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A Note on Extreme Points in the Closed Unit Ball of Upper Triangular 2×2 Matrices Over a C^*-Algebra

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Abstract. Given a unital C^*-algebra A, let $M_{m\times n}(A)$ be the set of all $m\times n$ matrices algebra over A and $\big(M_n(A)\big)_1$ be the closed unit ball of $M_{n\times n}(A)$. Let $x=\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}\in \big(M_{m+n}(A)\big)_1$ be determined by $a\in M_{m\times m}(A), b\in M_{m\times n}(A)$ and $c\in M_{n\times n}(A)$. Some characterizations are given such that the above upper triangular matrix x is an extreme point of $\big(M_{m+n}(A)\big)_1$ and $X_{m,n}(A)$ respectively, where $X_{m,n}(A)$ is the subset of $\big(M_{m+n}(A)\big)_1$ consisting of all upper triangular matrices.

1. Introduction

Throughout this paper, $\mathbb{N}$ is the set consisting of all natural numbers, $\mathbb{C}$ is the complex field, A is a nonzero unital C^*-algebra [6], and $M_{m\times n}(A)$ is the set of all $m\times n$ matrices algebra over A, which is simplified to $M_n(A)$ whenever $m=n$. Let $(A)_1$ and $\big(M_n(A)\big)_1$ be the closed unit ball of A and $M_n(A)$, respectively. The identity of $M_n(A)$ is denoted simply by 1 for all $n\in\mathbb{N}$. When $A=\mathbb{C}$, we use the notation $\mathbb{C}^{m\times n}$ instead of $M_{m\times n}(\mathbb{C})$.

Given $m,n\in\mathbb{N}$, let $X_{m,n}(A)$ be the subset of $\big(M_{m+n}(A)\big)_1$ consisting of all upper triangular matrices. More precisely, $x\in X_{m,n}(A)$ if and only if $\|x\|\le 1$, and x has the form

$$x=\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}, \tag{1}$$

where $a\in M_m(A), b\in M_{m\times n}(A)$ and $c\in M_n(A)$.

Recall that an element v in a convex set V is said to be an extreme point of V if for every $y,z\in V$ and $t\in(0,1)$, $v=ty+(1-t)z$ implies $v=y=z$. It is well-known that v is an extreme point of V if and only if for every $y,z\in V$, $v=\frac{1}{2}(y+z)$ implies $v=y=z$. Clearly, the closed unit ball $(A)_1$ of a C^*-algebra A is a convex set. To ensure the existence of an extreme point of $(A)_1$, it is necessary that A has a unit [6, Proposition 1.4.7]. So, all the C^*-algebras considered in this paper are assumed to be unital.

A useful characterization of the extreme points reads as follows.

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Lemma 1.1. (cf. [5, Theorem 1], [6, Proposition 1.4.7]) *Let A be a unital C^*-algebra. Then the extreme points of $(A)_1$ are precisely those elements v of A for which*

$$(1 - vv^*)A(1 - v^*v) = 0. \tag{2}$$

Remark 1.2. (1) Let $v \in A$ be such that (2) is satisfied. Due to

$$v^*(1 - vv^*)v(1 - v^*v) = 0,$$

it can be concluded that v^*v is a projection, i.e., v is a partial isometry, whence $\|v\| \le 1$. So, every element v of A satisfying (2) will be contained in $(A)_1$ automatically.

(2) Let $\mathcal{U}(A)$ be the set of all unitary elements in A. Obviously, (2) is satisfied for every $v \in \mathcal{U}(A)$. Thus, every element of $\mathcal{U}(A)$ is an extreme point of $(A)_1$. Furthermore, if x is an extreme point of $(A)_1$ and $u \in \mathcal{U}(A)$, then both ux and xu are extreme points of $(A)_1$. It follows that x is an extreme point of $(A)_1$ if and only if uxu^* is an extreme point of $(A)_1$.

(3) It is well-known that every square matrix has its Schur triangular form [2, P. 5]. That is, for every $x \in \mathbb{C}^{n\times n}$, there exists a unitary $u \in \mathbb{C}^{n\times n}$ such that uxu^* is an upper triangular matrix. So, to deal with the extreme points of $(\mathbb{C}^{n\times n})_1$, it needs only to consider the upper triangular matrices. Likewise, it seems interesting to deal with the extreme points given by upper triangular matrices over a unital C^*-algebra. However, as far as we know, little has been done in the literature on the extreme points of $\left(M_{m+n}(A)\right)_1$ and $X_{m,n}(A)$ by choosing elements in $X_{m,n}(A)$.

Based on Lemma 1.1 and the motivation described in Remark 1.2, we provide some direct characterizations of the extreme points in Section 2. Specifically, we will show that there exist a unital C^*-algebra A and non-zero elements $a, b, c \in A$ such that x given by (1) is an extreme point of $\left(M_2(A)\right)_1$ (see Theorem 2.6). In Section 3, we study the extreme points of $\left(M_{m+n}(A)\right)_1$ furthermore. In the case that x given by (1) is an extreme point of $\left(M_{m+n}(A)\right)_1$, some necessary and sufficient conditions are investigated under which $b = 0, c = 0$ and $a = 0$, respectively; see Theorems 3.2 and 3.6, and Proposition 3.8. Some specific unital C^*-algebras are considered; see Corollary 3.3, Proposition 3.4, Theorem 3.5 and Corollary 3.7. In Section 4, we focus on the study of the extreme points of $X_{m,n}(A)$. Some new results in this direction are obtained; see Proposition 4.1, Theorems 4.5 and 4.6, and Corollary 4.7.

2. Some direct characterizations of the extreme points

In this section, we provide some direct characterizations of the extreme points. We first give two definitions as follows.

Let A be a unital C^*-algebra. Recall that A is said to be finite [4, 7] if for every element $v \in A$, $vv^* = 1$ implies $v^*v = 1$, and A is said to be stably finite if $M_n(A)$ is finite for all $n \ge 1$. It is notable that there exists a finite but not stably finite C^*-algebra [4]. However, if A is a finite von Neumann algebra, then A is stably finite [7, Proposition 2.6.1] (see also [7, Theorem 2.5.4]).

Proposition 2.1. *Suppose that A is a finite von Neumann algebra and $v \in (A)_1$. Then v is an extreme point of $(A)_1$ if and only if v is a unitary element of A.*

Proof. $\Longleftarrow$. See Remark 1.2 (2).

$\Longrightarrow$. Suppose $v \in (A)_1$ is given such that (2) is satisfied. According to Remark 1.2 (1), v is a partial isometry, hence $vv^* \sim v^*v$. Since A is a finite von Neumann algebra, by [7, Proposition 2.4.2] we have $1 - vv^* \sim 1 - v^*v$. Thus, from [8, Lemma 5.2.5] we can conclude that there exists a unitary element $u \in A$ such that $u(1 - v^*v)u^* = 1 - vv^*$. This together with (2) gives

$$1 - vv^* = (1 - vv^*)^2 = (1 - vv^*)u(1 - v^*v)u^* = 0.$$

Therefore, v is a unitary element by the finiteness of A. □

A similar result can be obtained for commutative C^*-algebras.

Proposition 2.2. *Suppose that A is a unital commutative C^*-algebra. Then for every $v \in A$, v is an extreme point of $(A)_1$ if and only if $v \in \mathcal{U}(A)$.*

Proof. The proof of the sufficiency is the same as that of Proposition 2.1. Assume that (2) is satisfied. Since A is commutative and v is a partial isometry, we have

$$1 - vv^* = (1 - vv^*)(1 - v^*v) \cdot 1 = (1 - vv^*) \cdot 1 \cdot (1 - v^*v) = 0.$$

This shows that $v \in \mathcal{U}(A)$. □

Remark 2.3. For a characterization of an extreme point to be a unitary in a general C^*-algebra, the reader is referred to [1].

Our next result concerns the extreme points of the form (1).

Proposition 2.4. *Let A be a unital C^*-algebra, and let $x \in X_{m,n}(A)$ be given by* (1) *such that $a \in M_m(A)$, $b \in M_{m\times n}(A)$ and $c \in M_n(A)$. Then x is an extreme point of $\big(M_{m+n}(A)\big)_1$ if and only if the following conditions are all satisfied:*

$$\big[1 - (aa^* + bb^*)\big] M_m(A)(1 - a^*a) = 0, \tag{3}$$

$$\big[1 - (aa^* + bb^*)\big] M_m(A)a^*b = 0, \tag{4}$$

$$\big[1 - (aa^* + bb^*)\big] M_{m\times n}(A)b^*a = 0, \tag{5}$$

$$\big[1 - (aa^* + bb^*)\big] M_{m\times n}(A)\big[1 - (b^*b + c^*c)\big] = 0, \tag{6}$$

$$(1 - cc^*)M_{n\times m}(A)(1 - a^*a) = 0, \quad (1 - cc^*)M_{n\times m}(A)a^*b = 0, \tag{7}$$

$$(1 - cc^*)M_n(A)b^*a = 0, \quad (1 - cc^*)M_n(A)\big[1 - (b^*b + c^*c)\big] = 0, \tag{8}$$

$$bc^*M_{n\times m}(A)(1 - a^*a) = 0, \quad bc^*M_{n\times m}(A)a^*b = 0, \tag{9}$$

$$bc^*M_n(A)b^*a = 0, \quad bc^*M_n(A)\big[1 - (b^*b + c^*c)\big] = 0, \tag{10}$$

$$cb^*M_m(A)(1 - a^*a) = 0, \quad cb^*M_m(A)a^*b = 0, \tag{11}$$

$$cb^*M_{m\times n}(A)b^*a = 0, \quad cb^*M_{m\times n}(A)\big[1 - (b^*b + c^*c)\big] = 0. \tag{12}$$

Proof. Direct computation yields

$$1 - xx^* = \begin{pmatrix} 1 - (aa^* + bb^*) & -bc^* \\ -cb^* & 1 - cc^* \end{pmatrix}, \tag{13}$$

$$1 - x^*x = \begin{pmatrix} 1 - a^*a & -a^*b \\ -b^*a & 1 - (b^*b + c^*c) \end{pmatrix}. \tag{14}$$

Utilizing Lemma 1.1 we see that x is an extreme point of $\big(M_{m+n}(A)\big)_1$ if and only if

$$(1 - xx^*)M_{m+n}(A)(1 - x^*x) = 0. \tag{15}$$

Substituting (13) and (14) into (15) yields the equivalence of (15) with (3)–(12). □

An application of the preceding proposition is as follows.

Proposition 2.5. *Let A be a unital C^*-algebra, and let x be given by* (1). *Then x is an extreme point of $\big(M_{m+n}(A)\big)_1$ if and only if for every $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = 1$ $(i = 1, 2, 3)$, the element*

$$\begin{pmatrix} \lambda_1 a & \lambda_2 b \\ 0 & \lambda_3 c \end{pmatrix}$$

is an extreme point of $\big(M_{m+n}(A)\big)_1$.

Proof. Let $\lambda_i \in \mathbb{C}$ be given such that $|\lambda_i| = 1$ ($i = 1, 2, 3$). Clearly, equations (3)–(12) are satisfied for a, b and c if and only if these equations are satisfied for $\lambda_1 a$, $\lambda_2 b$ and $\lambda_3 c$. □

We end this section by another application of Proposition 2.4.

Theorem 2.6. *There exist a unital C^*-algebra A and non-zero elements $a, b, c \in A$ such that x given by* (1) *is an extreme point of $\big(M_2(A)\big)_1$.*

Proof. Let H be the separable Hilbert space $\ell^2(\mathbb{N})$ and $\{e_n : n \in \mathbb{N}\}$ be its usual orthonormal basis. Put $A = \mathbb{B}(H)$, the set of all bounded linear operator on H. Let a be the unilateral shift characterized by $ae_n = e_{n+1}$ for all $n \in \mathbb{N}$. Choose $b = 1 - aa^*$ and $c = a^*$. Then b is a projection whose range is spanned by e_1. Consequently, a, b and c are all nonzero, and

$$aa^* + bb^* = cc^* = 1, \quad bc^* = cb^* = 0,$$

which lead clearly to (3)–(12). □

3. Some special cases of the extreme points of $(M_{m+n}(A))_1$

Suppose that A is a unital C^*-algebra, and $x \in X_{m,n}(A)$ is given by (1) such that $a \in M_m(A), b \in M_{m\times n}(A)$ and $c \in M_n(A)$. When x is an extreme point of $\big(M_{m+n}(A)\big)_1$, by Remark 1.2 (1), x is a partial isometry, so according to (13) and (14), we have

$$aa^* + bb^* \leq 1, \quad b^*b + c^*c \leq 1. \tag{16}$$

In what follows, we investigate conditions under which $b = 0, c = 0$ and $a = 0$, respectively. To this end, we need a lemma as follows.

Lemma 3.1. [6, Proposition 1.4.5] *Let x and a be elements of a C^*-algebra A satisfying $x^*x \leq a$. Then for every $\alpha \in (0, \frac{1}{2})$, there exists $u \in A$ with $\|u\| \leq \big\|a^{\frac{1}{2}-\alpha}\big\|$ such that $x = ua^\alpha$.*

3.1. Characterizations of $b = 0$

Theorem 3.2. *Suppose that A is a unital C^*-algebra, and x given by* (1) *is an extreme point of $\big(M_{m+n}(A)\big)_1$. Then the following statements are all equivalent:*

(i) $b = 0$;

(ii) $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ *is an extreme point of $\big(M_{m+n}(A)\big)_1$;*

(iii) *a and c are extreme points of $\big(M_m(A)\big)_1$ and $\big(M_n(A)\big)_1$ respectively such that*

$$(1 - aa^*)M_{m\times n}(A)(1 - c^*c) = 0, \ (1 - cc^*)M_{n\times m}(A)(1 - a^*a) = 0; \tag{17}$$

(iv) $a^*a + bb^* \leq 1$ *and* $b^*b + cc^* \leq 1$.

Proof. The implication of (i)⟹(ii) is clear.

(ii)⟹(i). Let

$$y = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \quad z = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}. \tag{18}$$

By assumption z is an extreme point of $\big(M_{m+n}(A)\big)_1$, so we have $z \in \big(M_{m+n}(A)\big)_1$. It follows from Proposition 2.5 that y is also an extreme point of $\big(M_{m+n}(A)\big)_1$. Therefore, $x, y, z \in \big(M_{m+n}(A)\big)_1$ and $z = \frac{1}{2}(x + y)$, which gives $x = y = z$, hence $b = 0$.

(ii)$\Longleftrightarrow$(iii). Let z be defined by (18). By (3)–(12) with $b = 0$ therein, we know that z is an extreme point of $\left(M_{m+n}(A)\right)_1$ if and only if (17) as well as

$$(1 - aa^*)M_m(A)(1 - a^*a) = 0 \quad \text{and} \quad (1 - cc^*)M_n(A)(1 - c^*c) = 0$$

are satisfied. The latter two equations are exactly the characterizations of a and c to be the extreme points of $\left(M_m(A)\right)_1$ and $\left(M_n(A)\right)_1$, respectively (see Lemma 1.1).

(i)$\Longrightarrow$(iv). Since $b = 0$, we have $\max\{\|a\|, \|c\|\} = \|x\| \leq 1$, hence

$$a^*a + bb^* = a^*a \leq 1, \quad b^*b + cc^* = cc^* \leq 1.$$

(iv)$\Longrightarrow$(i). By assumption we have $bb^* \leq 1 - a^*a$, so according to Lemma 3.1 there exists $u \in M_m(A)$ such that

$$(bb^*)^{\frac{1}{2}} = u(1 - a^*a)^{\frac{1}{3}}.$$

Taking $*$-operation, we arrive at

$$(bb^*)^{\frac{1}{2}} = (1 - a^*a)^{\frac{1}{3}} u^*. \tag{19}$$

Note that the first equation in (7) implies that

$$(1 - cc^*)M_{n\times m}(A)(1 - a^*a)^{\frac{1}{3}} = 0,$$

so we may use (19) to obtain

$$(1 - cc^*)M_{n\times m}(A)(bb^*)^{\frac{1}{2}} = 0,$$

which clearly gives

$$(1 - cc^*)M_{n\times m}(A)bb^* = 0.$$

It follows that

$$(1 - cc^*)b^*b\,[(1 - cc^*)b^*b]^* = 0,$$

and thus

$$(1 - cc^*)b^*b = 0,$$

which leads furthermore to

$$(1 - cc^*)b^*\,[(1 - cc^*)b^*]^* = 0.$$

Consequently, $(1 - cc^*)b^* = 0$ and thus

$$b(1 - cc^*) = 0. \tag{20}$$

Similarly, there exists $v \in M_n(A)$ such that

$$(b^*b)^{\frac{1}{2}} = (1 - cc^*)^{\frac{1}{3}} v^*.$$

The equation above together with (20) yields $b(b^*b)^{\frac{1}{2}} = 0$, which leads furthermore to $(b^*b)^{\frac{3}{2}} = 0$, therefore $b = 0$. $\square$

The equivalence of items (i) and (iv) in the preceding theorem together with (16) gives a corollary immediately as follows.

Corollary 3.3. *Let a, b and c be elements of a unital commutative C^*-algebra A. Suppose that x given by* (1) *is an extreme point of $\left(M_2(A)\right)_1$, then $b = 0$.*

Inspired by (16) and Corollary 3.3, we give a new characterization of the commutativity as follows.

Proposition 3.4. *Let A be a unital C^*-algebra such that for all $a, b \in A$, the inequality $aa^* + bb^* \leq 1$ implies $a^*a + bb^* \leq 1$, then A is commutative.*

Proof. Given every $b \in (A)_1$ and $u \in \mathcal{U}(A)$, put $a = (1 - bb^*)^{\frac{1}{2}} u^*$. Then clearly $aa^* + bb^* = 1$, whence

$$u(1 - bb^*)u^* + bb^* = a^*a + bb^* \leq 1,$$

which gives $bb^* \leq u(bb^*)u^*$. Replacing u with u^*, we arrive at $bb^* \leq u^*(bb^*)u$, hence $u(bb^*)u^* \leq bb^*$. Consequently, $u(bb^*)u^* = bb^*$ and thus $u(bb^*) = (bb^*)u$.

Now, let A_+ denote the set of all positive elements in A. For every $x \in A_+ \setminus \{0\}$, put $b = \left(\frac{x}{\|x\|}\right)^{\frac{1}{2}}$. Then for every $u \in \mathcal{U}(A)$, from the equation $u(bb^*) = (bb^*)u$ we obtain $ux = xu$, which ensures the commutativity of A, since every unital C^*-algebra A is spanned by A_+ and $\mathcal{U}(A)$, respectively. □

Theorem 3.5. *Suppose that A is a finite von Neumann algebra and x given by* (1) *is an extreme point of $\left(M_{m+n}(A)\right)_1$, then $b = 0$.*

Proof. By assumption $M_k(A)$ is a finite von Neumann algebra for all $k \geq 1$, so Proposition 2.1 indicates that x is a unitary element of $\left(M_{m+n}(A)\right)_1$. Thus, $1 = xx^*$. Combining this equation with (13) yields $bc^* = 0$ and $1 = cc^*$, which yield $b = 0$, since by the finiteness of $M_n(A)$ we have $c^*c = 1$. □

3.2. *Characterizations of $c = 0$*

Theorem 3.6. *Let A be a unital C^*-algebra, and let x given by* (1) *be an extreme point of $\left(M_{m+n}(A)\right)_1$. Then $c = 0$ if and only if*

$$a^*a = 1, \quad b^*b = 1, \quad b^*a = 0. \tag{21}$$

Proof. Assume that $c = 0$. Then it follows from (7) and (8) that

$$M_{n\times m}(A)(1 - a^*a) = 0, \quad b^*a = 0, \quad b^*b = 1.$$

Let $z_1, z_2, \ldots, z_m \in M_{n\times m}(A)$ be defined by

$$z_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; z_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots,$$

$$z_{m-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; z_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In virtue of $z_i(1 - a^*a) = 0$ for $i = 1, 2, \cdots, m$, we arrive at $a^*a = 1$. This shows the validity of (21).

Conversely, assume that (21) is satisfied. Combining (21) with the second equation in (8) and the second equation in (10), we obtain

$$(1 - cc^*)c^*c = 0, \quad bc^*c^*c = 0.$$

Thus, $c^*c^*c = b^*bc^*c^*c = 0$, hence $c^*c = (1 - cc^*)c^*c = 0$. This shows that $c = 0$. □

Corollary 3.7. *There exist a unital C^*-algebra A and an element x given by* (1) *with $c = 0$ such that x is an extreme point of $\big(M_2(A)\big)_1$.*

Proof. For $n \in \mathbb{N}$ with $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ ([3, Section 1], [9, Section 5]) is the universal C^*-algebra generated by isometries $s_i (1 \leq i \leq n)$ such that $\sum\limits_{i=1}^{n} s_i s_i^* = 1$. Let $A = \mathcal{O}_n$. Choose any $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$, put $a = s_i$ and $b = s_j$. Then (21) is satisfied for such a pair of a and b, hence the element x given by (1) with $c = 0$ is an extreme point of $\big(M_2(A)\big)_1$. □

3.3. Characterizations of $a = 0$

The following propositions can be obtained by using the same method employed in the proof of Theorem 3.6 and Corollary 3.7.

Proposition 3.8. *Let x given by* (1) *be an extreme point of $\big(M_{m+n}(A)\big)_1$. Then $a = 0$ if and only if $bb^* = 1$, $cc^* = 1$ and $cb^* = 0$.*

Proposition 3.9. *There exist a unital C^*-algebra A and an element x given by* (1) *with $a = 0$ such that x is an extreme point of $\big(M_2(A)\big)_1$.*

4. Characterizations of the extreme points of $X_{m,n}(A)$

Given $m, n \in \mathbb{N}$, $X_{m,n}(A)$ is obviously a convex subset of $\big(M_{m+n}(A)\big)_1$, which however is not invariant under the $*$-operation. Due to the latter property of $X_{m,n}(A)$, some new phenomena may happen in dealing with the extreme points of $X_{m,n}(A)$. Our first result in this direction is as follows.

Proposition 4.1. *For every unital C^*-algebra A and $n \in \mathbb{N}$, there exists x given by* (1) *with $m = n$ such that x is an extreme point of $X_{n,n}(A)$, whereas neither a nor c is an extreme point of $\big(M_n(A)\big)_1$.*

Proof. Given $y, z \in M_n(A)$, let $s = \begin{pmatrix} y & 1 \\ 0 & z \end{pmatrix}$ and $t = \begin{pmatrix} y & 1 \\ 0 & 0 \end{pmatrix}$. Since $t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s$, we have

$$1 + \|y\|^2 = 1 + \|yy^*\| = \|1 + yy^*\| = \|tt^*\| = \|t\|^2 \leq \|s\|^2.$$

Similarly, $1 + \|z\|^2 \leq \|s\|^2$. It follows that $\|s\| \leq 1$ if and only if $y = z = 0$. Due to this observation and the fact that 1 is an extreme point of $\big(M_n(A)\big)_1$, we see that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an extreme point of $X_{n,n}(A)$. □

In the rest of this section, we study the extreme points of $X_{m,n}(A)$ under the restriction that a and c are extreme points of $\big(M_m(A)\big)_1$ and $\big(M_n(A)\big)_1$, respectively. For this, we provide a useful lemma as follows.

Lemma 4.2. *Suppose that A is a unital C^*-algebra. Let $a \in M_m(A), b \in M_{m\times n}(A)$ and $c \in M_n(A)$ be such that both a and c are nonzero partial isometries. Suppose that x given by* (1) *satisfies $\|x\| \leq 1$, then $b = (1 - aa^*)b(1 - c^*c)$.*

Proof. A simple computation shows that

$$xx^* = \begin{pmatrix} aa^* + bb^* & bc^* \\ cb^* & cc^* \end{pmatrix}. \tag{22}$$

So, if we put $s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} xx^*$, then we have

$$1 \geq \|ss^*\| = \|cb^*bc^* + cc^*\|.$$

Since cc^* is a nonzero projection and $e \triangleq cb^*bc^*$ is a positive element satisfying $cc^* \cdot e \cdot cc^* = e$, the inequality above implies that $e = 0$, or equivalently, $bc^* = 0$. Furthermore, from (22) we can obtain

$$\begin{aligned}1 &\geq \|xx^*\| \geq \|aa^* + bb^*\| \geq \|aa^*(aa^* + bb^*)aa^*\| \\ &= \|aa^* + (aa^*b)(aa^*b)^*\|,\end{aligned}$$

hence $aa^*b = 0$. It follows that

$$b = (1 - aa^*)b = (1 - aa^*)b(1 - c^*c).$$

Hence, b has the form as desired. □

A direct application of the preceding lemma is as follows.

Corollary 4.3. *Let A be a unital C^*-algebra. Suppose that $a \in M_m(A)$ and $c \in M_n(A)$ are nonzero partial isometries such that $(1 - aa^*)M_{m\times n}(A)(1 - c^*c) = 0$. Then for every $b \in M_{m\times n}(A) \setminus \{0\}$, the element x given by* (1) *satisfies $\|x\| > 1$.*

We provide an additional lemma for the sake of completeness.

Lemma 4.4. *Suppose that A is a unital C^*-algebra. Let x be given by* (1) *such that both $a \in M_m(A)$ and $c \in M_n(A)$ are partial isometries, and at least one of a and c is nonzero. Then $\|x\| = \max\{1, \|b\|\}$ for every $b \in (1-aa^*)M_{m\times n}(A)(1-c^*c)$.*

Proof. By the assumptions on a, b and c, we have

$$\begin{aligned}\|x\|^2 &= \|xx^*\| = \left\|\mathrm{diag}(aa^* + bb^*, cc^*)\right\| \\ &= \max\left\{\max\{\|aa^*\|, \|bb^*\|\}, \|cc^*\|\right\} = \max\{1, \|b\|\}^2.\end{aligned}$$

Thereby showing that $\|x\| = \max\{1, \|b\|\}$, as desired. □

Our first main result of this section reads as follows.

Theorem 4.5. *Suppose that A is a unital C^*-algebra. Let x be given by* (1) *such that a and c are extreme points of $\left(M_m(A)\right)_1$ and $\left(M_n(A)\right)_1$ respectively, and $(1 - aa^*)M_{m\times n}(A)(1 - c^*c) \neq \{0\}$. Then x is an extreme point of $X_{m,n}(A)$ if and only if b is an extreme point of the closed unit ball of $(1 - aa^*)M_{m\times n}(A)(1 - c^*c)$.*

Proof. For simplicity, we denote $(1 - aa^*)M_{m\times n}(A)(1 - c^*c)$ and its closed unit ball by B and $(B)_1$, respectively. Suppose that

$$y = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \quad z = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$$

are given in $X_{m,n}(A)$ such that $x = \frac{1}{2}(y + z)$. Then $a = \frac{1}{2}(a_1 + a_2)$, $c = \frac{1}{2}(c_1 + c_2)$, and

$$\max\left\{\|a_i\|, \|b_i\|, \|c_i\|\right\} \leq \max\{\|y\|, \|z\|\} \leq 1$$

for $i = 1, 2$, which lead to $a_1 = a_2 = a$, $c_1 = c_2 = c$, and $b_1, b_2 \in (B)_1$ by Lemma 4.2. Then the desired conclusion is immediate from Lemma 4.4. □

Our second main result of this section reads as follows.

Theorem 4.6. *Suppose that A is a unital C^*-algebra. Let x be given by* (1) *with $b = 0$. Then x is an extreme point of $X_{m,n}(A)$ if and only if the following conditions are both satisfied:*

(i) *a and c are extreme points of $\left(M_m(A)\right)_1$ and $\left(M_n(A)\right)_1$, respectively;*

(ii) $(1 - aa^*)M_{m\times n}(A)(1 - c^*c) = 0$.

Proof. Let Y be the C^*-subalgebra of $M_{m+n}(A)$ defined by $Y = M_m(A) \oplus M_n(A)$, and let $(Y)_1$ be the closed unit ball of Y. Obviously, we have

$$(Y)_1 \subseteq X_{m,n}(A) \subseteq \big(M_{m+n}(A)\big)_1.$$

By Lemma 1.1, it is easy to verify that $x = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ is an extreme point of $(Y)_1$ if and only if

$$(1 - aa^*)M_m(A)(1 - a^*a) = 0, \quad (1 - cc^*)M_n(A)(1 - c^*c) = 0,$$

or equivalently, a and c are extreme points of $\big(M_m(A)\big)_1$ and $\big(M_n(A)\big)_1$, respectively.

Assume that x is an extreme point of $X_{m,n}(A)$, then apparently x is an extreme point of $(Y)_1$, hence a and c are extreme points of $\big(M_m(A)\big)_1$ and $\big(M_n(A)\big)_1$, respectively. In particular, both a and c are nonzero isometries (see Remark 1.2 (1)). Suppose that $(1 - aa^*)M_{m\times n}(A)(1 - c^*c) \neq \{0\}$. Then there exists $w \in M_{m\times n}(A)$ such that

$$0 < \|w\| < 1, \quad (1 - aa^*)w(1 - c^*c) = w. \tag{23}$$

Let

$$y = \begin{pmatrix} a & w \\ 0 & c \end{pmatrix}, \quad z = \begin{pmatrix} a & -w \\ 0 & c \end{pmatrix}.$$

In virtue of (23) and Lemma 4.4, we have

$$\|y\| = \|z\| = \max\{1, \|w\|\} = 1.$$

It follows that $x = \frac{1}{2}(y + z)$ with $y, z \in (Y)_1$ such that $x \neq y$, which contradicts the fact that x is an extreme point of $(Y)_1$. This shows that $(1 - aa^*)M_{m\times n}(A)(1 - c^*c) = 0$.

Conversely, suppose that conditions (i) and (ii) are satisfied. Let

$$y = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \quad z = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$$

be chosen in $(Y)_1$ such that $x = \frac{1}{2}(y + z)$. Then $a = \frac{1}{2}(a_1 + a_2)$, $c = \frac{1}{2}(c_1 + c_2)$, and

$$\max\Big\{\|a_1\|, \|c_1\|, \|a_2\|, \|c_2\|\Big\} \le \max\{\|y\|, \|z\|\} \le 1,$$

which yield $a_1 = a_2 = a$ and $c_1 = c_2 = c$. Hence, we may combine $\|y\| \le 1$ and $\|z\| \le 1$ with Corollary 4.3 to conclude that $b_1 = b_2 = 0$, and thus $y = z = x$. This completes the proof that x is an extreme point of $X_{m,n}(A)$. □

Corollary 4.7. *There exist a unital C^*-algebra A and an element x given by* (1) *with $b = 0$ such that x is an extreme point of $X_{1,1}(A)$, whereas x fails to be an extreme point of $\big(M_2(A)\big)_1$.*

Proof. Let A be the Cuntz algebra O_3 generated by isometries s_1, s_2 and s_3 such that $\sum\limits_{i=1}^{3} s_i s_i^* = 1$. Put $a = s_1^*$ and $c = s_2$. Then $aa^* = c^*c = 1$ and $(1 - cc^*)(1 - a^*a) = s_3 s_3^* \neq 0$. Therefore, conditions (i) and (ii) stated in Theorem 4.6 are satisfied, whereas the second equation in (17) fails to be true for $m = n = 1$. Thus, the element x given by (1) with $b = 0$ and a, c be chosen as above meets the demanding. □

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