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Bézier-Baskakov-Beta Type Operators

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Abstract. In this study, we construct the Bézier-Baskakov-Beta type operators. We provide elements of Lipschitz type space, a direct approximation theorem by means of modulus of continuity $\omega_{\rho^\ell}(\zeta, t)$ $(0 \le \ell \le 1)$ and approximation rate for functions having derivatives of bounded variation. We support the theoretical parts by computer graphics.

1. Introduction

Aral and Erbay [5] introduced Baskakov operators based on $\alpha \in [0,1]$ as follows:

$$\mathcal{B}_m^{(\alpha)}(\zeta; x) = \sum_{j=0}^{\infty} b_{m,j}^{(\alpha)}(x)\zeta\left(\frac{j}{m}\right), \quad x \in [0,\infty). \tag{1}$$

Here

$$b_{m,j}^{(\alpha)}(x) = \frac{x^{j-1}}{(1+x)^{m+j-1}}\left[\frac{\alpha x}{(1+x)}\binom{m+j-1}{j} - (1-\alpha)(1+x)\binom{m+j-3}{j-2} + (1-\alpha)x\binom{m+j-1}{j}\right].$$

The operators $\mathcal{B}_m^{(\alpha)}(\zeta; x)$ reduce to Baskakov operators [6] for $\alpha = 1$.

Kajla et al. [19] considered a Durrmeyer type generalization of the operators (1) and gave the uniform convergence results. Gupta et al. [14] presented a family of approximation operators of exponential type and obtained some approximation theorem (See also [13]). In 2020, Mohiuddine et al. [21] Baskakov-Durrmeyer type operators based on the parameters and studied quantitative approximation results. Very recently, Mohiuddine et al. [20] introduced Stancu-Kantorovich form of the operators (1) and studied the direct results. Md. Nasiruzzaman et al. [22] constructed α-Schurer-Kantorovich operators and studied some direct results. N. L. Braha et al. [7] considered a Baskakov-Schurer-Szász-Stancu operators and established Korovkin type theorem, Grüss-Voronovskaya type theorem and the rate of convergence. H. M. Srivastava et al. in [26] proposed Szász-Mirakjan Beta-type operators and obtained the rate of convergence with the help of the classical and second moduli of continuity. Zeng and Chen [29] defined Bézier form of

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Bernstein-Durrmeyer operators and obtained the direct approximation theorem for functions of bounded variation. Srivastava and Gupta [25] obtained the convergence rate for Bézier form of BBH operators for functions of bounded variation. In 2007, Guo et al. [12] defined Baskakov-Bézier operators and studied the direct, inverse and equivalence approximation theorems with the help of Ditzian-Totik modulus of smoothness. Bézier type operators were studied by several researchers (cf. [1–3, 8, 15, 17, 27, 28, 31, 33]). Also uniform modulus of smoothness of certain operators were examined in [37–46].

Nasiruzzaman et al. [23] considered following Durrmeyer type generalization of the operators (1):

$$\mathcal{B}^*_{m,\alpha}(\zeta;x) = \sum_{j=0}^{\infty} b^{(\alpha)}_{m,j}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}\zeta(t)dt, \tag{2}$$

where $B(j+1,m)$ is the beta function defined as

$$B(r,s) = \int_0^{\infty}\frac{w^{r-1}}{(1+w)^{r+s}}dw = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad r,s>0.$$

Nasiruzzaman et al. [23] studied uniform convergence of these operators in weighted spaces. A Durrmeyer type generalization which includes Beta function can be found in [4], and α type generalization of operators can be found in [35, 36].

Main goal of this article is to define the Bézier variant of the operators (2) and study some approximation properties e. g. the elements of Lipschitz type space, direct approximation theorem and the rate of approximation for functions having derivatives of bounded variation.

Let $\mathfrak{L}$ denote the class of all Lebesgue measurable functions ζ on $[0,\infty)$ such that

$$\mathfrak{L} = \left\{\zeta : \int_0^{\infty}\frac{|\zeta(t)|}{(1+t)^k}dt < \infty \text{ , for some positive integer } k\right\}.$$

For $\theta \geq 1$, we now consider the Bézier variant $\mathcal{B}^*_{m,\alpha,\theta}$ of the operators $\mathcal{B}^*_{m,\alpha}$ as follows:

$$\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x) = \sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}\zeta(t)dt, \tag{3}$$

where $Q^{(\theta)}_{m,j,\alpha}(x) = \left[V_{m,j,\alpha}(x)\right]^{\theta} - \left[V_{m,j+1,\alpha}(x)\right]^{\theta}$ with $V_{m,j,\alpha}(x) = \sum_{v=j}^{\infty} b^{(\alpha)}_{m,v}(x)$.

Alternatively we may rewrite the operators (3) as

$$\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x) = \int_0^{\infty} U_{m,\alpha,\theta}(x,t)\zeta(t)dt, \ \ x\in[0,\infty), \tag{4}$$

where

$$U_{m,\alpha,\theta}(x,t) = \sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\frac{t^j}{(1+t)^{m+j+1}}.$$

For the sake of simplicity let $\bar{\nu} = (t-x)^2$ and $\nu = t-x$.

Lemma 1.1. *[23] Following relations are satisfied for* $\mathcal{B}^*_{m,\alpha}$

(a) $\mathcal{B}^*_{m,\alpha}(1;x) = 1;$

(b) $\mathcal{B}^*_{m,\alpha}(t;x) = \dfrac{x(m+2\alpha-2)}{(m-1)} + \dfrac{1}{(m-1)}, \;\; m > 1;$

(c) $\mathcal{B}^*_{m,\alpha}(t^2;x) = \dfrac{mx^2(m+4\alpha-3)}{(m-2)(m-1)} + \dfrac{x(4m+10(\alpha-1))}{(m-2)(m-1)}\dfrac{2}{(m-2)(m-1)}, \;\; m > 2.$

Lemma 1.2. *[23] Following relations are satisfied for* $\mathcal{B}^*_{m,\alpha}$

(a) $\mathcal{B}^*_{m,\alpha}(\nu;x) = \dfrac{x(2\alpha-1)}{(m-1)} + \dfrac{1}{(m-1)}, \;\; m > 1;$

(b) $\mathcal{B}^*_{m,\alpha}(\bar{\nu};x) = \dfrac{2x^2(m+4\alpha-3)}{(m-2)(m-1)} + \dfrac{2x(m+5\alpha-3)}{(m-2)(m-1)} + \dfrac{2}{(m-2)(m-1)}, \;\; m > 2.$

Corollary 1.3. *For* $\lambda > 2$ *and* m *sufficiently large, we have*

$$\mathcal{B}^*_{m,\alpha}(\bar{\nu};x) \leq \frac{\lambda\rho^2(x)}{m}, \tag{5}$$

where $\rho(x) = \sqrt{x(1+x)}$.

Lemma 1.4. *The inequality* $\| \mathcal{B}^*_{m,\alpha}(\zeta) \| \leq \| \zeta \|$ *is satisfied for* $\zeta \in C_B[0,\infty)$.

Lemma 1.5. *The inequality* $\| \mathcal{B}^*_{m,\alpha,\theta}(\zeta) \| \leq \theta \| \zeta \|$ *is satisfied for* $\zeta \in C_B[0,\infty)$.

Proof. Applying the inequality $|a^\beta - b^\beta| \leq \beta|a-b|$ with $0 \leq a,b \leq 1, \beta \geq 1$ and the definition of $Q^{(\theta)}_{m,j,\alpha}$, we obtain,

$$0 < [V_{m,j,\alpha}(x)]^\theta - [V_{m,j+1,\alpha}(x)]^\theta \leq \theta[V_{m,j,\alpha}(x) - V_{m,j+1,\alpha}(x)] = \theta\, b^{(\alpha)}_{m,j}(x). \tag{6}$$

From the operators $\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)$ and Lemma 1.4, we get

$$\| \mathcal{B}^*_{m,\alpha,\theta}(\zeta) \| \leq \theta \| \mathcal{B}^*_{m,\alpha}(\zeta) \| \leq \theta \| \zeta \| .$$

□

2. Direct Theorem

From [30], we have

$$1 = V_{m,0,\alpha}(x) > V_{m,1,\alpha}(x) > \cdots > V_{m,j,\alpha}(x) > V_{m,j+1,\alpha}(x) > \cdots$$
$$0 < \left[V_{m,j,\alpha}(x)\right]^\theta - \left[V_{m,j+1,\alpha}(x)\right]^\theta \leq \theta b^{(\alpha)}_{m,j}(x), \;\; \theta \geq 1. \tag{7}$$

We need following definitions to prove our results. Let $\rho(x) = \sqrt{x(1+x)}, 0 \leq \ell \leq 1$, Ditizian-Totik modulus $\omega_{\rho^\ell}(\zeta,t)$ [9] is

$$\omega_{\rho^\ell}(\zeta,t) = \sup_{0<h\leq t}\; \sup_{x\pm h\rho^\ell(x)/2\geq 0} \left\{ \left| \zeta\Big(x + \frac{h\rho^\ell(x)}{2}\Big) - \zeta\Big(x - \frac{h\rho^\ell(x)}{2}\Big) \right| \right\},$$

and the appropriate Petree's K-functional is defined by

$$\overline{K}_{\rho^\ell}(\zeta,t) = \inf_{g\in W_\ell}\{\|\zeta - g\| + t\|\rho^\ell g'\|\}, \;\; t > 0,$$

where $W_\ell = \{g : g \in AC_{loc}, \|\rho^\ell g'\| < \infty\}$. The relation $\overline{K}_{\rho^\ell}(\zeta,t) \sim \omega_{\rho^\ell}(\zeta,t)$ is well known [9, Theorem 3.1.2]. This implies there is a constant $M > 0$ such that

$$M^{-1}\omega_{\rho^\ell}(\zeta,t) \leq \overline{K}_{\rho^\ell}(\zeta,t) \leq M\omega_{\rho^\ell}(\zeta,t). \tag{8}$$

Lemma 2.1. *For $\zeta \in W_\ell$, $\rho(x) = \sqrt{x(1+x)}$, $0 \leq \ell \leq 1$ and $t, x > 0$, one has*

$$\left|\int_x^t \zeta'(u)du\right| \leq 2^\ell \left(x^{-\ell/2}(1+t)^{-\ell/2} + \rho^{-\ell}(x)\right)|\nu| \left\|\rho^\ell \zeta'\right\|.$$

Proof. On an application of Hölder's inequality, one has

$$\begin{aligned}\left|\int_x^t \zeta'(u)du\right| &\leq \|\rho^\ell \zeta'\| \left|\int_x^t \frac{du}{\rho^\ell(u)}\right| \\ &\leq \left\|\rho^\ell \zeta'\right\| |\nu|^{1-\ell} \left|\int_x^t \frac{du}{\rho(u)}\right|^\ell.\end{aligned}$$

Taking into account following two inequalities

$$\left|\int_x^t \frac{du}{\rho(u)}\right| \leq \left|\int_x^t \frac{du}{\sqrt{u}}\right| \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+t}}\right)$$

and

$$\left|\int_x^t \frac{du}{\sqrt{u}}\right| \leq \frac{2|\nu|}{\sqrt{x}}$$

one achieve following relations

$$\begin{aligned}\left|\int_x^t \zeta'(u)du\right| &\leq \left\|\rho^\ell \zeta'\right\| |\nu| \frac{2^\ell}{x^{\ell/2}} \left|\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+t}}\right|^\ell \\ &\leq \left\|\rho^\ell \zeta'\right\| |\nu| \frac{2^\ell}{x^{\ell/2}} \left((1+t)^{-\ell/2} + (1+x)^{-\ell/2}\right)\end{aligned}$$

via inequality $|a+b|^p \leq |a|^p + |b|^p$ for $0 \leq p \leq 1$. □

Lemma 2.2. *Following inequality is satisfied*

$$\mathcal{B}^*_{m,\alpha,\theta}((1+t)^{-k}; x) \leq C_k (1+x)^{-k} \tag{9}$$

for any non negative real number k, where C_k is a constant dependent on k.

Proof. For each $x \in [0, \infty)$, the result holds from (2). From (7), we have

$$\begin{aligned}\mathcal{B}^*_{m,\alpha,\theta}((1+t)^{-k}; x) &= \sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x) B(j+1, m) \int_0^\infty \frac{t^j}{(1+t)^{m+j+k+1}} dt \\ &= \sum_{j=0}^{\infty} \frac{Q^{(\theta)}_{m,j,\alpha}(x)\Gamma(m+j+1)\Gamma(m+k)}{\Gamma(m+k+j+1)\Gamma(m)} \\ &\leq \theta(1+x)^{-k} \frac{x^{j-1}}{(1+x)^{m+k+j-1}} \\ &\quad \times \left[\frac{\alpha x}{(1+x)}\binom{m+j-1}{j} - (1-\alpha)(1+x)\binom{m+j-3}{j-2} + (1-\alpha)x\binom{m+j-1}{j}\right] \\ &\quad \times \frac{\Gamma(m+j+1)\Gamma(m+k)}{\Gamma(m+k+j+1)\Gamma(m)}.\end{aligned} \tag{10}$$

Applying D'Alembert's ratio test, the series on the right hand side (10) becomes convergent. This completes the proof. □

For $c_1 \geq 0$ and $c_2 > 0$, Lipschitz-type space in two parameters [24] is given as

$$Lip_M^{(c_1,c_2)}(\vartheta) := \left\{\zeta \in C_B[0,\infty) : |\zeta(t)-\zeta(x)| \leq M\frac{|\nu|^{\vartheta}}{(t+c_1x^2+c_2x)^{\frac{\vartheta}{2}}}; x \in (0,\infty)\right\},$$

where M is a positive constant and $\vartheta \in (0,1]$.

Theorem 2.3. *Let $\zeta \in Lip_M^{(c_1,c_2)}(\vartheta)$. Then, one has*

$$|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)| \leq M\left(\theta\frac{\mu_{m,\alpha,2}(x)}{(c_1x^2+c_2x)}\right)^{\frac{\vartheta}{2}}$$

$\forall x > 0$, where $\mu_{m,\alpha,2}(x) = \mathcal{B}^*_{m,\alpha,\theta}(\bar{\nu};x)$.

Proof. For $\vartheta = 1$, one has

$$\begin{aligned}|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)| &\leq \mathcal{B}^*_{m,\alpha,\theta}(|\zeta(t)-\zeta(x)|;x)\\ &\leq M\mathcal{B}^*_{m,\alpha,\theta}\left(\frac{|\nu|}{\sqrt{t+c_1x^2+c_2x}};x\right). \qquad (11)\end{aligned}$$

Using the fact that $\dfrac{1}{\sqrt{t+c_1x^2+c_2x}} < \dfrac{1}{\sqrt{c_1x^2+c_2x}}$, the Cauchy-Schwarz inequality and applying Lemma 1.1 and (6), inequality (11) implies that

$$\begin{aligned}|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)| &\leq M\frac{1}{\sqrt{c_1x^2+c_2x}}\mathcal{B}^*_{m,\alpha,\theta}(|\nu|;x) \leq \frac{M}{\sqrt{c_1x^2+c_2x}}(\mathcal{B}^*_{m,\alpha,\theta}(\bar{\nu};x))^{1/2}\\ &\leq M\left(\sqrt{\theta\frac{\mu_{m,\alpha,2}(x)}{c_1x^2+c_2x}}\right).\end{aligned}$$

By the aid of Hölder inequality with $p = \frac{1}{\vartheta}$ and $q = \frac{1}{1-\vartheta}$, Lemma 1.1 and (6), one has

$$\begin{aligned}|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)| &\leq \sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}|\zeta(t)-\zeta(x)|dt\\ &\leq \left\{\sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\left(\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}|\zeta(t)-\zeta(x)|dt\right)^{\frac{1}{\vartheta}}\right\}^{\vartheta}\\ &\leq \left\{\sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}|\zeta(t)-\zeta(x)|^{\frac{1}{\vartheta}}dt\right\}^{\vartheta}\\ &\leq M\left\{\sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}\frac{|\nu|}{\sqrt{t+c_1x^2+c_2x}}dt\right\}^{\vartheta}\\ &\leq \frac{M}{(c_1x^2+c_2x)^{\frac{\vartheta}{2}}}\left\{\sum_{j=0}^{\infty} Q^{(\theta)}_{m,j,\alpha}(x)\frac{1}{B(j+1,m)}\int_0^{\infty}\frac{t^j}{(1+t)^{m+j+1}}|\nu|dt\right\}^{\vartheta}\\ &\leq \frac{M}{(c_1x^2+c_2x)^{\frac{\vartheta}{2}}}(\mathcal{B}^*_{m,\alpha,\theta}(\bar{\nu};x))^{\vartheta/2} \leq M\theta\frac{\mu^{\frac{\vartheta}{2}}_{m,\alpha,2}(x)}{(c_1x^2+c_2x)^{\frac{\vartheta}{2}}},\end{aligned}$$

where $0 < \vartheta < 1$. □

Next, we achieve a direct approximation theorem for Bézier-Baskakov-Beta type operators concerning Ditzian-Totik modulus of smoothness.

Theorem 2.4. *Following inequality is satisfied for $\zeta \in C_B[0,\infty)$*

$$\left|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)\right| \le C\omega_{\rho^\ell}\left(\zeta,\frac{\rho^{1-\ell}(x)}{\sqrt{m}}\right). \tag{12}$$

Proof. Bearing in mind definition of $\overline{K}_{\rho^\ell}(\zeta,t)$, for fixed m,x,ℓ one can take $g=g_{m,x,\ell}\in W_\ell$ so that

$$\|\zeta-g\|+\frac{\rho^{1-\ell}(x)}{\sqrt{m}}\|\rho^\ell g'\| \le \overline{K}_{\rho^\ell}\left(\zeta,\frac{\rho^{1-\ell}(x)}{\sqrt{m}}\right). \tag{13}$$

Since $\mathcal{B}^*_{m,\alpha,\theta}(1;x)=1$, one has

$$\left|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)\right| \le 2\|\zeta-g\|+|\mathcal{B}^*_{m,\alpha,\theta}(g;x)-g(x)|. \tag{14}$$

Taking into account Lemma 2.1 and the representation $g(t)=g(x)+\int_x^t g'(u)du$, we get

$$\begin{aligned}\left|\mathcal{B}^*_{m,\alpha,\theta}(g;x)-g(x)\right| &= \left|\mathcal{B}^*_{m,\alpha,\theta}\left(\int_x^t g'(u)du;x\right)\right|\\ &\le 2^\ell\|\rho^\ell g'\|\left\{\rho^{-\ell}(x)\mathcal{B}^*_{m,\alpha,\theta}(|\nu|;x)+x^{-\ell/2}\mathcal{B}^*_{m,\alpha,\theta}\left(\frac{|\nu|}{(1+t)^{\ell/2}};x\right)\right\}.\end{aligned} \tag{15}$$

By the help of (7), Cauchy-Schwarz inequality and Corollary 1.3, we have

$$\begin{aligned}\mathcal{B}^*_{m,\alpha,\theta}(|\nu|;x) &\le \left(\mathcal{B}^*_{m,\alpha,\theta}(\bar{\nu};x)\right)^{1/2}\\ &\le \frac{\sqrt{\theta\lambda}\rho(x)}{\sqrt{m}}.\end{aligned} \tag{16}$$

Similarly using Lemma 2.2, we get

$$\begin{aligned}\mathcal{B}^*_{m,\alpha,\theta}\left(\frac{|\nu|}{(1+t)^{\ell/2}};x\right) &\le \theta\mathcal{B}^*_{m,\alpha}\left(\frac{|\nu|}{(1+t)^{\ell/2}};x\right)\\ &\le \theta\left(\mathcal{B}^*_{m,\alpha}(\bar{\nu};x)\right)^{1/2}\left(\mathcal{B}^*_{m,\alpha}((1+t)^{-\ell};x)\right)^{1/2}\\ &\le C_1\frac{\sqrt{\theta\lambda}\rho(x)}{\sqrt{m}}(1+x)^{-\ell/2}.\end{aligned} \tag{17}$$

From (15)-(17), we get

$$\left|\mathcal{B}^*_{m,\alpha,\theta}(g;x)-g(x)\right| \le C\|\rho^\ell g'\|\frac{\rho^{1-\ell}(x)}{\sqrt{m}}. \tag{18}$$

Using $\overline{K}_{\rho^\ell}(\zeta,t)\sim\omega_{\rho^\ell}(\zeta,t)$, (13), (14) and (18), we obtain (12). □

3. Rate of Convergence

Suppose $\zeta \in DBV_\gamma(0,\infty)$, $\gamma\ge 0$, is the class of all functions defined on $(0,\infty)$, having a derivative of bounded variation on every finite subinterval of $(0,\infty)$, $\forall t>0$ and $|\zeta(t)|\le Mt^\gamma$.
Let $g(t)$ be a function of bounded variation on each finite subinterval of $(0,\infty)$. One has

$$\zeta(x)=\int_0^x g(t)dt+\zeta(0)$$

since $\zeta\in DBV_\gamma(0,\infty)$.

Lemma 3.1. *Following equalities are satisfied*

(a) $\beta_{m,\alpha,\theta}(x,y) = \int_0^y U_{m,\alpha,\theta}(x,t)dt \leq \frac{\theta\lambda}{m}\frac{\rho^2(x)}{(x-y)^2}$, $0 \leq y < x$,

(b) $1 - \beta_{m,\alpha,\theta}(x,z) = \int_z^\infty U_{m,\alpha,\theta}(x,t)dt \leq \frac{\theta\lambda}{m}\frac{\rho^2(x)}{(z-x)^2}, x < z < \infty$

for sufficiently large m, $x \in (0,\infty)$ *and* $\theta \geq 1, \lambda > 2$.

Proof. Taking into account (7) and Corollary 1.3, one has

$$\begin{aligned}
\beta_{m,\alpha,\theta}(x,y) &= \int_0^y U_{m,\alpha,\theta}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 U_{m,\alpha,\theta}(x,t)dt \\
&\leq \mathcal{B}^*_{m,\alpha,\theta}(\bar{\upsilon};x)\,(x-y)^{-2} \leq \theta\mathcal{B}^*_{m,\alpha}(\bar{\upsilon};x)(x-y)^{-2} \\
&\leq \theta\frac{\lambda}{m}\frac{\rho^2(x)}{(x-y)^2}
\end{aligned}$$

which completes proof of (a). Using the same manner one can prove part (b), so we omit this. □

Theorem 3.2. *Suppose that* $\upsilon_a^b(\zeta_x')$ *is the total variation of* ζ_x' *on* $[a,b] \subset (0,\infty)$, $\lambda > 2$ *and* $\zeta \in DBV_\gamma(0,\infty), \theta \geq 1$. *Then, for every* $x \in (0,\infty)$ *and sufficiently large* m, *we have*

$$\begin{aligned}
\left|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x) - \zeta(x)\right| &\leq \frac{\sqrt{\theta}}{\theta+1}\left|\zeta'(x+) + \theta\zeta'(x-)\right|\sqrt{\frac{\lambda}{m}}\rho(x) \\
&\quad + \sqrt{\frac{\lambda}{m}}\,\rho(x)\frac{\theta^{3/2}}{\theta+1}\left|\zeta'(x+) - \zeta'(x-)\right|
\end{aligned}$$

$$+\frac{\theta\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\ ]} \upsilon_{x-x/j}^{x}(\zeta_x') + \frac{x}{\sqrt{m}}\upsilon_{x-x/\sqrt{m}}^{x}(\zeta_x')$$

$$+\frac{\theta\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\ ]} \upsilon_{x}^{x+x/j}(\zeta_x') + \frac{x}{\sqrt{m}}\upsilon_{x}^{x+x/\sqrt{m}}(\zeta_x').$$

Here auxiliary function ζ_x' *is defined as*

$$\zeta_x'(t) = \begin{cases} \zeta'(t) - \zeta'(x-), & 0 \leq t < x, \\ 0, & t = x, \\ \zeta'(t) - \zeta'(x+) & x < t \leq 1. \end{cases}$$

Proof. Bearing in mind the result $\int_0^\infty U_{m,\theta,a}(x,t)dt = \mathcal{B}^*_{m,\alpha,\theta}(1;x) = 1$, one has

$$\begin{aligned}
\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x) - \zeta(x) &= \int_0^\infty [\zeta(t) - \zeta(x)]U_{m,\alpha,\theta}(x,t)dt \\
&= \int_0^\infty \left(\int_x^t \zeta'(u)du\right)U_{m,\alpha,\theta}(x,t)dt. \qquad (19)
\end{aligned}$$

By definition of function ζ'_x, for any $\zeta \in DBV_\gamma(0,\infty)$, one can have

$$\begin{aligned} \zeta'(t) &= \frac{1}{\theta+1}\Big(\zeta'(x+)+\theta\zeta'(x-)\Big)+\zeta'_x(t) \\ &+\frac{1}{2}\Big(\zeta'(x+)-\zeta'(x-)\Big)\Big(sgn(\nu)+\frac{\theta-1}{\theta+1}\Big) \\ &+\delta_x(t)\Big(\zeta'(x)-\frac{1}{2}\Big(\zeta'(x+)+\zeta'(x-)\Big)\Big), \end{aligned} \tag{20}$$

where

$$\delta_x(t)=\begin{cases} 1\ , & x=t \\ 0\ , & x\neq t \end{cases}.$$

The following result is clear

$$\int_0^\infty U_{m,\alpha,\theta}(x,t)\int_x^t\Big[\zeta'(x)-\frac{1}{2}\Big(\zeta'(x+)+\zeta'(x-)\Big)\Big]\delta_x(u)dudt=0.$$

By Cauchy-Schwarz inequality, (6) and simple computations, we have

$$\begin{aligned} E_1 &= \int_0^\infty\Big(\int_x^t\frac{1}{\theta+1}\Big(\zeta'(x+)+\theta\zeta'(x-)\Big)du\Big)U_{m,\alpha,\theta}(x,t)dt \\ &= \frac{1}{\theta+1}\Big|\zeta'(x+)+\theta\zeta'(x-)\Big|\int_0^\infty|\nu|U_{m,\alpha,\theta}(x,t)dt \\ &\leq \frac{1}{\theta+1}\Big(\zeta'(x+)+\theta\zeta'(x-)\Big)\Big(\mathcal{B}^*_{m,\alpha,\theta}(\bar{\nu};x)\Big)^{1/2} \\ &\leq \frac{\sqrt{\theta}}{\theta+1}\Big|\zeta'(x+)+\theta\zeta'(x-)\Big|\sqrt{\frac{\lambda}{m}}\rho(x) \end{aligned} \tag{21}$$

and

$$\begin{aligned} E_2 &= \int_0^\infty\Big(\int_x^t\frac{1}{2}\Big(\zeta'(x+)-\zeta'(x-)\Big)\Big(sgn(u-x)+\frac{\theta-1}{\theta+1}\Big)du\Big)U_{m,\theta,a}(x,t)dt \\ &\leq \frac{\theta}{\theta+1}\Big|\zeta'(x+)-\zeta'(x-)\Big|\int_0^\infty|\nu|\,U_{m,\alpha,\theta}(x,t)dt \\ &= \frac{\theta}{\theta+1}\Big|\zeta'(x+)-\zeta'(x-)\Big|\mathcal{B}^*_{m,\alpha,\theta}\,(|\nu|\,;x) \\ &\leq \frac{\theta}{\theta+1}\Big|\zeta'(x+)-\zeta'(x-)\Big|\Big(\mathcal{B}^*_{m,\alpha,\theta}\,(\bar{\nu};x)\Big)^{1/2} \\ &\leq \frac{\theta^{3/2}}{\theta+1}\Big|\zeta'(x+)-\zeta'(x-)\Big|\sqrt{\frac{\lambda}{m}}\rho(x). \end{aligned} \tag{22}$$

Taking into account (19)-(22), Lemma 1.2 and Corollary 1.3 following estimate is obtained

$$\begin{aligned} \Big|\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)-\zeta(x)\Big| &\leq \Big|H_{m,\alpha,\theta}(\zeta'_x,x)+I_{m,\alpha,\theta}(\zeta'_x,x)\Big| \\ &+\frac{\sqrt{\theta}}{\theta+1}\Big|\zeta'(x+)+\theta\zeta'(x-)\Big|\sqrt{\frac{\lambda}{m}}\rho(x) \\ &+\frac{\theta^{3/2}}{\theta+1}\,|\zeta'(x+)-\zeta'(x-)|\,\sqrt{\frac{\lambda}{m}}\,\rho(x), \end{aligned} \tag{23}$$

where

$$H_{m,\alpha,\theta}(\zeta'_x, x) = \int_0^x \Big(\int_x^t \zeta'_x(u)du\Big) U_{m,\alpha,\theta}(x,t)dt,$$

$$I_{m,\alpha,\theta}(\zeta'_x, x) = \int_x^\infty \Big(\int_x^t \zeta'_x(u)du\Big) U_{m,\alpha,\theta}(x,t)dt.$$

The rest of the proof is to estimate the terms $H_{m,\theta,a}(\zeta'_x, x)$ and $I_{m,\alpha,\theta}(\zeta'_x, x)$. Following relations are satisfied

$$\begin{aligned}
\left|H_{m,\alpha,\theta}(\zeta'_x, x)\right| &= \left|\int_0^x \Big(\int_x^t \zeta'_x(u)du\Big) d_t\beta_{m,\alpha,\theta}(x,t)\right| \\
&= \left|\int_0^x \beta_{m,\alpha,\theta}(x,t)\zeta'_x(t)dt\right| \\
&\leq \Big(\int_0^y + \int_y^x\Big) \left|\zeta'_x(t)\right| \left|\beta_{m,\alpha,\theta}(x,t)\right| dt \\
&\leq \theta\frac{\lambda\rho^2(x)}{m}\int_0^y \upsilon_t^x(\zeta'_x)(x-t)^{-2}dt + \int_y^x \upsilon_t^x(\zeta'_x)dt \\
&\leq \theta\frac{\lambda\rho^2(x)}{m}\int_0^y \upsilon_t^x(\zeta'_x)(x-t)^{-2}dt + \frac{x}{\sqrt{m}}\upsilon^x_{x-x/\sqrt{m}}(\zeta'_x)
\end{aligned}$$

by the help of Lemma 3.1 with $y = x - x/\sqrt{m}$, the inequality $\int_a^b d_t\beta_{m,\alpha,\theta}(x,t) \leq 1$ for all $[a,b] \subseteq (0,\infty)$ and integration by parts.

Putting $u = x/(x-t)$ following relations are satisfied

$$\begin{aligned}
\theta\frac{\lambda\rho^2(x)}{m}\int_0^{x-x/\sqrt{m}} (x-t)^{-2}\upsilon_t^x(\zeta'_x)dt &= \theta\frac{\lambda(1+x)}{m}\int_1^{\sqrt{m}} \upsilon^x_{x-x/u}(\zeta'_x)du \\
&\leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\ ]}\int_j^{j+1} \upsilon^x_{x-x/u}(\zeta'_x)du \\
&\leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\ ]} \upsilon^x_{x-x/j}(\zeta'_x).
\end{aligned}$$

So the following inequality is satisfied

$$\left|H_{m,\alpha,\theta}(\zeta'_x, x)\right| \leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\ ]} \upsilon^x_{x-x/j}(\zeta'_x) + \frac{x}{\sqrt{m}}\upsilon^x_{x-x/\sqrt{m}}(\zeta'_x). \tag{24}$$

Applying Lemma 3.1 with $z = x + x/\sqrt{m}$ and via integration by parts, we have

$$\begin{aligned}
|I_{m,\theta,a}(\zeta'_x, x)| &= \left|\int_x^\infty \Big(\int_x^t \zeta'_x(u)du\Big) U_{m,\theta,a}(x,t)dt\right| \\
&= \left|\int_x^z \Big(\int_x^t \zeta'_x(u)du\Big) d_t(1-\beta_{m,\alpha,\theta}(x,t)) + \int_z^\infty \Big(\int_x^t \zeta'_x(u)du\Big) d_t(1-\beta_{m,\alpha,\theta}(x,t))\right|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg|\Bigg[\left(\int_x^t \zeta_x'(u)du\right)(1-\beta_{m,\alpha,\theta}(x,t))\Bigg]_x^z - \int_x^z \zeta_x'(t)(1-\beta_{m,\alpha,\theta}(x,t))dt \\
&\quad + \int_z^\infty \left(\int_x^t \zeta_x'(u)du\right) d_t(1-\beta_{m,\alpha,\theta}(x,t))\Bigg| \\
&= \Bigg|\left(\int_x^z \zeta_x'(u)du\right)(1-\beta_{m,\alpha,\theta}(x,z)) - \int_x^z \zeta_x'(t)(1-\beta_{m,\alpha,\theta}(x,t))dt \\
&\quad + \Bigg[\left(\int_x^t \zeta_x'(u)du\right)(1-\beta_{m,\alpha,\theta}(x,t))\Bigg]_z^\infty - \int_z^\infty \zeta_x'(t)(1-\beta_{m,\alpha,\theta}(x,t))dt\Bigg| \\
&= \Bigg|\int_x^z \zeta_x'(t)(1-\beta_{m,\alpha,\theta}(x,t))dt + \int_z^\infty \zeta_x'(t)(1-\beta_{m,\alpha,\theta}(x,t))dt\Bigg| \\
&< \theta\frac{\lambda\rho^2(x)}{m}\int_z^\infty \upsilon_x^t(\zeta')_x(\nu)^{-2}dt + \int_x^z \upsilon_x^t(\zeta_x')dt \\
&\leq \theta\frac{\lambda\rho^2(x)}{m}\int_{x+x/\sqrt{m}}^\infty \upsilon_x^t(\zeta_x')(\nu)^{-2}dt + \frac{x}{\sqrt{m}}\upsilon_x^{x+x/\sqrt{m}}(\zeta_x'). \qquad (25)
\end{aligned}
$$

Putting $u = x/\nu$ following relations are satisfied

$$
\begin{aligned}
\theta\frac{\lambda\rho^2(x)}{m}\int_{x+x/\sqrt{m}}^\infty \upsilon_x^t(\zeta_x')(\nu)^{-2}dt &= \theta\frac{\lambda\rho^2(x)}{xm}\int_0^{\sqrt{m}} \upsilon_x^{x+x/u}(\zeta_x')du \\
&\leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\,]}\int_j^{j+1} \upsilon_x^{x+x/u}(\zeta_x')du \\
&\leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\,]} \upsilon_x^{x+x/j}(\zeta_x'). \qquad (26)
\end{aligned}
$$

Again combining (25)-(26), we get

$$
|I_{m,\theta,a}(\zeta_x', x)| \leq \theta\frac{\lambda(1+x)}{m}\sum_{j=1}^{[\sqrt{m}\,]} \upsilon_x^{x+x/j}(\zeta_x') + \frac{x}{\sqrt{m}}\upsilon_x^{x+x/\sqrt{m}}(\zeta_x'). \qquad (27)
$$

As a result, we get the required result combining (23), (24) and (27). □

4. Computer graphics

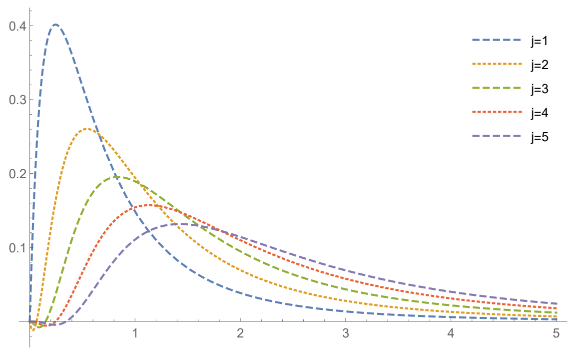
In this part, we provide some numerical results to see convergence of our operators. We first demonstrate behavior of polynomials $b_{m,j}^{(\alpha)}(x)$ for some j and α values, and $x \in [0,5]$ in Figure 1 and Figure 2. We consider convergence and errors of convergence of our operators to the basic function

$$\zeta(x) = \sqrt{x}$$

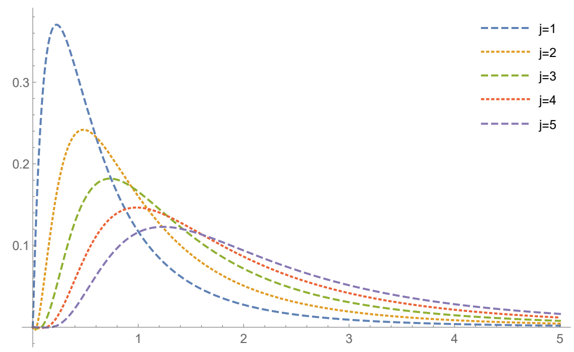
in Figure 3 and Figure 4, respectively. We also consider

$$\zeta(x) = \frac{x+4}{x^2+6}$$

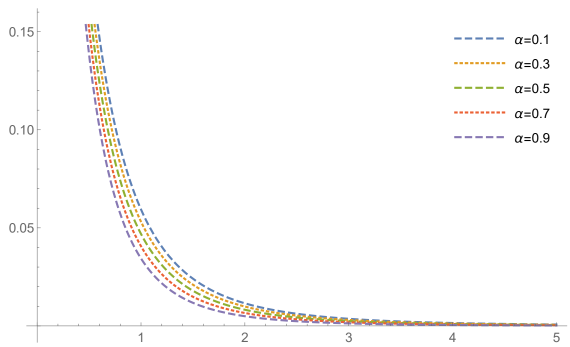
to see effectiveness of our operators. In Figure 5 (a), we choose $m = 20, \alpha = 0.5$ and $\theta = 2$ to see convergence of $\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)$, where $x \in [0,3]$. We select $m = 5, \alpha = 0.9$ and $\theta = 2$ in Figure 5 (b), $m = 10, \alpha = 0.8$ and $\theta = 3$ in Figure 5 (c), $m = 10, \alpha = 0.9$ and $\theta = 2$ in Figure 5 (d), $m = 15, \alpha = 0.9$ and $\theta = 2$ in Figure 5 (e) and $m = 25, \alpha = 0.5$ and $\theta = 2$ in Figure 5 (f) to observe approximation of $\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)$ and corresponding errors. We can clearly say that, the operators $\mathcal{B}^*_{m,\alpha,\theta}(\zeta;x)$ have less errors even for small values of m.



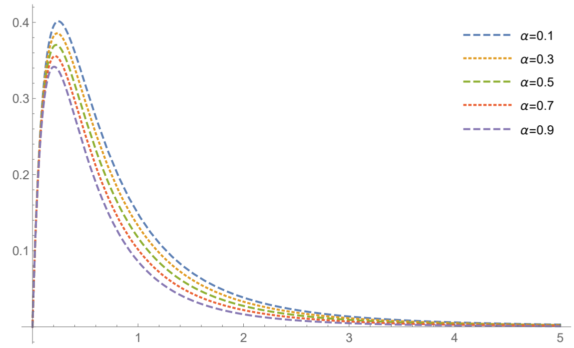
(a) $b^{(0.1)}_{5,j}(x)$ with some j values



(b) $b^{(0.5)}_{5,j}(x)$ with some j values

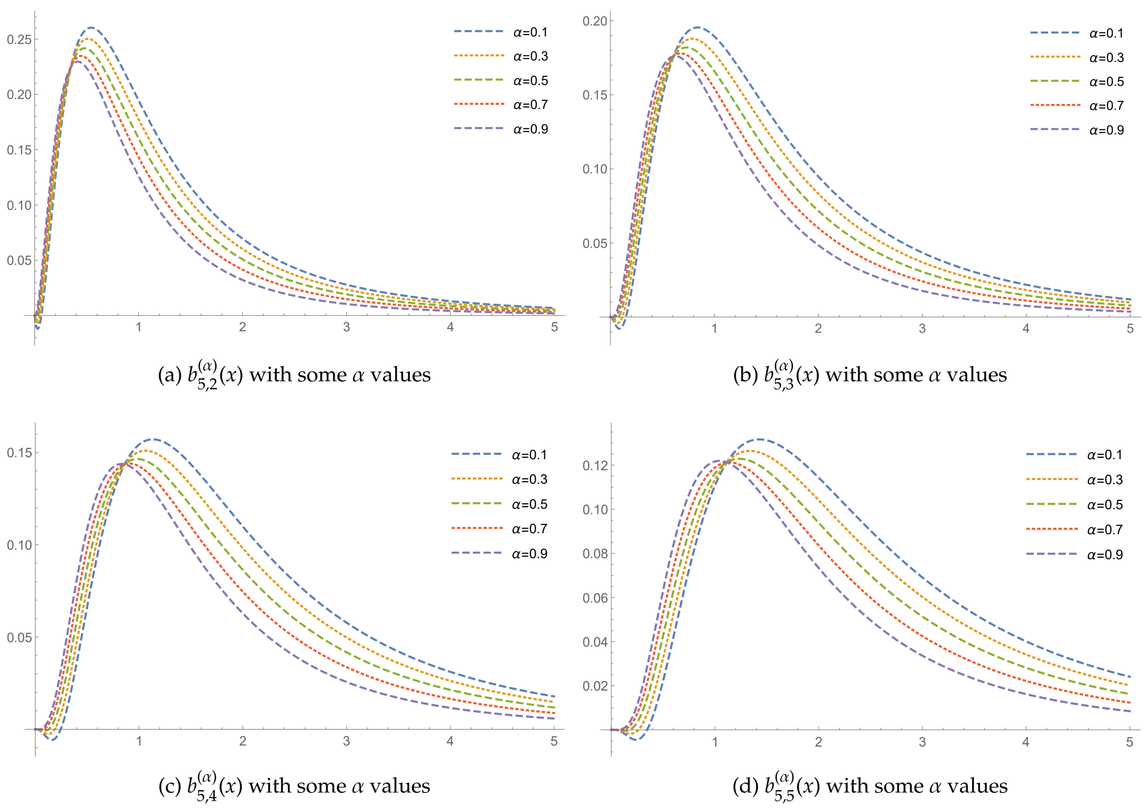


(c) $b^{(\alpha)}_{5,0}(x)$ with some α values



(d) $b^{(\alpha)}_{5,1}(x)$ with some α values

Figure 1: Behavior of polynomials $b^{(\alpha)}_{m,j}(x)$ for $x \in [0, 5]$



(a) $b_{5,2}^{(\alpha)}(x)$ with some α values

(b) $b_{5,3}^{(\alpha)}(x)$ with some α values

(c) $b_{5,4}^{(\alpha)}(x)$ with some α values

(d) $b_{5,5}^{(\alpha)}(x)$ with some α values

Figure 2: Behavior of polynomials $b_{m,j}^{(\alpha)}(x)$ for some j and α values, and $x \in [0, 5]$

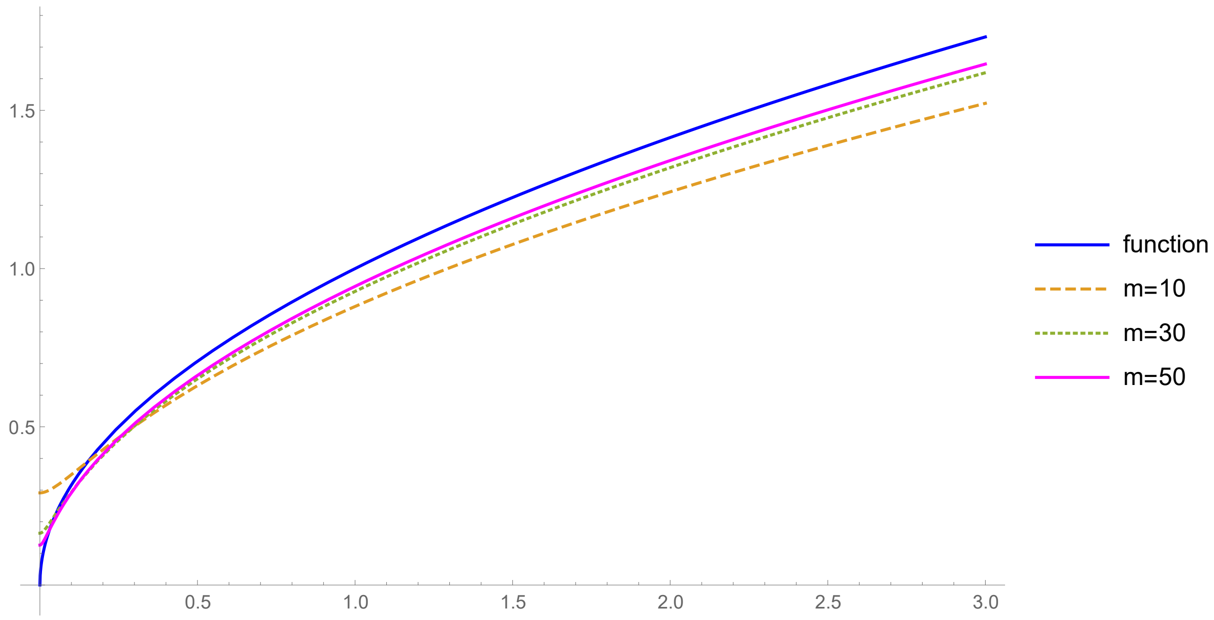


Figure 3: Approximation of operators for some m values

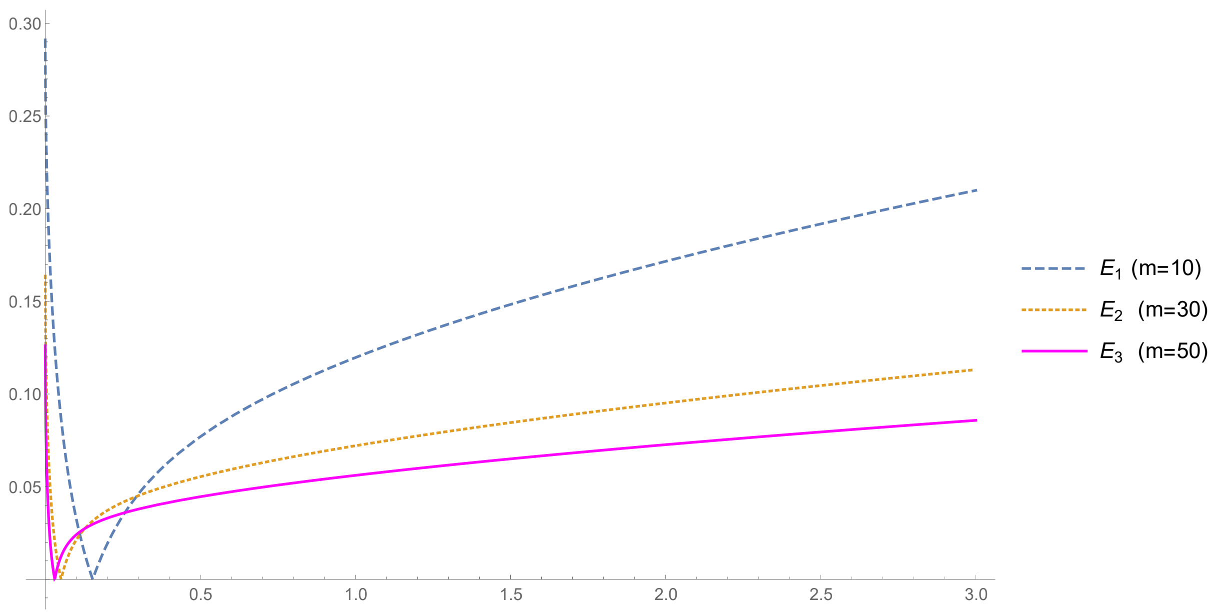
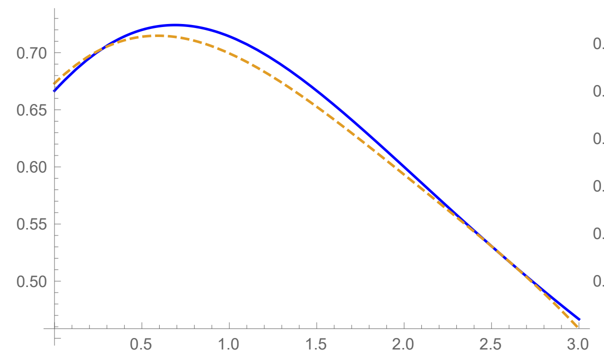
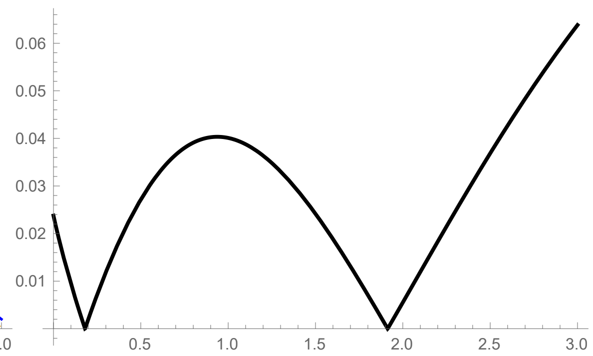


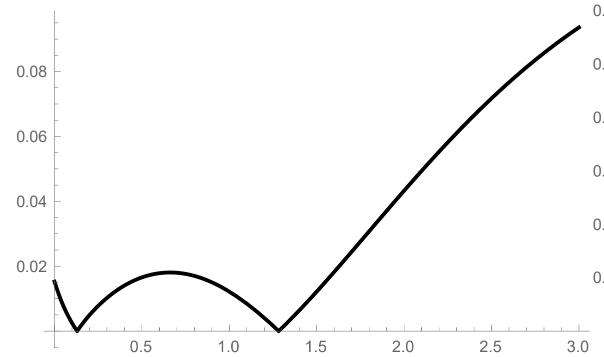
Figure 4: Errors of approximation for some m values



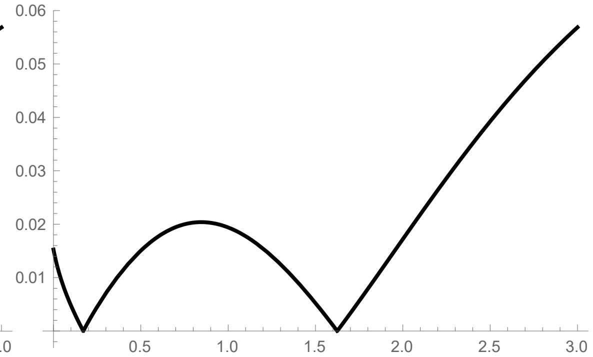
(a) Convergence of $\mathcal{B}^*_{m,\alpha,\theta}$ to $\zeta = \frac{x+4}{x^2+6}$



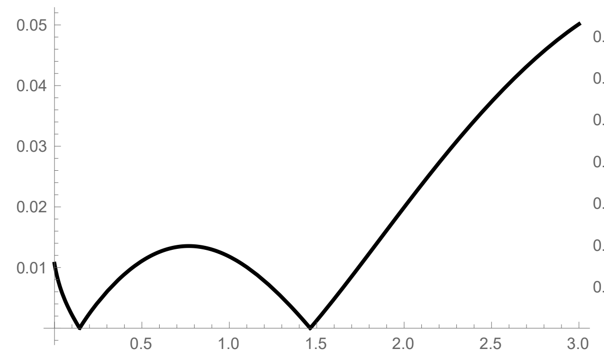
(b) Error for $m = 5, \alpha = 0.9, \theta = 2$



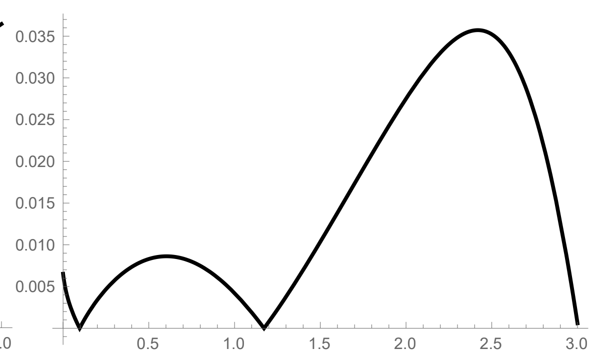
(c) Error for $m = 10, \alpha = 0.8, \theta = 3$



(d) Error for $m = 10, \alpha = 0.9, \theta = 2$



(a) Error for $m = 15, \alpha = 0.9, \theta = 2$



(b) Error for $m = 25, \alpha = 0.5, \theta = 2$

Figure 5: Convergence of operators and errors of approximation

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