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Bivariate Bernstein-Kantorovich Operators with a Summability Method and Related GBS Operators

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Abstract. In this paper, we apply four-dimensional infinite matrices to newly constructed original extension of bivariate Bernstein-Kantorovich type operators based on multiple shape parameters. We also use Bögel continuity to construct the GBS (Generalized Boolean Sum) operators for defined bivariate Kantorovich type. Moreover, we demonstrate certain illustrative graphs to show the applicability and validity of proposed operators.

1. Introduction

A double sequence $\lambda = (\lambda_{c,d})$ is said to be convergent to M in Pringsheim sense (P-convergent), and it is denoted by $P - \lim_{c,d} \lambda_{c,d} = M$ if there exists $T = T(\sigma) \in \mathbb{N}$ for all $\sigma > 0$, such that $\left|\lambda_{c,d} - M\right| < \sigma$ whenever $c, d > T$. The double sequence $\lambda = (\lambda_{c,d})$ is called bounded if there exists a positive number D so that $\left|\lambda_{c,d}\right| \leq D$ for all $(c, d) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.

Let F be a four-dimensional summability method. The F transform of double sequence $\lambda = (\lambda_{a,b})$ (denoted by $F\lambda := ((F\lambda)_{a,b})$) is defined as

$$(F\lambda)_{a,b} = \sum_{c,d=1}^{\infty} f_{a,b,c,d} \lambda_{c,d},$$

and the given double series is P-convergent for any $(a, b) \in \mathbb{N}^2$.

A four-dimensional matrix $F = (f_{a,b,c,d})$ is called $RH-$*regular* (shortly RHR, please see [4]) if it transforms each bounded $P-$convergent sequence into a $P-$convergent sequence preserving the same $P-$limit. Four-dimensional infinite matrices have been used in recent summability papers (please see, [1, 2]).

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Assume that $F = (f_{a,b,c,d})$ is a nonnegative RHR matrix, and $A \subset \mathbb{N}^2$, then $F-$density of A is defined as

$$\delta_F(A) := P - \lim_{a,b} \sum_{(c,d)\in A} f_{a,b,c,d} \quad (P - \text{limit exists}).$$

The notion of statistical convergence for sequences of real and complex numbers was given and discussed in the studies [3, 5–12]. A real double sequence $\lambda = (\lambda_{c,d})$ is called $F-$*statistically convergent* to M and denoted by $st_F - \lim_{c,d} \lambda_{c,d} = M$ if, for every $\sigma > 0$,

$$\delta_F(\{(c,d) \in \mathbb{N}^2 : |\lambda_{c,d} - M| \geq \sigma\}) = 0.$$

Constructing a novel extension of bivariate operators based on certain multiple shape parameters in the next section, we obtain Korovkin type (see, [13]) theorems via the four-dimensional summability method and statistical convergence. Using the notion of four-dimensional summability method, we obtain the rates of convergence in Section 3. Moreover, we construct the GBS version of proposed operators and estimate the rate of convergence for them. In the final section, we demonstrate certain computer graphics to see and understand the convergence of our operators.

2. Construction of operators and Korovkin type theorems

Bernstein polynomials of degree r are defined [14] on $[0,1] = \mathcal{I}$ as follows:

$$B_r(\vartheta; z) = \sum_{s=0}^{r} \vartheta\left(\frac{s}{r}\right) b_{r,s}(z), \tag{2.1}$$

where $\vartheta \in C[0,1] = \mathcal{C}$, $b_{r,s}(z) = \binom{r}{s} z^s (1-z)^{r-s}$, $s = 0, \ldots, r$, and $b_{r,s}(z) = 0$, $r < 0$ or $s > r$.

Kantorovich [15] presented an approximation process for Lebesgue integrable real-dalued functions defined on $\mathcal{I}$ by replacing sample values $\vartheta\left(\frac{s}{r}\right)$ with the mean values of ϑ in the interval $\left[\frac{s}{r}, \frac{s+1}{r}\right]$. It is well known that these operators involving Lebesgue integrable functions on $\mathcal{I}$ can be expressed by means of Bernstein basis functions $b_{r,s}(z)$,

$$K_r(\vartheta; z) = (r+1) \sum_{s=0}^{r} b_{r,s}(z) \int_{\frac{s}{r+1}}^{\frac{s+1}{r+1}} \vartheta(t)\, dt.$$

A new class of Bernstein basis functions including adjustable shape parameters was proposed by Han et al. [16] as follows: For r arbitrarily selected real values of μ_s, where $r \geq 2$, $s = 1, 2, \ldots, r$, the following polynomial functions in $z \in \mathcal{I}$

$$\begin{cases} a_{r,0}(\mu; z) = (1-z)^r (1 - \mu_1 z), \\ a_{r,s}(\mu; z) = \left(\binom{r}{s} + \mu_s - \mu_s z - \mu_{s+1} z\right) z^s (1-z)^{r-s}, & s = 1, 2, \ldots, [\frac{r}{2}] - 1, \\ a_{r,[\frac{r}{2}]}(\mu; z) = \left(\binom{r}{s} + \mu_s - \mu_s z + \mu_{s+1} z\right) z^s (1-z)^{r-s}, & s = [\frac{r}{2}], \\ a_{r,s}(\mu; z) = \left(\binom{r}{s} - \mu_s + \mu_s z + \mu_{s+1} z\right) z^s (1-z)^{r-s}, & s = [\frac{r}{2}] + 1, \ldots, r-1, \\ a_{r,r}(\mu; z) = z^r (1 - \mu_r + \mu_r z) \end{cases} \tag{2.2}$$

are called the generalized Bernstein polynomials of degree r with shape parameters μ_s, $s = 1, 2, \ldots, r$ such that

$$\begin{cases}\mu_s \in [-\binom{r}{s}, \binom{r}{s-1}], & s = 1, 2, \ldots, [\frac{r}{2}],\\ \mu_s \in [-\binom{r}{s-1}, \binom{r}{s}], & s = [\frac{r}{2}] + 1, \ldots, r,\end{cases} \quad \text{where} \quad \begin{cases}[\frac{r}{2}] = \frac{r}{2}, & \text{if } r \text{ is even},\\ [\frac{r}{2}] = \frac{r-1}{2}, & \text{if } r \text{ is odd}.\end{cases} \tag{2.3}$$

In particular, for $\mu_s = 0$ ($s = 1, 2, \ldots, r$), the generalized Bernstein basis functions defined by (2.2) are reduced to the classical Bernstein basis functions in (2.1).

Hu et al. [17] obtained the following equations using the degree elevation formula for Bernstein basis polynomials:

$$a_{r,s}(\mu; z) = \begin{cases} \frac{\binom{r}{s}+\mu_s}{\binom{r+1}{s}} b_{r+1,s}(z) + \frac{\binom{r}{s}-\mu_{s+1}}{\binom{r+1}{s+1}} b_{r+1,s+1}(z), & s = 0, 1, 2\ldots, \left[\frac{r}{2}\right] - 1,\\ \frac{\binom{r}{[\frac{r}{2}]}+\mu_{[\frac{r}{2}]}}{\binom{r+1}{[\frac{r}{2}]}} b_{r+1,[\frac{r}{2}]}(z) + \frac{\binom{r}{[\frac{r}{2}]}+\mu_{[\frac{r}{2}]+1}}{\binom{r+1}{[\frac{r}{2}]+1}} b_{r+1,[\frac{r}{2}]+1}(z), & s = [\frac{r}{2}],\\ \frac{\binom{r}{s}-\mu_s}{\binom{r+1}{s}} b_{r+1,s}(z) + \frac{\binom{r}{s}+\mu_{s+1}}{\binom{r+1}{s+1}} b_{r+1,s+1}(z), & s = \left[\frac{r}{2}\right] + 1, \ldots, r,\end{cases}$$

where $b_{r,s}(z)$ are the classical Bernstein basis functions for $s = 0, \ldots, r$.

In [18], the authors constructed following operators using (2.2):

$$K_r(\vartheta; z; \mu) = (r+1) \sum_{s=0}^{r} a_{r,s}(\mu; z) \int_{\frac{s}{r+1}}^{\frac{s+1}{r+1}} \vartheta(t)\, dt. \tag{2.4}$$

The moments and central moments of operators (2.4) are given below, respectively:

Lemma 1. *[18, Lemma 1] The operators (2.4) satisfy*

$$\begin{aligned}
K_r(1; z; \mu) &= 1;\\
K_r(t; z; \mu) &= \frac{r}{r+1}z + \frac{1}{2(r+1)} + \frac{(1-z)\varphi_0(z)}{r+1};\\
K_r(t^2; z; \mu) &= \frac{r^2 - r}{(r+1)^2}z^2 + \frac{2r}{(r+1)^2}z + \frac{1}{3(r+1)^2} + \frac{(2-2z)\varphi_1(z)}{(r+1)^2};\\
K_r(t^3; z; \mu) &= \frac{r(r-1)(r-2)}{(r+1)^3}z^3 + \frac{9r(r-1)}{2(r+1)^3}z^2 + \frac{7r}{2(r+1)^3}z + \frac{1}{4(r+1)^3} + \frac{(1-z)(6\varphi_2(z) + \varphi_0(z))}{2(r+1)^3};\\
K_r(t^4; z; \mu) &= \frac{(r-1)(r-2)(r-3)r}{(r+1)^4}z^4 + \frac{8r(r-1)(r-2)}{(r+1)^4}z^3 + \frac{15r(r-1)}{(r+1)^4}z^2\\
&\quad + \frac{6r}{(r+1)^4}z + \frac{1}{5(r+1)^4} + \frac{(4\varphi_3(z) + 2\varphi_1(z))(1-z)}{(r+1)^4},
\end{aligned}$$

where

$$\varphi_i(z) = \sum_{m=1}^{[\frac{r}{2}]} m^i z^m (1-z)^{r-m} \mu_m - \sum_{m=[\frac{r}{2}]+1}^{r} m^i z^m (1-z)^{r-m} \mu_m \qquad (i \in \mathbb{N}_0).$$

Lemma 2. *[18, Lemma 2] Following identities hold true for K_r:*

$$
\begin{aligned}
K_r(t-z,z;\mu) &= -\frac{z}{r+1}+\frac{1}{2(r+1)}+\frac{(1-z)\varphi_0(z)}{r+1};\\
K_r((t-z)^2;z;\mu) &= \frac{1-r}{(r+1)^2}z^2+\frac{r-1}{(r+1)^2}z+\frac{1}{3(r+1)^2}+\frac{2(1-z)\varphi_1(z)}{(r+1)^2}-\frac{2z(1-z)\varphi_0(z)}{r+1};\\
K_r((t-z)^3;z;\mu) &= \frac{5r-1}{(r+1)^3}z^3-\frac{15r-3}{2(r+1)^3}z^2+\frac{5r-2}{2(r+1)^3}z+\frac{3(1-z)\varphi_2(z)}{2(r+1)^3}\\
&\quad -\frac{6(z-z^2)\varphi_1(z)}{(r+1)^2}+\frac{(6z^2(r+1)^2+1)(1-z)\varphi_0(z)}{2(r+1)^3};\\
K_r((t-z)^4;z;\mu) &= \frac{3r^2-20r+1}{(r+1)^4}z^4-\frac{6r^2-4r+2}{(r+1)^4}z^3+\frac{3r^2-25r+2}{(r+1)^4}z^2+\frac{5r-1}{(r+1)^4}z\\
&\quad +\frac{1}{5(z+1)^4}+\frac{(1-z)(4\varphi_3(z)+2\varphi_1(z))}{(r+1)^4}-\frac{4z(1-z)(6\varphi_2(z)+\varphi_0(z))}{2(z+1)^3}\\
&\quad +\frac{12z^2(1-z)\varphi_1(z)}{(r+1)^2}-\frac{4z^3(1-z)\varphi_0(z)}{r+1},
\end{aligned}
$$

where $\varphi_i(z)$ is given for $i=0,1,2,3$ in Lemma 1.

Let $z,y\in\mathcal{I}$, we define following operators

$$\mathcal{K}_{c,d}^{p,q}(\vartheta;z,y)=(c+1)(d+1)\sum_{m=0}^{c}\sum_{n=0}^{d}a_{c,m}(p;z)a_{d,n}(q;y)\int_{\frac{m}{c+1}}^{\frac{m+1}{c+1}}\int_{\frac{n}{d+1}}^{\frac{n+1}{d+1}}\vartheta(t,s)\,dtds, \tag{2.5}$$

where shape parameters p_m and q_n satisfy the conditions (2.3), and call them as generalized bivariate Bernstein-Kantorovich operators. We refer to certain recent papers about approximation of functions by positive linear operators [18–38].

Let $\mathcal{A}=[0,1]\times[0,1]$ and $C(\mathcal{A})=\mathbf{C}$ throughout the paper.

We refer to certain recent papers about approximation of functions by positive linear operators [18–38].

Lemma 3. *Let $e_{cd}(z,y)=z^cy^d$, $c,d\in\mathbb{N}$. The operators (2.5) satisfy*

$$
\begin{aligned}
&\mathcal{K}_{c,d}^{p,q}(e_{00};z,y)=1;\\
&\mathcal{K}_{c,d}^{p,q}(e_{10};z,y)=\frac{2cz+1+2(1-z)\varphi_0(z)}{2(c+1)},\quad \mathcal{K}_{c,d}^{p,q}(e_{01};z,y)=\frac{2dy+1+2(1-y)\varphi_0(y)}{2(d+1)};\\
&\mathcal{K}_{c,d}^{p,q}(e_{20};z,y)=\frac{c^2-c}{(c+1)^2}z^2+\frac{2c}{(c+1)^2}z+\frac{1}{3(c+1)^2}+\frac{(2-2z)\varphi_1(z)}{(c+1)^2};\\
&\mathcal{K}_{c,d}^{p,q}(e_{02};z,y)=\frac{d^2-d}{(d+1)^2}y^2+\frac{2d}{(d+1)^2}y+\frac{1}{3(d+1)^2}+\frac{(2-2y)\varphi_1(y)}{(d+1)^2};\\
&\mathcal{K}_{c,d}^{p,q}(e_{30};z,y)=\frac{c(c-1)(c-2)}{(c+1)^3}z^3+\frac{9c(c-1)}{2(c+1)^3}z^2+\frac{7c}{2(c+1)^3}z+\frac{1}{4(c+1)^3}+\frac{(1-z)(6\varphi_2(z)+\varphi_0(z))}{2(c+1)^3};\\
&\mathcal{K}_{c,d}^{p,q}(e_{03};z,y)=\frac{d(d-1)(d-2)}{(d+1)^3}y^3+\frac{9d(d-1)}{2(d+1)^3}y^2+\frac{7d}{2(d+1)^3}y+\frac{1}{4(d+1)^3}\\
&\qquad +\frac{(1-y)(6\varphi_2(y)+\varphi_0(y))}{2(d+1)^3};
\end{aligned}
$$

$$\mathcal{K}^{p,q}_{c,d}(e_{40};z,y) = \frac{(c-1)(c-2)(c-3)c}{(c+1)^4}z^4 + \frac{8c(c-1)(c-2)}{(c+1)^4}z^3 + \frac{15c(c-1)}{(c+1)^4}z^2 + \frac{6c}{(c+1)^4}z + \frac{1}{5(c+1)^4} + \frac{(4\varphi_3(z)+2\varphi_1(z))(1-z)}{(c+1)^4};$$

$$\mathcal{K}^{p,q}_{c,d}(e_{04};z,y) = \frac{(d-1)(d-2)(d-3)d}{(d+1)^4}y^4 + \frac{8d(d-1)(d-2)}{(d+1)^4}y^3 + \frac{15d(d-1)}{(d+1)^4}y^2 + \frac{6v}{(d+1)^4}y + \frac{1}{5(d+1)^4} + \frac{(4\varphi_3(y)+2\varphi_1(y))(1-y)}{(d+1)^4},$$

where

$$\varphi_i(z) = \sum_{m=1}^{[\frac{c}{2}]} m^i z^m (1-z)^{c-m} p_m - \sum_{m=[\frac{c}{2}]+1}^{c} m^i z^m (1-z)^{c-m} p_m \qquad (i \in \mathbb{N}_0),$$

$$\varphi_j(y) = \sum_{n=1}^{[\frac{d}{2}]} n^j y^n (1-y)^{d-n} q_n - \sum_{n=[\frac{d}{2}]+1}^{d} n^j y^n (1-y)^{d-n} q_n \qquad (j \in \mathbb{N}_0).$$

Proof. The proof is based on the linearity of operators $\mathcal{K}^{p,q}_{c,d}$ and Lemma 1. □

Lemma 4. *Following identities hold true for $\mathcal{K}^{p,q}_{c,d}$:*

$$\mathcal{K}^{p,q}_{c,d}(e_{10}-z,z,y) = -\frac{z}{c+1} + \frac{1}{2(c+1)} + \frac{(1-z)\varphi_0(z)}{c+1};$$

$$\mathcal{K}^{p,q}_{c,d}(e_{01}-y,z,y) = -\frac{y}{d+1} + \frac{1}{2(d+1)} + \frac{(1-y)\varphi_0(y)}{d+1};$$

$$\mathcal{K}^{p,q}_{c,d}((e_{10}-z)^2;z,y) = \frac{1-c}{(c+1)^2}z^2 + \frac{c-1}{(c+1)^2}z + \frac{1}{3(c+1)^2} + \frac{2(1-z)\varphi_1(z)}{(c+1)^2} - \frac{2z(1-z)\varphi_0(z)}{c+1};$$

$$\mathcal{K}^{p,q}_{c,d}((e_{01}-y)^2;z,y) = \frac{1-d}{(d+1)^2}y^2 + \frac{d-1}{(d+1)^2}y + \frac{1}{3(d+1)^2} + \frac{2(1-y)\varphi_1(y)}{(d+1)^2} - \frac{2y(1-y)\varphi_0(y)}{d+1};$$

$$\mathcal{K}^{p,q}_{c,d}((e_{30}-z)^3;z,y) = \frac{5c-1}{(c+1)^3}z^3 - \frac{15c-3}{2(c+1)^3}z^2 + \frac{5c-2}{2(c+1)^3}z + \frac{3(1-z)\varphi_2(z)}{2(c+1)^3} - \frac{6(z-z^2)\varphi_1(z)}{(c+1)^2} + \frac{(6z^2(c+1)^2+1)(1-z)\varphi_0(z)}{2(c+1)^3};$$

$$\mathcal{K}^{p,q}_{c,d}((e_{03}-y)^3;z,y) = \frac{5d-1}{(d+1)^3}y^3 - \frac{15d-3}{2(d+1)^3}y^2 + \frac{5d-2}{2(d+1)^3}y + \frac{3(1-y)\varphi_2(y)}{2(d+1)^3} - \frac{6(y-y^2)\varphi_1(y)}{(d+1)^2} + \frac{(6y^2(d+1)^2+1)(1-y)\varphi_0(y)}{2(d+1)^3};$$

$$\mathcal{K}^{p,q}_{c,d}((e_{40}-z)^4;z,y) = \frac{3c^2-20c+1}{(c+1)^4}z^4 - \frac{6c^2-4c+2}{(c+1)^4}z^3 + \frac{3c^2-25c+2}{(c+1)^4}z^2 + \frac{5c-1}{(c+1)^4}z + \frac{1}{5(z+1)^4} + \frac{(1-z)(4\varphi_3(z)+2\varphi_1(z))}{(c+1)^4} - \frac{4z(1-z)(6\varphi_2(z)+\varphi_0(z))}{2(z+1)^3} + \frac{12z^2(1-z)\varphi_1(z)}{(c+1)^2} - \frac{4z^3(1-z)\varphi_0(z)}{c+1};$$

$$\mathcal{K}_{c,d}^{p,q}((e_{04}-y)^4;z,y)=\frac{3d^2-20d+1}{(d+1)^4}y^4-\frac{6d^2-40d+2}{(d+1)^4}y^3+\frac{3d^2-25d+2}{(d+1)^4}y^2+\frac{5d-1}{(d+1)^4}y$$
$$+\frac{1}{5(y+1)^4}+\frac{(1-y)(4\varphi_3(y)+2\varphi_1(y))}{(d+1)^4}-\frac{4y(1-y)(6\varphi_2(y)+\varphi_0(y))}{2(y+1)^3}$$
$$+\frac{12y^2(1-y)\varphi_1(y)}{(d+1)^2}-\frac{4y^3(1-y)\varphi_0(y)}{d+1}.$$

Proof. The proof is based on the linearity of operators $\mathcal{K}_{c,d}^{p,q}$ and Lemma 2. □

Using this lemma we provide following theorem to give Korovkin type approximation for F-statistical convergence.

Theorem 1. *Let $\vartheta \in \boldsymbol{C}$, then*

$$st_F-\lim_{c,d}\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_C=0.$$

Proof. Considering the criteria in [39, Theorem 2.1] we claim that

$$st_F-\lim_{c,d}\left\|\mathcal{K}_{c,d}^{p,q}\left(\vartheta_j\right)-\vartheta_j\right\|_{\mathbf{C}}=0, \tag{2.6}$$

where $\vartheta_0=1$, $\vartheta_1=z$, $\vartheta_2=y$ and $\vartheta_3=z^2+y^2$.

The following equality is satisfied by Lemma 3:

$$st_F-\lim_{c,d}\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_0)-\vartheta_0\right\|_{\mathbf{C}}=0.$$

And this result guarantees that (2.6) holds true for $j=0$. By Lemma 3, we obtain

$$\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_1)-\vartheta_1\right\|_{\mathbf{C}}=\sup_{(z,y)\in[0,1]\times[0,1]}\left|\frac{2cz+1+2(1-z)\varphi_0(z)}{2(c+1)}-z\right|\le\frac{3}{c+1}.$$

For a given $\epsilon'>0$, we choose a number $\epsilon>0$ such that $\epsilon<\epsilon'$. Let us define the following sets:

$$\mathcal{S}\ :\ =\left\{(c,d):\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_1)-\vartheta_1\right\|_{C(\mathcal{A})}\ge\epsilon'\right\},$$
$$\mathcal{S}_1\ :\ =\left\{(c,d):\frac{3}{c+1}\ge\epsilon-\epsilon'\right\}.$$

We see that $\mathcal{S}\subseteq\mathcal{S}_1$. Hence, $\delta_F(\mathcal{S})\le\delta_F(\mathcal{S}_1)$ and one obtains

$$st_F-\lim_{c,d}\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_1)-\vartheta_1\right\|_{\mathbf{C}}=0.$$

Similary we have

$$st_F-\lim_{c,d}\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_2)-\vartheta_2\right\|_{\mathbf{C}}=0,$$

that is (2.6) holds true for $j = 2$. Finally, since

$$\begin{aligned} & \left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_3) - \vartheta_3\right\|_{\mathbf{C}} \\ \leq & \left\|\mathcal{K}_{c,d}^{p,q}(e_{20}) - e_{20}\right\|_{\mathbf{C}} + \left\|\mathcal{K}_{c,d}^{p,q}(e_{02}) - e_{02}\right\|_{\mathbf{C}} \\ \leq & \left|\frac{-3c-1}{(c+1)^2} + \frac{6c+1}{3(c+1)^2} + \frac{4(c-1)}{c(c+1)} + \frac{4}{c(c+1)}\right| \\ + & \left|\frac{-3d-1}{(d+1)^2} + \frac{6d+1}{3(d+1)^2} + \frac{4(d-1)}{d(d+1)} + \frac{4}{d(d+1)}\right| \\ \leq & \frac{3c+2}{3(c+1)^2} + \frac{4}{c+1} + \frac{3d+2}{3(d+1)^2} + \frac{4}{d+1} \end{aligned}$$

and taking F-statistical limit in both-sides of last inequality, we get

$$st_F - \lim_{c,d} \left\|\mathcal{K}_{c,d}^{p,q}(\vartheta_3) - \vartheta_3\right\|_{\mathbf{C}} = 0,$$

that is (2.6) holds true for $j = 3$. Since $\mathcal{K}_{c,d}^{p,q}$ is a sequence of linear positive operators, we obtain the desired result by [39, Theorem 2.1]. □

3. Rates of convergence via a summability method

In this section, we calculate rates of convergence via four-dimensional summability matrix.

We need the notion of modulus of continuity which is defined as

$$\omega(\vartheta,\rho) = \sup_{\sqrt{(s-y)^2+(t-z)^2}\leq\rho} \left|\vartheta(s,t) - \vartheta(z,y)\right| \quad (\rho > 0),\ \vartheta \in \mathbf{C}$$

to obtain rate of convergence results. We know that, for any $\gamma > 0$ and for all $\vartheta \in \mathbf{C}$

$$\omega(\vartheta,\gamma\rho) \leq (1 + [\gamma])\,\omega(\vartheta,\rho),$$

where $[\gamma]$ is greatest integer less than or equal to γ.

Using the concept of four-dimensional summability matrix F, we give a rate of convergence result by following theorem.

Theorem 2. *Let F be a nonnegative RHR matrix and $(\lambda_{c,d})$ be a positive double sequence so that $\omega(\vartheta,\rho_{c,d}) = st_F - o(\lambda_{c,d})$, then*

$$\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta) - \vartheta\right\|_{C} = st_F - o(\lambda_{c,d}),$$

where $\vartheta \in C$ and

$$\rho_{c,d} := \left\{\frac{4(2c+2)}{c(c+1)} + \frac{6(1-c)+1}{3(c+1)^2} + \frac{4(2d+2)}{d(d+1)} + \frac{6(1-d)+1}{3(d+1)^2}\right\}^{\frac{1}{2}}.$$

Proof. Assume that our hypotheses are satisfied, then the following inequalities are obtained because $\mathcal{K}_{c,d}^{p,q}$ is positive

$$\begin{aligned} \left|\mathcal{K}_{c,d}^{p,q}(\vartheta;z,y) - \vartheta(z,y)\right| & \leq \mathcal{K}_{c,d}^{p,q}\left(\left|\vartheta(s,t) - \vartheta(z,y)\right|;z,y\right) \\ & \leq \mathcal{K}_{c,d}^{p,q}\left(\left(1 + \frac{(s-y)^2 + (t-z)^2}{\rho^2}\right)\omega(\vartheta,\rho);z,y\right) \\ & = \omega(\vartheta,\rho) + \frac{\omega(\vartheta,\rho)}{\rho^2}\mathcal{K}_{c,d}^{p,q}\left((s-y)^2 + (t-z)^2;z,y\right). \end{aligned}$$

Then, taking supremum over $(z,y) \in \mathcal{A}$, we have

$$\begin{aligned}&\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{\mathbf{C}}\\&\leq \omega(\vartheta,\rho)+\frac{\omega(\vartheta,\rho)}{\rho^2}\left\{\left\|\mathcal{K}_{c,d}^{p,q}\left((s-.)^2\right)\right\|_{\mathbf{C}}+\left\|\mathcal{K}_{c,d}^{p,q}\left((t-.)^2\right)\right\|_{\mathbf{C}}\right\}\\&\leq \omega(\vartheta,\rho)+\frac{\omega(\vartheta,\rho)}{\rho^2}\left\{\frac{4(2c+2)}{c(c+1)}+\frac{6(1-c)+1}{3(c+1)^2}+\frac{4(2d+2)}{d(d+1)}+\frac{6(1-d)+1}{3(d+1)^2}\right\}.\end{aligned}$$

Choosing ρ as

$$\rho_{c,d} := \left\{\frac{4(2c+2)}{c(c+1)}+\frac{6(1-c)+1}{3(c+1)^2}+\frac{4(2d+2)}{d(d+1)}+\frac{6(1-d)+1}{3(d+1)^2}\right\}^{\frac{1}{2}},$$

we obtain following inequality for any positive integers c, d

$$\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{\mathbf{C}} \leq 2\omega(\vartheta,\rho_{c,d}).$$

Hence, we obtain

$$\frac{1}{\lambda_{c,d}}\sum_{\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{\mathbf{C}}\geq\sigma} f_{a,b,c,d} \leq \frac{1}{\lambda_{c,d}}\sum_{\omega(\vartheta,\rho_{c,d})\geq\frac{\sigma}{2}} f_{a,b,c,d}$$

for any $\sigma > 0$ and from our hypothesis it follows that

$$\left\|\mathcal{K}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{\mathbf{C}} = st_F - o(\lambda_{c,d}).$$

□

4. Approximation degree via GBS operators

Continuous functions have been used in most approximation theorems. However, the considered approximation processes are often meaningful for a bigger class of functions. This is why we consider Bögel continuity (or, simply, B-continuity) in our approximation theorem, and we construct the GBS operators of bivariate Bernstein-Kantorovich type and estimate the rate of convergence for these operators.

The B-continuity was introduced by Bögel (see [40–42]) and given as follows:

Let $\mathcal{X} = I \times J$ and I, J be compact subsets of the real numbers. Then, a function $\vartheta : \mathcal{X} \to \mathbb{R}$ is called B-continuous at a point $(s,t) \in \mathcal{X}$ if, for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$\Delta_{(s,t)}\vartheta(z,y) < \varepsilon,$$

for any $(z,y) \in \mathcal{X}$ with $|z-s| < \delta$, $|y-t| < \delta$, where the the symbol $\Delta_{(s,t)}\vartheta(z,y)$ denotes the mixed difference of ϑ defined by

$$\Delta_{(s,t)}\vartheta(z,y) = \vartheta(z,y) - \vartheta(z,t) - \vartheta(s,y) + \vartheta(s,t).$$

We denote the space of all B-continuous functions on $\mathcal{X}$ by $C_b(\mathcal{X})$. The function $\vartheta : \mathcal{X} \to \mathbb{R}$ is B-bounded on $\mathcal{X}$ if there exists $K > 0$ such that $\Delta_{(s,t)}\vartheta(z,y) \leq K$ for any $(s,t),(z,y) \in \mathcal{X}$. Here since $\mathcal{X}$ is a compact subset, each B-continuous function is B-bounded on $\mathcal{X}$. We denote the set of all B-bounded functions on $\mathcal{X}$ equipped with the norm

$$\|\vartheta\|_{B_b(\mathcal{X})} = \sup_{(s,t),(z,y)\in\mathcal{X}}\left|\Delta_{(s,t)}\vartheta(z,y)\right|$$

by $B_b(\mathcal{X})$. In order to get approximation degree of a B-continuous function, it is important to give the mixed modulus of smoothness. Let $\vartheta \in C_b(\mathcal{X})$, then mixed modulus of smoothness of ϑ, denoted by $\omega_{mixed}(\vartheta;\delta_1,\delta_2)$, is defined to be

$$\omega_{mixed}(\vartheta;\delta_1,\delta_2)=\sup\left\{\left|\Delta_{(s,t)}\vartheta(z,y)\right| : |z-s|\le\delta_1, \left|y-t\right|\le\delta_2\right\}$$

for $\delta_1,\delta_2>0$. (for more information about modulus of smoothness see, [43–45]). To obtain our result, we use of the elementary inequality

$$\omega_{mixed}(\vartheta;\tau_1\delta_1,\tau_2\delta_2)\le(\tau_1+\delta_1)(\tau_2+\delta_2)\,\omega_{mixed}(\vartheta;\delta_1,\delta_2)$$

for $\tau_1,\tau_2>0$.

Let $L : C_b(\mathcal{X}) \to B_b(\mathcal{X})$ be a linear positive operator. The operator $UL : C_b(\mathcal{X}) \to B_b(\mathcal{X})$ defined for any function $\vartheta \in C_b(\mathcal{X})$ and $(z,y)\in\mathcal{X}$ by

$$UL(\vartheta(s,t);z,y)=L(\vartheta(y,t)+\vartheta(s,z)-\vartheta(s,t);z,y)$$

are called GBS operators associated to the operator L.

Now, we define GBS operators of $\mathcal{K}_{c,d}^{p,q}$ for any $\vartheta\in C(\mathcal{A})$ and $c,d\in\mathbb{N}$, by

$$\mathcal{T}_{c,d}^{p,q}(\vartheta(s,t);z,y):=\mathcal{K}_{c,d}^{p,q}(\vartheta(y,t)+\vartheta(s,z)-\vartheta(s,t);z,y),$$

for all $(z,y)\in\mathcal{X}$.

More precisely for any $\vartheta\in C_b(\mathcal{A})$, GBS operators of proposed bivariate type are given by

$$\begin{aligned}\mathcal{T}_{c,d}^{p,q}(\vartheta(s,t);z,y) &= (c+1)(d+1)\sum_{m=0}^{c}\sum_{n=0}^{d}a_{c,m}(p;z)a_{d,n}(q;y)\\ &\quad\times\int_{\frac{m}{c+1}}^{\frac{m+1}{c+1}}\int_{\frac{n}{d+1}}^{\frac{n+1}{d+1}}[\vartheta(z,t)+\vartheta(s,y)-\vartheta(s,t)]\,dsdt.\end{aligned}$$

Here the operators $\mathcal{T}_{c,d}^{p,q}$ are linear and positive.

Theorem 3. *Let $F=(f_{a,b,c,d})$ be a nonnegative RHR summability matrix method. Also, let $\vartheta\in C_b(\mathcal{A})$ and $(\lambda_{c,d})$ be a positive double sequence such that $\omega_{mixed}(\vartheta;\gamma_{c,d},\delta_{c,d})=st_F-o(\lambda_{c,d})$, then*

$$\left\|\mathcal{T}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{C(\mathcal{A})}=st_F-o(\lambda_{c,d}),$$

where

$$\begin{aligned}\gamma_{c,d} &:= \left\{\frac{4(2c+2)}{c(c+1)}+\frac{6(1-c)+1}{3(c+1)^2}\right\}^{\frac{1}{2}},\\ \delta_{c,d} &:= \left\{\frac{4(2d+2)}{d(d+1)}+\frac{6(1-d)+1}{3(d+1)^2}\right\}^{\frac{1}{2}},\end{aligned}$$

for any positive integers $c,d\in\mathbb{N}$.

Proof. Let $\vartheta\in C(\mathcal{A})$ and $(z,y)\in\mathcal{A}$ be fixed. Using the properties of ω_{mixed}, we get

$$\begin{aligned}\left|\Delta_{(s,t)}\vartheta(z,y)\right| &\le \omega_{mixed}(\vartheta;|z-s|,\left|y-t\right|)\\ &\le \left(1+\frac{1}{\delta_1}|z-s|\right)\left(1+\frac{1}{\delta_2}\left|y-t\right|\right)\omega_{mixed}(\vartheta;\delta_1,\delta_2)\end{aligned}\tag{4.1}$$

for any $\delta_1,\delta_2>0$ and from the definition $\Delta_{(s,t)}\vartheta(z,y)$, we can write

$$\vartheta(s,t)-\vartheta(z,t)+\vartheta(z,y)=\vartheta(s,t)-\Delta_{(s,t)}\vartheta(z,y). \tag{4.2}$$

Now, using the positivity and monotonicity of $\mathcal{K}_{c,d}^{p,q}$ and in view of (4.1) and (4.2) we get

$$\begin{aligned}
&\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right|\\
\leq\ & \mathcal{K}_{c,d}^{p,q}\left(\left|\Delta_{(s,t)}\vartheta(z,y)\right|;z,y\right)\\
\leq\ & \mathcal{K}_{c,d}^{p,q}\left(\left(1+\frac{1}{\delta_1}\left|y-t\right|\right)\left(1+\frac{1}{\delta_2}\left|z-s\right|\right)\omega_{mixed}(\vartheta;\delta_1,\delta_2);z,y\right)\\
=\ & \omega_{mixed}(\vartheta;\delta_1,\delta_2)\left\{1+\frac{1}{\delta_1}\mathcal{K}_{c,d}^{p,q}\left(\left|y-t\right|;z,y\right)+\frac{1}{\delta_2}\mathcal{K}_{c,d}^{p,q}\left(\left|z-s\right|;z,y\right)\right.\\
&\left.+\frac{1}{\delta_1\delta_2}\mathcal{K}_{c,d}^{p,q}\left(\left|y-t\right|\left|z-s\right|;z,y\right)\right\}.
\end{aligned}$$

By Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right|\\
\leq\ & \omega_{mixed}(\vartheta;\delta_1,\delta_2)\left\{1+\frac{1}{\delta_1}\sqrt{\mathcal{K}_{c,d}^{p,q}\left((y-t)^2;z,y\right)}+\frac{1}{\delta_2}\sqrt{\mathcal{K}_{c,d}^{p,q}\left((z-s)^2;z,y\right)}\right.\\
&\left.+\frac{1}{\delta_1\delta_2}\sqrt{\mathcal{K}_{c,d}^{p,q}\left((y-t)^2;z,y\right)}\sqrt{\mathcal{K}_{c,d}^{p,q}\left((z-s)^2;z,y\right)}\right\}.
\end{aligned}$$

Then, taking supremum over $(z,y)\in\mathcal{A}$, we get

$$\left\|\mathcal{T}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{C(\mathcal{A})}\leq 4\omega_{mixed}(\vartheta;\gamma_{c,d},\delta_{c,d}),$$

where

$$\begin{aligned}
\delta_1 &= \gamma_{c,d}:=\left\{\frac{4(2c+2)}{c(c+1)}+\frac{6(1-c)+1}{3(c+1)^2}\right\}^{\frac{1}{2}},\\
\delta_2 &= \delta_{c,d}:=\left\{\frac{4(2d+2)}{d(d+1)}+\frac{6(1-d)+1}{3(d+1)^2}\right\}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have for any $\varepsilon>0$ that

$$\frac{1}{\lambda_{c,d}}\sum_{\left\|\mathcal{T}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{C(\mathcal{A})}\geq\varepsilon}f_{j,k,c,d}\leq\frac{1}{\lambda_{c,d}}\sum_{\omega_{mixed}(\vartheta;\gamma_{c,d},\delta_{c,d})\geq\frac{\varepsilon}{4}}f_{j,k,c,d}$$

and from the hypothesis it follows that

$$\left\|\mathcal{T}_{c,d}^{p,q}(\vartheta)-\vartheta\right\|_{C(\mathcal{A})}=st_F-o(\lambda_{c,d}).$$

Hence, we arrive at the desired result. □

Defining the Lipschitz class $Lip_M(\mu,\nu)$ for B-continuous functions as

$$Lip_M(\mu,\nu)=\left\{\vartheta\in C_b(\mathcal{X}):\left|\Delta_{(s,t)}\vartheta(z,y)\right|\leq M\left|y-t\right|^{\mu}\left|z-s\right|^{\nu},\ \text{for } (s,t),(z,y)\in\mathcal{X}\right\},$$

where $\vartheta\in C_b(\mathcal{X})$ and $\mu,\nu\in(0,1]$ we give following theorem to obtain the degree of approximation for operators $\mathcal{T}_{c,d}^{p,q}$ by means of Lipschitz class of Bögel continuous functions.

Theorem 4. *Let $\vartheta \in Lip_M(\mu,\nu)$, then*

$$\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right| \le M\delta_c^{\mu/2}\delta_d^{\nu/2},$$

where $M>0$, $\mu,\nu \in (0,1]$.

Proof. By linearity of the operators $\mathcal{K}_{c,d}^{p,q}$ and definition of operators $\mathcal{T}_{c,d}^{p,q}$, we have

$$\begin{aligned}\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right| &\le \mathcal{K}_{c,d}^{p,q}\left(\left|\Delta_{(s,t)}\vartheta(z,y)\right|;z,y\right)\\ &\le M\mathcal{K}_{c,d}^{p,q}\left(|z-s|^{\mu}\left|y-t\right|^{\nu};z,y\right)\\ &= M\mathcal{K}_{c,d}^{p,q}\left(|z-s|^{\mu};y\right)\mathcal{K}_{c,d}^{p,q}\left(\left|y-t\right|^{\nu};z\right).\end{aligned}$$

Using the Hölders' inequality with $a_1=\frac{2}{\mu}$, $b_1=\frac{2}{2-\mu}$ and $a_2=\frac{2}{\nu}$, $b_2=\frac{2}{2-\nu}$, we have

$$\begin{aligned}\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right| \le\ & M\left(\mathcal{K}_{c,d}^{p,q}\left((z-s)^2;z,y\right)\right)^{\mu/2}\mathcal{K}_{c,d}^{p,q}(1;z,y)^{(2-\mu)/2}\\ &\times\mathcal{K}_{c,d}^{p,q}((y-t)^2;z,y)\mathcal{K}_{c,d}^{p,q}(1;z,y)^{(2-\nu)/2}.\end{aligned}$$

Considering Lemma 4 and choosing $\delta_c(y)=\mathcal{K}_{c,d}^{p,q}\left((z-s)^2;z,y\right)$ and $\delta_d(z)=\mathcal{K}_{c,d}^{p,q}((y-t)^2;z,y)$, we get

$$\left|\mathcal{T}_{c,d}^{p,q}(\vartheta;z,y)-\vartheta(z,y)\right| \le M\delta_c^{\mu/2}\delta_d^{\nu/2}$$

that implies the degree of local approximation for B-continuous functions belonging to $Lip_M(\mu,\nu)$. □

5. Convergence of operators via graphics

In this part, we give some graphics which validate the convergence of the proposed operators to the following functions

$$\vartheta_1(z,y)=\left|3z^3-1\right|\left|3y^3-1\right|e^{\frac{(8-z)(8-y)}{zy}}$$

and

$$\vartheta_2(z,y)=\frac{|z-1|\cos(\pi y)}{y-2}$$

on the interval $(z,y)\in\mathcal{A}$ with following shape parameters:

$$\begin{aligned}&p_i=-\binom{c}{i},\ i=1,2,\ldots,\left[\frac{c}{2}\right],\ q_j=-\binom{d}{j},\ j=1,2,\ldots,\left[\frac{d}{2}\right],\\ &p_i=-\binom{c}{i-1},\ i=\left[\frac{c}{2}\right]+1,\ldots,c,\ q_j=-\binom{d}{j-1},\ j=\left[\frac{d}{2}\right]+1,\ldots,d.\end{aligned}$$

In Figures 1-3, we demonstrate convergence of proposed operators to the functions for $c=d=10$ and $c=d=5$, respectively. In Figures 2-4, we demonstrate corresponding errors of approximations.

These graphics show that proposed bivariate operators are well defined and approximate complicated functions even for small values of c and d.

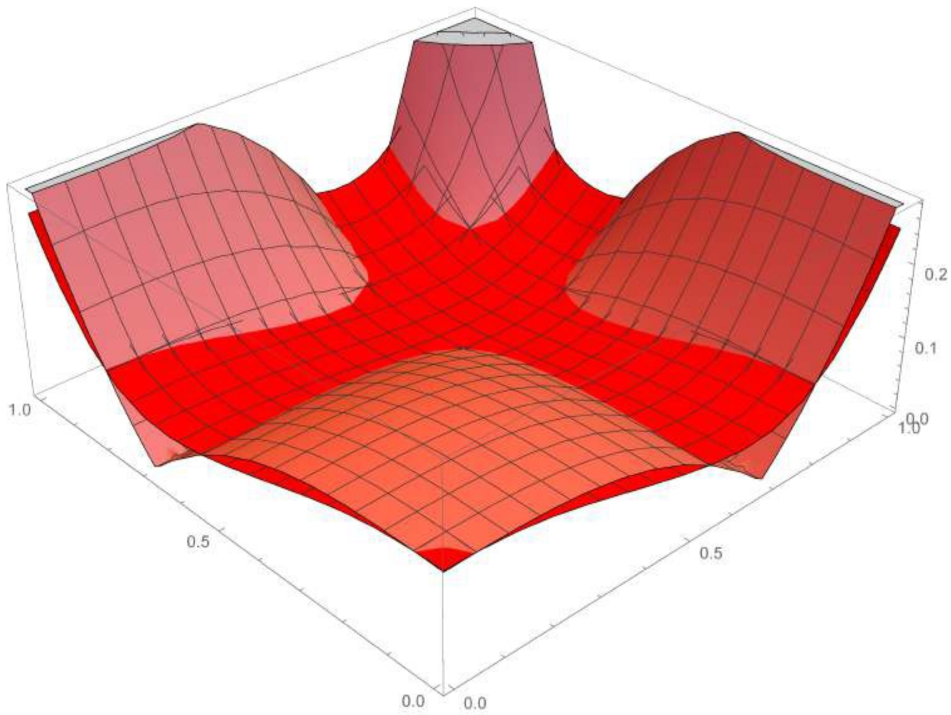

Figure 1: Approximation of $\mathcal{K}_{c,d}^{p,q}$ operators to the function ϑ_1 for $c = d = 10$

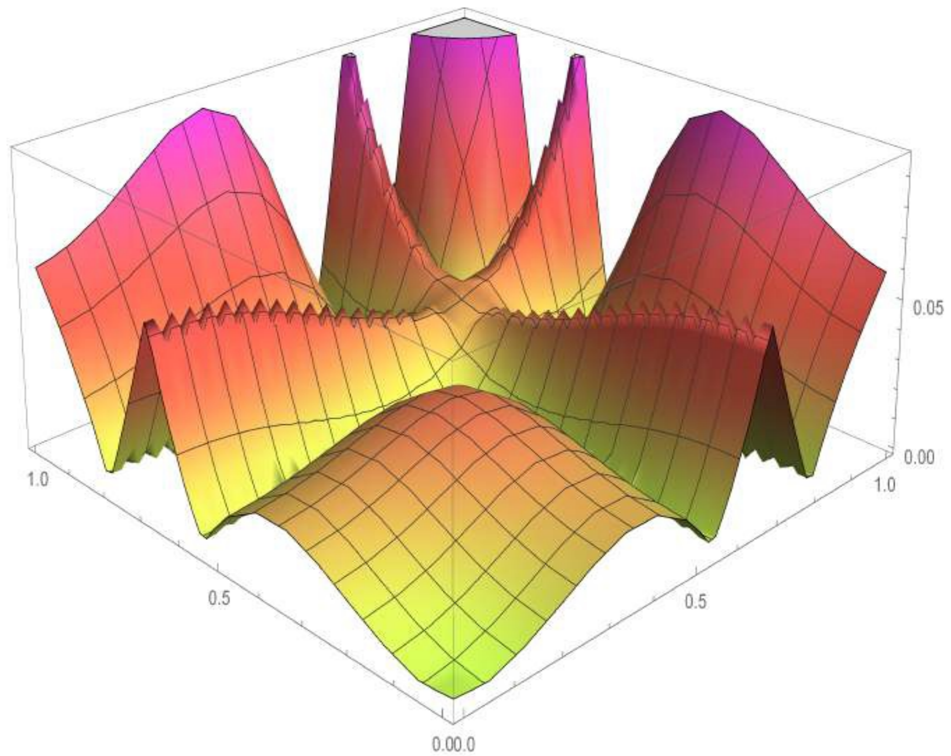

Figure 2: Error of approximation for function ϑ_1

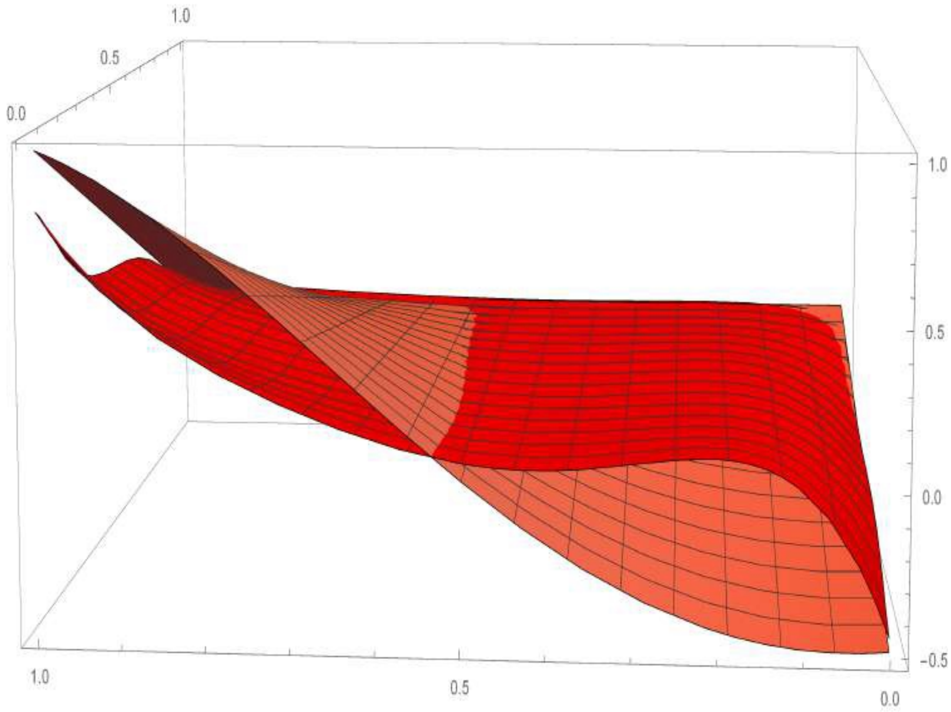

Figure 3: Approximation of $\mathcal{K}_{c,d}^{p,q}$ operators to the function ϑ_2 for $c = d = 5$

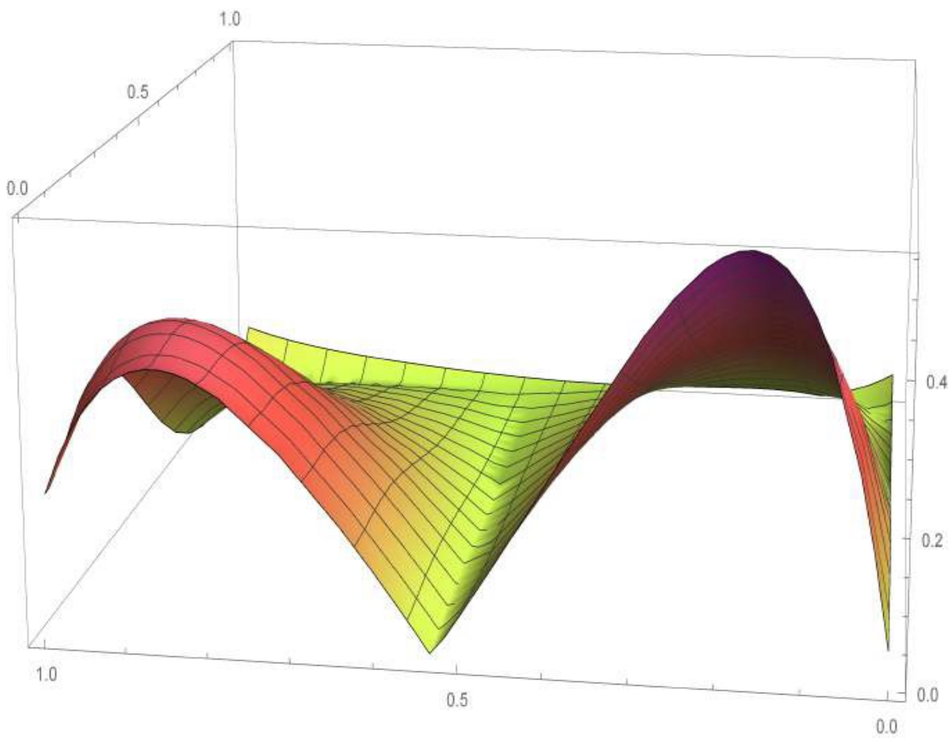

Figure 4: Error of approximation for function ϑ_2

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