type contraction in various distinct directions, see e.g. [2, 6, 16, 19]. We recall the notion of Ćirić type contraction as follows.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Ćirić type contraction (or called quasi-contraction) if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

In 1969, Nadler [18] generalized the Banach contraction principle to set-valued mappings by the Hausdorff metric. The theory of set-valued mappings has many applications and a lot of authors investigated the fixed point theorem for set-valued contraction, see e.g. [3–5, 7, 8, 10, 13–15, 17, 20]. The relative concepts are introduced as below.

Throughout this paper, let $\mathbb{N}$ and $\mathbb{N}^+$ denote the nonnegative integers, the positive integers, respectively. Let (X, d) be a metric space. We denote by 2^X and $CB(X)$, the collection of all nonempty subsets of X, the collection of all nonempty closed bounded subsets of (X, d), respectively. Let $T : X \to 2^X$ be a multi-valued mapping. We say that $x \in X$ is a fixed point of T if $x \in Tx$.

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Definition 1.1. *Let (X, d) be a metric space. For any $x \in X$ and $A, B \in CB(X)$, denote*

$$d(x, A) = \inf_{y \in A} d(x, y) \quad \text{and} \quad \delta(A, B) = \sup_{a \in A} d(a, B).$$

We say that $H(A, B) : CB(X) \times CB(X) \to \mathbb{R}^+$ is a Hausdorff metric *on $CB(X)$ induced by d if*

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

In what follows, the set-valued version of Ćirić type fixed point theorem is considered. Let (X, d) be a metric space. A set-valued mapping $T : X \to CB(X)$ is called a *set-valued quasi-contraction* if there exists $\lambda \in [0, 1)$ such that

$$H(Tx, Ty) \leq \lambda M(x, y) \tag{1}$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty),$
$d(y, Tx)\}$.

In 2011, Amini-Harandi [3] proved a set-valued quasi-contraction fixed point theorem as follows.

Theorem 1.2 ([3]). *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a set-valued quasi-contraction with constant $\lambda < \frac{1}{2}$. Then T has a fixed point.*

And then, a question was raised following the above theorem in [3].

Problem 1.3 ([3]). *Does the conclusion of [3, Theorem 2.2] remain true for any $\frac{1}{2} \leq \lambda < 1$?*

In the last decade, many scholars devoted to investigating set-valued version of quasi-contraction mappings and obtain some valuable results, see e.g. [11, 13, 17, 20]. In [11], Haghi et al. gave a similar result called quasi-contraction type, but Mohammadi et al. [17] showed that a set-valued quasi-contraction need not be quasi-contraction type. Up to now, the question above is still open.

On the other hand, the cyclic version of set-valued contraction fixed point theorem was studied by some scholars in recent years. In 2018, Sridarat and Suantai [21] investigated and gave a theorem for nonlinear cyclic set-valued mapping. In 2020, Ahmadi et al. [1] obtained some results on cyclic set-valued contraction in metric spaces.

In this paper, we establish some results on set-valued mapping for quasi-contraction, quasi-contraction type and weak contraction, as well as the corresponding cyclic version. In Section 3, we partially answer Question 1.3 by extending the range of constant λ from $\left[0, \frac{1}{2}\right)$ to $\left[0, \frac{1}{\sqrt[3]{3}}\right)$. To this end, we prove an essential lemma and apply new technique for the proof of Cauchy sequence in our theorem. Also, we prove again the result of quasi-contraction type in [11] by our new lemma and technique, and an example is given to verify our results. In Section 4, we give a counterexample to show that cyclic set-valued mapping for quasi-contraction and quasi-contraction type fail to hold with $\lambda \geq \frac{1}{2}$. On the other hand, the result of cyclic set-valued weak contraction is established, which extends the results proposed by Ahmadi et al. [1] and Khojasteh et al. [13].

2. Preliminaries

In this section, we introduce some useful lemmas concerning Hausdorff metric.

Lemma 2.1 ([18]). *Let (X, d) be a metric space and $A, B \in CB(X)$ be two nonempty sets. Then for any $b \in B$ and any $\alpha > 0$, there exists $a \in A$ such that*

$$d(a, b) \leq H(A, B) + \alpha.$$

Lemma 2.2 ([18]). *Let (X, d) be a metric space, $a, b \in X$ be two points and $A, B, C \in CB(X)$ be three sets. Then the following hold:*

(1) $d(a, A) \leq d(a, b) + d(b, A)$;

(2) $d(a, B) \leq d(a, A) + H(A, B)$;

(3) $H(A, C) \leq H(A, B) + H(B, C)$.

Lemma 2.3 ([18]). *Let (X, d) be a metric space, $\{A_n\} \subset CB(X)$ be a sequence of set and $A^* \in CB(X)$. Let $\{a_n\} \subset X$ be a sequence such that $a_n \in A_n$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} H(A_n, A^*) = 0 \tag{2}$$

and

$$\lim_{n\to\infty} d(a_n, a^*) = 0 \tag{3}$$

for some $a^* \in X$, then $a^* \in A^*$.

3. Quasi-contraction and quasi-contraction type

In this section, we give a partial answer to Question 1.3 and give a new proof for set-valued quasi-contraction type fixed point theorem. First, we prove a crucial lemma for our theorems.

Lemma 3.1. *Let (X, d) be a metric space, $\{x_n\} \subset X$ be a sequence. If there exist $\alpha < 1$ and a positive integer p such that*

$$d(x_n, x_{n+1}) \leq \alpha \max\{d(x_{n-i}, x_{n-i+1}) : 1 \leq i \leq p\} \tag{4}$$

for all $n \in \mathbb{N}$ with $n \geq p$, then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $\lfloor a \rfloor = \max\{n \in \mathbb{N} : n \leq a\}$ for all $a > 0$. Denote that

$$Q = \max\{d(x_i, x_{i+1}) : 0 \leq i \leq p-1\}.$$

By (4), we can see that $d(x_p, x_{p+1}) \leq \alpha Q$. Note that $d(x_p, x_{p+1}) \leq Q$. Then, applying (4) again, we have $d(x_{p+1}, x_{p+2}) \leq \alpha Q$. Continuing inductively, we obtain that

$$d(x_{p+k}, x_{p+k+1}) \leq \alpha Q$$

for all $0 \leq k \leq p-1$. It follows that $\max\{d(x_{p+k}, x_{p+k+1}) : 0 \leq k \leq p-1\} \leq \alpha Q$. Similarly, we can obtain $d(x_{2p+k}, x_{2p+k+1}) \leq \alpha^2 Q$ for all $0 \leq k < p$.

Proceeding inductively, we deduce that

$$\max\{d(x_{mp+k}, x_{mp+k+1}) : 0 \leq k \leq p-1\} \leq \alpha^m Q$$

for all $m \in \mathbb{N}$. Note that if $mp + k = n$, we have $m = \lfloor \frac{n}{p} \rfloor$. Then, we can see that

$$d(x_n, x_{n+1}) \leq \alpha^{\lfloor \frac{n}{p} \rfloor} Q$$

for all $n \in \mathbb{N}$. Hence, for any $m, n \in \mathbb{N}$ and $m < n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \sum_{i=m}^{n-1} \alpha^{\lfloor \frac{i}{p} \rfloor} Q \\ &\leq p \sum_{k=\lfloor \frac{m}{p} \rfloor}^{\lfloor \frac{n-1}{p} \rfloor} \alpha^k Q \\ &\leq p \sum_{k=\lfloor \frac{m}{p} \rfloor}^{\infty} \alpha^k Q = \frac{p\alpha^{\lfloor \frac{m}{p} \rfloor} Q}{1-\alpha}. \end{aligned}$$

Since $\alpha < 1$, letting $m \to \infty$, we can see that $d(x_m, x_n) \to 0$, which implies $\{x_n\}$ is a Cauchy sequence. □

Remark 3.2. *As shown in Lemma 3.1, the constant p should be independent of the index n in* (4). *If they are correlative, the sequence $\{x_n\}$ could not be Cauchy. In fact, let $X = [0, +\infty)$ be a complete metric space with the standard metric, $\{x_n\} \subset X$ be a sequence such that $x_n = \sum_{i=1}^{n} \frac{1}{i}$ for $n \geq 1$ and $x_0 = 0$. Suppose that $p = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then, we have*

$$\begin{aligned} d(x_n, x_{n+1}) = \frac{1}{n+1} &\leq \frac{1}{2} \frac{1}{\left\lfloor \frac{n+1}{2} \right\rfloor} \\ &\leq \frac{1}{2} \max \left\{ d(x_i, x_{i+1}) : n - \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq i \leq n - 1 \right\} \end{aligned}$$

for all $n \geq 1$. However, it is clear that $\{x_n\}$ is not convergent in X and so it is not Cauchy.

Now, we give the following theorem to answer Question 1.3.

Theorem 3.3. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a set-valued quasi-contraction with constant λ. If the constant λ satisfies $\lambda < \frac{1}{\sqrt[3]{3}}$, then T has a fixed point.*

Proof. Since $\lambda < \frac{1}{\sqrt[3]{3}}$, there exists $\beta \in \mathbb{R}$ such that $\lambda < \beta < \frac{1}{\sqrt[3]{3}}$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_0 \in Tx_0$, then x_0 is the fixed point. So we assume that $x_0 \notin Tx_0$, which implies that $x_0 \neq x_1$ and $d(x_0, Tx_0) > 0$. From Lemma 2.1 and (1), there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + (\beta - \lambda) M(x_0, x_1) \\ &\leq \beta M(x_0, x_1). \end{aligned}$$

Similarly, assume that $x_1 \notin Tx_1$. Then there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \beta M(x_1, x_2).$$

Proceeding inductively, we can obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and

$$d(x_{n+1}, x_{n+2}) \leq \beta M(x_n, x_{n+1}) \tag{5}$$

for all $n \in \mathbb{N}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let $n \in \mathbb{N}$ be such that $n \geq 4$. From (5) and $x_{n+1} \in Tx_n$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta M(x_{n-1}, x_n) \\ &= \beta \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &\leq \beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_n)\}. \end{aligned}$$

Note that if $d(x_n, x_{n+1}) \leq \beta d(x_n, x_{n+1})$, we have $x_n = x_{n+1}$, which contradicts the fact $x_{n+1} \in Tx_n$ and $x_n \notin Tx_n$. So, we obtain that

$$d(x_n, x_{n+1}) \leq \beta \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n)\}. \tag{6}$$

Since $d(x_{n-1}, Tx_n) \leq H(Tx_{n-2}, Tx_n)$, by (1), we deduce that

$$\begin{aligned} d(x_{n-1}, Tx_n) &\leq \lambda M(x_{n-2}, x_n) \\ &\leq \beta M(x_{n-2}, x_n) \\ &\leq \beta \max\{d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-2}, Tx_n), d(x_{n-1}, x_n)\}. \end{aligned} \tag{7}$$

Combining (6) and (7), we can see

$$d(x_n, x_{n+1}) \leq \max\{\beta d(x_{n-1}, x_n), \beta^2 d(x_{n-2}, x_n), \beta^2 d(x_{n-2}, x_{n-1}), \beta^2 d(x_{n-2}, Tx_n)\}.$$

Similarly, we can obtain that

$$d(x_{n-2}, Tx_n) \leq \beta \max\{d(x_{n-3}, x_n), d(x_{n-3}, x_{n-2}), d(x_n, x_{n+1}), d(x_{n-3}, Tx_n), d(x_{n-2}, x_n)\},$$

leading to that

$$\begin{aligned} d(x_n, x_{n+1}) \leq \max\{&\beta d(x_{n-1}, x_n), \beta^2 d(x_{n-2}, x_n), \beta^2 d(x_{n-2}, x_{n-1}), \\ &\beta^3 d(x_{n-3}, x_n), \beta^3 d(x_{n-3}, x_{n-2}), \beta^3 d(x_{n-3}, Tx_n)\}. \end{aligned} \tag{8}$$

As in the proof of (7), we have

$$d(x_{n-3}, Tx_n) \leq \beta \max\{d(x_{n-4}, x_n), d(x_{n-4}, x_{n-3}), d(x_n, x_{n+1}), d(x_{n-4}, Tx_n), d(x_{n-3}, x_n)\}.$$

If $d(x_{n-3}, Tx_n) \leq \beta d(x_{n-4}, Tx_n)$, then from the triangle inequality, we have

$$d(x_{n-3}, Tx_n) \leq \beta[d(x_{n-4}, x_{n-3}) + d(x_{n-3}, Tx_n)],$$

which implies that $d(x_{n-3}, Tx_n) \leq \frac{\beta}{1-\beta} d(x_{n-4}, x_{n-3})$. So, we can see

$$d(x_{n-3}, Tx_n) \leq \beta \max\left\{d(x_{n-4}, x_n), d(x_n, x_{n+1}), \frac{1}{1-\beta} d(x_{n-4}, x_{n-3}), d(x_{n-3}, x_n)\right\}. \tag{9}$$

Therefore, by (8) and (9), we conclude that

$$\begin{aligned} &d(x_n, x_{n+1}) \\ \leq &\max\Big\{\beta d(x_{n-1}, x_n), \beta^2 d(x_{n-2}, x_n), \beta^2 d(x_{n-2}, x_{n-1}), \\ &\beta^3 d(x_{n-3}, x_n), \beta^3 d(x_{n-3}, x_{n-2}), \beta^4 d(x_{n-4}, x_n), \\ &\frac{\beta^4}{1-\beta} d(x_{n-4}, x_{n-3})\Big\} \\ \leq &\max\Big\{\beta d(x_{n-1}, x_n), 2\beta^2 \frac{d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)}{2}, \beta^2 d(x_{n-2}, x_{n-1}), \\ &3\beta^3 \frac{d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)}{3}, \beta^3 d(x_{n-3}, x_{n-2}), \\ &4\beta^4 \frac{d(x_{n-4}, x_{n-3}) + d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)}{4}, \\ &\frac{\beta^4}{1-\beta} d(x_{n-4}, x_{n-3})\Big\} \\ \leq &\max\left\{\beta, 2\beta^2, 3\beta^3, 4\beta^4, \frac{\beta^4}{1-\beta}\right\} \max\{d(x_{n-4}, x_{n-3}), d(x_{n-3}, x_{n-2}), \\ &d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Since $\beta < \frac{1}{\sqrt[3]{3}}$, we can see that

$$\alpha = \max\left\{\beta, 2\beta^2, 3\beta^3, 4\beta^4, \frac{\beta^4}{1-\beta}\right\} < 1.$$

Then, from lemma 3.1, $\{x_n\}$ is a Cauchy sequence in X.

Since (X,d) is complete, there exists a $x^* \in X$ such that $\{x_n\}$ converges to x^*. Then we show that x^* is a fixed point of T. By (1), we have

$$\begin{aligned} H(Tx_n, Tx^*) &\le \lambda M(x_n, x^*) \\ &= \lambda \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \\ &\quad d(x_n, Tx^*), d(Tx_n, x^*)\}. \end{aligned}$$

If $M(x_n, x^*) = d(x^*, Tx^*)$ for some $n \in \mathbb{N}$, then by Lemma 2.2 (2), we have

$$H(Tx_n, Tx^*) \le \lambda d(x^*, Tx^*) \le \lambda[d(x^*, Tx_n) + H(Tx_n, Tx^*)],$$

which implies that

$$H(Tx_n, Tx^*) \le \frac{\lambda}{1-\lambda} d(x^*, Tx_n) \le \frac{\lambda}{1-\lambda} d(x^*, x_{n+1}).$$

Similarly, if $M(x_n, x^*) = d(x_n, Tx^*)$ for some $n \in \mathbb{N}$, then we obtain that

$$H(Tx_n, Tx^*) \le \frac{\lambda}{1-\lambda} d(x_n, Tx_n) \le \frac{\lambda}{1-\lambda} d(x_n, x_{n+1}).$$

Thus, for every $n \in \mathbb{N}$, we can see that

$$\begin{aligned} &H(Tx_n, Tx^*) \\ &\le \lambda \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), \frac{d(x^*, x_{n+1})}{1-\lambda}, \frac{d(x_n, x_{n+1})}{1-\lambda}, d(x_{n+1}, x^*)\right\}. \end{aligned}$$

Letting $n \to \infty$, since $\{x_n\}$ converges to x^*, we conclude that $H(Tx_n, Tx^*) \to 0$. Note that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Then, from Lemma 2.3 we can obtain that $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. □

Next, by Lemma 3.1, we give a new proof for set-valued quasi-contraction type fixed point theorem, which is simpler than that of [11, Theorem 2.2]. For convenience, the notion of set-valued quasi-contraction type is reviewed as follows.

A set-valued mapping $T : X \to CB(X)$ is called a *set-valued quasi-contraction type* if there exists $\lambda \in [0, 1)$ such that

$$H(Tx, Ty) \le \lambda N(x, y), \tag{10}$$

for all $x, y \in X$, where $N(x, y) = \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Theorem 3.4. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ a set-valued quasi-contraction type. Then T has a fixed point.*

Proof. Since $\lambda < 1$, there exists β such that $\lambda < \beta < 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. From a similar argument in the proof of Theorem 3.3, we can obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and

$$d(x_{n+1}, x_{n+2}) \le \beta N(x_n, x_{n+1}) \tag{11}$$

for all $n \in \mathbb{N}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\beta < 1$, there exists $\rho \in \mathbb{N}^+$ such that

$$\frac{\beta^{\rho+1}}{1-\beta} < 1. \tag{12}$$

Let $n \in \mathbb{N}$ be such that $n \ge \rho + 1$. From (11) and $x_{n+1} \in Tx_n$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\le \beta M(x_{n-1}, x_n) \\ &= \beta \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), \\ &\quad d(x_n, Tx_{n-1})\} \\ &= \beta \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n)\}. \end{aligned}$$

If $d(x_n, x_{n+1}) \leq \beta d(x_n, Tx_n) \leq \beta d(x_n, x_{n+1})$, then we have $x_n = x_{n+1}$, which contradicts the fact $x_{n+1} \in Tx_n$ and $x_n \notin Tx_n$. So, we obtain that

$$d(x_n, x_{n+1}) \leq \max\{\beta d(x_{n-1}, x_n), \beta d(x_{n-1}, Tx_n)\}. \tag{13}$$

Since $d(x_{n-1}, Tx_n) \leq H(Tx_{n-2}, Tx_n)$, by (10), we have

$$\begin{aligned} d(x_{n-1}, Tx_n) &\leq \lambda N(x_{n-2}, x_n) \leq \beta N(x_{n-2}, x_n) \\ &= \beta \max\{d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), d(x_{n-2}, Tx_n), \\ &\quad d(x_n, Tx_{n-2})\} \\ &\leq \beta \max\{d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-2}, Tx_n), \\ &\quad d(x_n, x_{n-1})\}. \end{aligned}$$

Combining (13), we obtain that

$$d(x_n, x_{n+1}) \leq \max\{\beta d(x_{n-1}, x_n), \beta d(x_{n-2}, x_{n-1}), \beta^2 d(x_{n-2}, Tx_n)\}. \tag{14}$$

Similarly, we can deduce

$$\begin{aligned} d(x_{n-2}, Tx_n) &\leq H(Tx_{n-3}, Tx_n) \leq \beta N(x_{n-3}, x_n) \\ &= \beta \max\{d(x_{n-3}, Tx_{n-3}), d(x_n, Tx_n), d(x_{n-3}, Tx_n), \\ &\quad d(x_n, Tx_{n-3})\} \\ &\leq \beta \max\{d(x_{n-3}, x_{n-2}), d(x_n, x_{n+1}), d(x_{n-3}, Tx_n), \\ &\quad d(x_n, Tx_{n-3})\}, \end{aligned}$$

which implies that

$$\begin{aligned} d(x_n, x_{n+1}) \leq \max\{&\beta d(x_{n-1}, x_n), \beta d(x_{n-2}, x_{n-1}), \beta d(x_{n-3}, x_{n-2}) \\ &\beta^3 d(x_{n-3}, Tx_n), \beta^3 d(x_n, Tx_{n-3})\}. \end{aligned}$$

Proceeding inductively, we can conclude that

$$d(x_n, x_{n+1}) \leq \max C_\rho \cup D_\rho, \tag{15}$$

where

$$C_\rho = \{\beta d(x_{n-i}, x_{n-i+1}) : 1 \leq i \leq \rho\}$$

and

$$D_\rho = \{\beta^\rho d(x_{n-i}, Tx_{n-j}) : 0 \leq i, j \leq \rho \text{ and } i + 1 \neq j\}.$$

Then, applying (10), we can see

$$\begin{aligned} d(x_{n-i}, Tx_{n-j}) &\leq H(Tx_{n-i-1}, Tx_{n-j}) \leq \beta N(x_{n-i-1}, x_{n-j}) \\ &\leq \beta \max\{d(x_{n-i-1}, x_{n-i}), d(x_{n-j}, x_{n-j+1}), \\ &\quad d(x_{n-i-1}, Tx_{n-j}), d(x_{n-j}, Tx_{n-i-1})\}. \end{aligned}$$

If $H(Tx_{n-i-1}, Tx_{n-j}) \leq \beta d(x_{n-i-1}, Tx_{n-j})$, then from Lemma 2.2 (2), we have

$$H(Tx_{n-i-1}, Tx_{n-j}) \leq \beta[d(x_{n-i-1}, Tx_{n-i-1}) + H(Tx_{n-i-1}, Tx_{n-j})],$$

which implies that

$$H(Tx_{n-i-1}, Tx_{n-j}) \leq \frac{\beta}{1-\beta} d(x_{n-i-1}, Tx_{n-i-1}) \leq \frac{\beta}{1-\beta} d(x_{n-i-1}, x_{n-i}).$$

If $H(Tx_{n-i-1}, Tx_{n-j}) \le \beta d(x_{n-j}, Tx_{n-i-1})$, then by Lemma 2.2 (2), we can see that

$$H(Tx_{n-i-1}, Tx_{n-j}) \le \frac{\beta}{1-\beta} d(x_{n-j}, Tx_{n-j}) \le \frac{\beta}{1-\beta} d(x_{n-j}, x_{n-j+1}).$$

Thus, from $\beta < \frac{\beta}{1-\beta}$, we obtain that

$$\begin{aligned} d(x_{n-i}, Tx_{n-j}) &\le H(Tx_{n-i-1}, Tx_{n-j}) \\ &\le \frac{\beta}{1-\beta} \max\{d(x_{n-i-1}, x_{n-i}), d(x_{n-j}, x_{n-j+1})\}. \end{aligned} \tag{16}$$

Combining (12), (15) and (16), we can deduce that

$$d(x_n, x_{n+1}) \le \gamma \max\{d(x_{n-i}, x_{n-i+1}) : 1 \le i \le \rho + 1\},$$

where $\gamma = \max\left\{\beta, \frac{\beta^{\rho+1}}{1-\beta}\right\} < 1$. By Lemma 3.1 with $p = \rho + 1$, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$. By (10), we have

$$\begin{aligned} H(Tx_n, Tx^*) &\le \lambda \max\{d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(Tx_n, x^*)\} \\ &\le \lambda \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, x^*) + d(x^*, Tx^*), \\ &\quad d(x_{n+1}, x^*)\}. \end{aligned}$$

Since the metric d is continuous, we deduce that

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) \le \overline{\lim_{n\to\infty}} H(Tx_n, Tx^*) \le \lambda d(x^*, Tx^*).$$

Thus, $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

Finally, an example is given to verify Theorem 3.3. Note that there are two existing results, quasi-contraction type [11, Theorem 2.2] and weak contraction [13, Theorem 2.2], which are similar to this theorem. In [17], Mohammadi et al. gave an example [17, Example 2.1], where the mapping T is a set-valued quasi-contraction but not a quasi-contraction type. Next, we show that set-valued quasi-contraction need not be a weak contraction in the following example.

Firstly, we review the notion of weak contraction. A mapping $T : X \to CB(X)$ is said to be a *set-valued weak contraction* if there exists $\alpha \in [0, 1)$ such that for any $x, y \in X$,

$$H(Tx, Ty) \le \alpha K(x, y), \tag{17}$$

where $K(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Example 3.5. *Let $X = \mathbb{R}^+$ be equipped with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define a set-valued mapping $T : X \to CB(X)$ by*

$$T(x) = \begin{cases} \left[\frac{1}{3}x, \frac{2}{3}x\right], & x \ge 1; \\ \left[\frac{1}{4}x, \frac{1}{2}x\right], & 0 < x < 1; \\ \{0\}, & x = 0. \end{cases}$$

Then the following hold:

1. *T is a set-valued quasi-contraction with $\lambda = \frac{2}{3}$;*
2. *all the conditions in Theorem 3.3 are satisfied, and $x = 0$ is a fixed point for T;*
3. *T is not a set-valued weak contraction.*

Proof. **(1)** It is sufficient to prove that for any $x, y \in X$,

$$\begin{aligned} H(Tx, Ty) &\leq \frac{2}{3}M(x, y) \\ &= \frac{2}{3}\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned} \tag{18}$$

Indeed, let $x, y \in X$ be arbitrarily given. Without loss of generality, we suppose that $x > y$. By the construction of T, we consider the following two cases.

Case 1. Assume that $x \geq 1$. Then, we have $Tx = \left[\frac{1}{3}x, \frac{2}{3}x\right]$. If $y \geq 1$, from the definition of Hausdorff metric we obtain that

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[\frac{1}{3}x, \frac{2}{3}x\right], \left[\frac{1}{3}y, \frac{2}{3}y\right]\right) = \frac{2}{3}(x - y) \\ &= \frac{2}{3}d(x, y) \leq \frac{2}{3}M(x, y). \end{aligned}$$

If $0 < y < 1$, then we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[\frac{1}{3}x, \frac{2}{3}x\right], \left[\frac{1}{4}y, \frac{1}{2}y\right]\right) = \frac{2}{3}x - \frac{1}{2}y \\ &\leq \frac{2}{3}\left(x - \frac{1}{2}y\right) = \frac{2}{3}d(x, Ty) \leq \frac{2}{3}M(x, y). \end{aligned}$$

If $y = 0$, it is easy to see that $H(Tx, Ty) = H\left(\left[\frac{1}{3}x, \frac{2}{3}x\right], \{0\}\right) = \frac{2}{3}d(x, Ty) \leq \frac{2}{3}M(x, y)$.

Case 2. Assume that $0 \leq x < 1$. Since $x > y$ and $y \geq 0$, we have $x > 0$. Then, we obtain that

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[\frac{1}{4}x, \frac{1}{2}x\right], \left[\frac{1}{4}y, \frac{1}{2}y\right]\right) = \frac{1}{2}(x - y) \\ &= \frac{1}{2}d(x, y) \leq \frac{2}{3}M(x, y) \end{aligned}$$

for any $0 < y < x$ and $H(Tx, Ty) = H\left(\left[\frac{1}{4}x, \frac{1}{2}x\right], \{0\}\right) = \frac{1}{2}d(x, Ty) \leq \frac{2}{3}M(x, y)$ for $y = 0$.

Therefore, we completely prove that (18) holds for all $x, y \in X$.

(2) It is clear that (X, d) is a complete metric space and $\lambda = \frac{2}{3} < \frac{1}{\sqrt[3]{3}}$. Note that $0 \in T0 = \{0\}$. Then $x = 0$ is a fixed point for mapping T.

(3) Let $x_0 = 1$ and $y_0 = \frac{2}{3}$. Then ,we can see that

$$H(Tx_0, Ty_0) = H\left(\left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{1}{6}, \frac{1}{3}\right]\right) = \frac{1}{3}$$

and

$$\begin{aligned} &\max\left\{d(x_0, y_0), d(x_0, Tx_0), d(y_0, Ty_0), \frac{d(x_0, Ty_0) + d(y_0, Tx_0)}{2}\right\} \\ &= \max\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{\frac{2}{3} + 0}{2}\right\} = \frac{1}{3} = H(Tx_0, Ty_0). \end{aligned}$$

Therefore, we obtain that T is not a set-valued weak contraction. □

4. Cyclic set-valued contraction

Let $A_1, A_2, \ldots, A_r$ be nonempty sets of a metric space (X, d), where r is a given integer. A mapping $T : \bigcup_{i=1}^{r} A_i \to CB(X)$ is called to be a *cyclic set-valued mapping* if

$$T(A_i) \subset A_{i+1}$$

for $i = 1, 2, \dots, r$ with $A_{r+1} = A_1$, where $T(A_i) = \bigcup_{a \in A_i} T(a)$.

In this section, we will use the following definitions of cyclic set-valued version of quasi-contraction, quasi-contraction type and weak contraction.

Definition 4.1. *Let (X, d) be a metric space and $A_1, A_2, \dots, A_r$ be nonempty sets of (X, d), where r is a given integer. We call that $T : \bigcup_{i=1}^{r} A_i \to CB(X)$ is*

(i) a cyclic set-valued quasi-contraction *if T is a cyclic set-valued mapping and there exists $\lambda \in [0, 1)$ satisfying* (1) *for all $x \in A_i$ and $y \in A_{i+1}$ with $i = 1, 2, \dots, r$;*

(ii) a cyclic set-valued quasi-contraction type *if T is a cyclic set-valued mapping and there exists $\lambda \in [0, 1)$ satisfying* (10) *for all $x \in A_i$ and $y \in A_{i+1}$ with $i = 1, 2, \dots, r$;*

(iii) a cyclic set-valued weak contraction *if T is a cyclic set-valued mapping and there exists $\alpha \in [0, 1)$ satisfying* (17) *for all $x \in A_i$ and $y \in A_{i+1}$ with $i = 1, 2, \dots, r$.*

It is worth mentioning that the set of fixed points for cyclic mappings of quasi-contraction may be empty in metric spaces. In [12, Example 2.1], He et al. gave a counterexample to show this fact. It is clear that this fact can be extended to set-valued version. Note that every quasi-contraction type is a quasi-contraction. Thus, we just need to give a counterexample, where the mapping T is a cyclic set-valued quasi-contraction type but has no fixed point.

Example 4.2. *Let $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ be a nonempty set, $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let $d : X \times X \to \mathbb{R}^+$ be such that*

$$d(x, y) = \begin{cases} 2, & x, y \in A, \\ 2, & x, y \in B, \\ 1, & x \in A, y \in B \text{ or } x \in B, y \in A. \end{cases}$$

Define a mapping $T : A \cup B \to CB(X)$ by

$$Ta_1 = \{b_1\},\ Ta_2 = \{b_2\},\ Tb_1 = \{a_2\},\ Tb_2 = \{a_1\}$$

and

$$Ta_3 = \{b_1, b_2\},\ Tb_3 = \{a_1, a_2\}.$$

Then the following hold:

1. *(X, d) is a complete metric space and A, B are two nonempty closed sets of (X, d);*
2. *T is a cyclic set-valued quasi-contraction type with constant $\lambda = \frac{1}{2}$;*
3. *T has no fixed point in X.*

Proof. **(1)** First, we show that (X, d) is a metric space. It is sufficient to prove the triangle inequality for all $x, y, z \in X$. Note that

$$\min_{x \neq y} d(x, y) = 1 \quad \text{and} \quad \max_{x \neq y} d(x, y) = 2.$$

Then, we have

$$d(x, y) \leq 2 = 1 + 1 \leq d(x, y) + d(z, y)$$

for all distinct points $x, y, z \in X$. If any two points of x, y, z are equal, the triangle inequality holds obviously. Thus, (X, d) is a metric space. On the other hand, since (X, d) is discrete, we can see that it is complete, and A, B are two nonempty closed sets of (X, d).

(2) It is clear that $T(A) \subset B$ and $T(B) \subset A$, so T is a cyclic set-valued mapping. Next, we show that (10) holds for all $x \in A$, $y \in B$ or $x \in B$, $y \in A$, where $\lambda = \frac{1}{2}$. Without loss of generality, let $x \in A$, $y \in B$ be given. We consider the following three cases.

Case 1. Assume that $x = a_1$. Then, we have $Tx = \{b_1\}$. If $y = b_1$, then $Ty = \{a_2\}$ and $d(x, Ty) = 2$. It follows that

$$\begin{aligned} H(Tx, Ty) &= H(\{a_2\}, \{b_1\}) = 1 = \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} d(x, Ty) \leq \frac{1}{2} N(x, y). \end{aligned}$$

If $y = b_2$, then we have $d(y, Tx) = 2$. It follows that

$$\begin{aligned} H(Tx, Ty) &= H(\{a_1\}, \{b_1\}) = 1 = \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} d(y, Tx) \leq \frac{1}{2} N(x, y). \end{aligned}$$

If $y = b_3$, then $Ty = \{a_1, a_2\}$. Note that $d(y, Tx) = d(b_1, b_3) = 2$. Then, we have

$$\begin{aligned} H(Tx, Ty) &= H(\{a_1, a_2\}, \{b_1\}) = 1 = \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} d(y, Tx) \leq \frac{1}{2} N(x, y). \end{aligned}$$

Case 2. Assume that $x = a_2$. Then, we have $Tx = \{b_2\}$. If $y = b_1$ or $y = b_3$, then we get $d(y, Tx) = 2$, leading to that

$$H(Tx, Ty) = 1 = \frac{1}{2} \cdot 2 = \frac{1}{2} d(y, Tx) \leq \frac{1}{2} N(x, y).$$

If $y = b_2$, then $Ty = a_1$ and $d(x, Ty) = d(a_1, a_2) = 2$. It follows that

$$\begin{aligned} H(Tx, Ty) &= H(\{a_1\}, \{b_2\}) = 1 = \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} d(x, Ty) \leq \frac{1}{2} N(x, y). \end{aligned}$$

Case 3. Suppose that $x = a_3$. Then, we get $Tx = \{b_1, b_2\}$. If $y = b_1$, then we have $Ty = \{a_2\}$ and $d(x, Ty) = 2$. If $y = b_2$, then $Ty = \{b_1\}$ and $d(x, Ty) = 2$. In these cases, we can see that

$$H(Tx, Ty) = 1 = \frac{1}{2} \cdot 2 = \frac{1}{2} d(x, Ty) \leq \frac{1}{2} N(x, y).$$

If $y = b_3$, then $Ty = \{a_1, a_2\}$. Note that $d(y, Tx) = d(b_3, \{b_1, b_2\}) = 2$. Then, we have

$$\begin{aligned} H(Tx, Ty) &= H(\{a_1, a_2\}, \{b_1, b_2\}) = 1 = \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} d(y, Tx) \leq \frac{1}{2} N(x, y). \end{aligned}$$

From the above three cases, we obtain that (10) holds and so T is a cyclic set-valued quasi-contraction type with constant $\lambda = \frac{1}{2}$.

(3) Note that $T(A) \subset B$, $T(B) \subset A$ and $A \cup B = \emptyset$. Then, it is clear that T has no fixed point in X. □

On the other hand, the fixed point theorem for cyclic set-valued weak contraction is true in metric spaces. We give the following theorem to show this result.

Theorem 4.3. *Let (X, d) be a complete metric space, $\{A_i\}_{i=1}^r$ be nonempty sets of (X, d) and $T : \bigcup_{i=1}^r A_i \to CB(X)$ be a cyclic set-valued weak contraction. Suppose that there exists $i_0 \in \{1, 2, \ldots, r\}$ such that A_{i_0} is closed. Then T has a fixed point.*

Proof. Since $\alpha < 1$, there exists $\gamma \in \mathbb{R}$ such that $\alpha < \gamma < 1$. Let $x_0 \in \bigcup_{i=1}^{r} A_i$. Without loss of generality, suppose that $x_0 \in A_1$. By the definition of T, we have $Tx_0 \subset A_2$. If $x_0 \in Tx_0$, then x_0 is the fixed point. So we assume that $x_0 \notin Tx_0$.

Take $x_1 \in Tx_0$. Then we have $x_0 \neq x_1$ and $d(x_0, Tx_0) > 0$. Since $x_1 \in A_2$, by Lemma 2.1 and (17), there exists $x_2 \in Tx_1 \subset A_3$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + (\gamma - \alpha)K(x_0, x_1) \\ &\leq \gamma K(x_0, x_1). \end{aligned}$$

Similarly, assume that $x_1 \notin Tx_1$. Then there exists $x_3 \in Tx_2 \subset A_4$ such that

$$d(x_2, x_3) \leq \gamma K(x_1, x_2).$$

Proceeding inductively, we can obtain a sequence $\{x_n\}$ such that $x_n \in A_i$, $x_{n+1} \in Tx_n \subset A_{i+1}$, $x_n \notin Tx_n$ and

$$d(x_{n+1}, x_{n+2}) \leq \gamma K(x_n, x_{n+1}), \tag{19}$$

for all $n \in \mathbb{N}$, where $i \in \{1, 2, \ldots, r\}$ satisfies $i \equiv n + 1 \pmod r$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. By (19), we have

$$\begin{aligned} & d(x_n, x_{n+1}) \leq \gamma K(x_{n-1}, x_n) \\ &= \gamma \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ & \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} \Big\} \\ &\leq \gamma \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2} \right\}. \end{aligned}$$

Since $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}x_n) + d(x_nx_{n+1})$, we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \gamma \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}x_n) + d(x_nx_{n+1})}{2} \right\} \\ &= \gamma \max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}. \end{aligned}$$

If $d(x_n, x_{n+1}) \leq \gamma d(x_n, x_{n+1})$, then we have $d(x_n, x_{n+1}) = 0$ and so $x_n = x_{n+1}$, which contradict our assumption. Thus, we deduce that

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. By Lemma 3.1 with $p = 1$, we conclude that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Note that $x_{rk+i_0-1} \in A_{i_0}$ for all $k \in \mathbb{N}$. Let $n_k = rk + i_0 - 1$, then we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subset A_{i_0}$ and $x_{n_k} \to x^*$ as $k \to \infty$. Since A_{i_0} is a closed set, we can see $x^* \in A_{i_0}$. Note that $x_{n_k-1} \in A_{i_0-1}$ for $k \in \mathbb{N}$. Then, from (17),

$$\begin{aligned} H(Tx_{n_k-1}, Tx^*) &\leq \alpha \max \Big\{ d(x_{n_k-1}, x^*), d(x_{n_k-1}, Tx_{n_k-1}), d(x^*, Tx^*), \\ & \frac{d(x_{n_k-1}, Tx^*) + d(Tx_{n_k-1}, x^*)}{2} \Big\} \\ &\leq \alpha \max \Big\{ d(x_{n_k-1}, x^*), d(x_{n_k-1}, x_{n_k}), d(x^*, Tx^*), \\ & \frac{d(x_{n_k-1}, x^*) + d(x^*, Tx^*) + d(x_{n_k}, x^*)}{2} \Big\}. \end{aligned}$$

Remarking that the subsequence $\{x_{n_k-1}\}$ converges to x^*, we obtain that

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n_k}, Tx^*) \le \overline{\lim_{n\to\infty}} H(Tx_{n_k-1}, Tx^*) \le \alpha d(x^*, Tx^*).$$

Thus, $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

It is easy to see that every Banach contraction is a weak contraction. Then, we can obtain the following result which is studied in [1].

Corollary 4.4 ([1]). *Let (X, d) be a complete metric space, $\{A_i\}_{i=1}^r$ be nonempty sets of (X, d) such that at least one of which is closed. Suppose that $T : \bigcup_{i=1}^r A_i \to CB(X)$ be a cyclic set-valued contraction, that is, T is a cyclic set-valued mapping and there exists $\alpha \in [0, 1)$ such that*

$$H(Tx, Ty) \le \alpha d(x, y)$$

for all $x \in A_i$ and $y \in A_{i+1}$ with $i = 1, 2, \ldots, r$. Then T has a fixed point.

Note that non-cyclic version is a special case of cyclic version. Then, we can obtain the following corollary which extends the result [13, Theorem 3.1] to cyclic version.

Corollary 4.5. *Let (X, d) be a complete metric space, $\{A_i\}_{i=1}^r$ be nonempty sets of (X, d) such that at least one of which is closed. Suppose that $T : \bigcup_{i=1}^r A_i \to CB(X)$ be a cyclic set-valued mapping. If there exist $\frac{1}{2} \le c < 1$ and $0 \le \lambda < \frac{1}{4c^2}$ such that*

$$H(Tx, Ty) \le \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), cd(x, Ty), cd(y, Tx)\}$$

for all $x \in A_i$ and $y \in A_{i+1}$ with $i = 1, 2, \ldots, r$, then T has a fixed point.

Proof. Since $c \ge \frac{1}{2}$, we have $\lambda < \frac{1}{4c^2} \le \frac{1}{2c} \le 1$, leading to that

$$2\lambda c < \frac{1}{2c} \cdot 2c = 1.$$

Let $\alpha = 2\lambda c$. Then, we get $\lambda \le \alpha < 1$. It follows that

$$\begin{aligned}
H(Tx, Ty) &\le \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), cd(x, Ty), cd(y, Tx)\} \\
&= \max\left\{\lambda d(x, y), \lambda d(x, Tx), \lambda d(y, Ty), \frac{\alpha}{2} d(x, Ty), \frac{\alpha}{2} d(y, Tx)\right\} \\
&\le \max\left\{\alpha d(x, y), \alpha d(x, Tx), \alpha d(y, Ty), \frac{\alpha}{2} d(x, Ty), \frac{\alpha}{2} d(y, Tx)\right\} \\
&\le \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} d(x, Ty), \frac{1}{2} d(y, Tx)\right\} \\
&\le \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \\
&= \alpha K(x, y).
\end{aligned}$$

Thus, T is a cyclic set-valued weak contraction and so the conclusion holds by applying Theorem 4.3. □

Remark 4.6. *Note that the conclusion of [13, Theorem 3.1] is that at least one of the following conditions holds:*

(i) T has a fixed point,

(ii) T^2 has a fixed point (that is, there exists $z \in X$ such that $z \in T^2 z$, where $T^2 z = \bigcup_{\omega \in Tz} T\omega$).

However, the conclusion of Corollary 4.5 can deduce all of (i) and (ii). That is, (i) implies (ii). In fact, if T has a fixed point $x^* \in X$, then $x^* \in Tx^*$ and

$$Tx^* \subset \bigcup_{\omega \in Tx^*} T\omega = T^2 x^*,$$

which implies that x^* is a fixed point for T^2.

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