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Riemannian Concircular Structure Manifolds

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Abstract. In this manuscript, we give the definition of Riemannian concircular structure manifolds. Some basic properties and integrability condition of such manifolds are established. It is proved that a Riemannian concircular structure manifold is semisymmetric if and only if it is concircularly flat. We also prove that the Riemannian metric of a semisymmetric Riemannian concircular structure manifold is a generalized soliton. In this sequel, we show that a conformally flat Riemannian concircular structure manifold is a quasi-Einstein manifold and its scalar curvature satisfies the partial differential equation $\triangle r = \frac{\partial^2 r}{\partial t^2} + \alpha(n-1)\frac{\partial r}{\partial t}$. To validate the existence of Riemannian concircular structure manifolds, we present some non-trivial examples. In this series, we show that a quasi-Einstein manifold with a divergence free concircular curvature tensor is a Riemannian concircular structure manifold.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold. To study the properties of Riemannian manifolds, the vector fields (likes, torse-forming, torqued, concircular, recurrent, parallel vector fields, etc.) play an important role, and therefore they attract the researchers to work in this area. In 1963, Tashiro [30] classified the Riemannian manifolds with concircular, special concircular, nonisometric concircular, nonisometric conformal and nonaffine projective vector fields. Mihai and Mihai [22] studied the properties of Riemannian manifolds with torse-forming and exterior concurrent vector fields. Chen [13] explored the properties of concircular vector field and established some results of Ricci solitons. He also studied the properties of torqued and parallel vector fields in [14] and [15], respectively, with his co-authors. The Riemannian manifolds with different vector fields have been studied by several authors. For instance, we refer [6], [16], [20], [21], [23], [30], [34], [35] and their references. In this manuscript, we start our study with torse-forming vector field in Riemannian setting and prove that it is a concircular one. We also introduce a new class of Riemannian manifold, named as Riemannian concircular structure (briefly, $(RCS)_n$-)manifold.

The study of symmetric space is a very keen and interesting research field of differential geometry. Semisymmetric space ($R(\vartheta_1, \vartheta_2) \cdot R = 0$) [29] is the generalization of locally symmetric space ($\nabla R = 0$), and it has been studied by several geometers. Here R denotes the non-vanishing curvature tensor of the Riemannian manifold M, ∇ is Levi-Civita connection of the Riemannian metric g and $R(\vartheta_1, \vartheta_2)$ acts as a

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derivation on R for all vector fields ϑ_1 and ϑ_2 on M. If the non-vanishing Ricci tensor S of M satisfies the curvature condition $R(\vartheta_1, \vartheta_2) \cdot S = 0$, then M is called a Ricci semisymmetric. Remark that the class of Ricci semisymmetric manifold is included in the class of Ricci symmetric manifold ($\nabla S = 0$), but the converse part is not true in general. Every semisymmetric manifold is Ricci semisymmetric, but its converse part is not true (in general). The properties of symmetric spaces (for example, locally symmetric, semisymmetric, Ricci semisymmetric, etc.) have been studied by several authors in Riemannian and semi-Riemannian setting. We refer [2], [3], [11], [19], [23], [28] and their references for more details.

In 1987, Koiso [18] introduced the notion of quasi-Einstein metric, as a generalization of Kähler-Einstein metric, on Fano manifolds. Since then, the properties of quasi-Einstein metrics have been studied by many academicians. The work of Chave and Valent [12] on quasi-Einstein metrics, motivates Chaki and Maity [7] to study the properties of Ricci tensor S of an n-dimensional Riemannian manifold M that satisfies the relation

$$S = ag + bu \otimes u, \tag{1}$$

where a and b are non-zero smooth functions on M and u is a 1-form. If the non-vanishing Ricci tensor S of M satisfies equation (1), then M is said to be a quasi-Einstein manifold (briefly, $(QE)_n$-manifold) [7]. Particularly, M together with $b = 0$ and $a = b = 0$ reduces to the Einstein manifold and Ricci flat manifold, respectively. In [7], Chaki and Maity considered the following assumptions:

- a and b are constants and the generator of $(QE)_n$-manifold is recurrent,
- $a + b = 0$, $\vartheta_4 = \frac{1}{2a} grad\, a$ and $\nabla_{\vartheta_1}\vartheta_4 = -\vartheta_1 + A(\vartheta_1)\vartheta_4$, where A is a dual 1-form of ϑ_4

for $(QE)_n$-manifolds, and they proved that in both the cases $(QE)_n$-manifolds are conformally conservative. In this manuscript, we are going to study the properties of $(QE)_n$-manifolds in general setting and therefore our work will be the generalization of Chaki and Maity work [7]. We also give some clue for selection of the smooth functions a and b.

We arrange our work in the following manner. In Section 2, we give the definition of Riemannian concircular structure manifold and prove its some basic properties. We also establish that the $(RCS)_n$-manifold is integrable. Section 3 deals with the study of semisymmetric $(RCS)_n$-manifolds. The properties of conformally flat $(RCS)_n$-manifolds are studied in Section 4. We construct some non-trivial examples to prove the existence of $(RCS)_n$-manifolds in Section 5.

2. Riemannian manifolds and torse-forming vector field

This section is dedicated to study the properties of Riemannian manifolds endowed with a torse-forming vector field.

Let M be an n-dimensional Riemannian manifold endowed with a Riemannian metric g. The curvature tensor R of a Riemannian manifold M satisfies

$$(divR)(\vartheta_1, \vartheta_2)\vartheta_3 = (\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) - (\nabla_{\vartheta_2} S)(\vartheta_1, \vartheta_3), \ \ \forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in \mathfrak{X}(M), \tag{2}$$

where div stands for divergence and $\mathfrak{X}(M)$ is the collection of all smooth vector fields of M. If M has a divergence free Riemann curvature tensor R then $divR = 0$ and vice-versa. The Ricci tensor S of M is said to be of Codazzi type if its covariant derivative is symmetric, that is, $(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) = (\nabla_{\vartheta_2} S)(\vartheta_1, \vartheta_3)$, $\forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in \mathfrak{X}(M)$. Thus from equation (2), we remark that the curvature tensor of M is divergence free if and only if S is of Codazzi type. For more details, we refer to [26].

The notion of torse-forming vector field on Riemannian spaces was introduced by Yano [35], and its properties have been studied by several academicians in Riemannian and semi-Riemannian setting (see, [6], [20], [21], [22], [23]). A smooth vector field ξ defined on M is said to be a torse-forming vector field [35] if

$$(\nabla_{\vartheta_1} u)(\vartheta_2) = \alpha g(\vartheta_1, \vartheta_2) + \pi(\vartheta_1)u(\vartheta_2), \ \ \forall\, \vartheta_1, \vartheta_2 \in \mathfrak{X}(M), \tag{3}$$

where $u(\cdot) = g(\cdot, \xi)$ is a 1-form associated with ξ and π is a 1-form. If 1-form π is closed on M, then ξ is said to be a concircular vector field [16, 34]. As a particular, the torse-forming vector field ξ on M reduces to

* torqued vector field [14] if $\pi(\xi) = 0$,
* concircular vector field (in Fialkow's sense) [13, 16] if $\pi = 0$,
* concircular vector field (in Yano's sense) [34] if the 1-form π is closed,
* recurrent vector field [27] if $\alpha = 0$,
* concurrent vector field [15] if $\pi = 0$, and $\alpha = 1$,
* parallel vector field [15, 16] if $\pi = 0$, and $\alpha = 0$.

These vectors are capable to address several issues of science and technology, especially they play a peculiar role in geometry and physics, therefore the study of geometric structures with these vectors attract researchers. In this manuscript, we classify the Riemannian manifolds with concircular vector field (in Yano's sense).

Let M admit a unit torse-forming vector field ξ, that is, $g(\xi, \xi) = 1 \implies g(\nabla_{\vartheta_1}\xi, \xi) = 0$. Set $\vartheta_2 = \xi$ in equation (3), we find

$$\alpha u(\vartheta_1) + \pi(\vartheta_1) = 0, \tag{4}$$

since $u(\xi) = 1$ and $g(\nabla_{\vartheta_1} u)(\xi) = 0$. Using equation (4) in equation (3), we obtain

$$(\nabla_{\vartheta_1} u)(\vartheta_2) = \alpha\{g(\vartheta_1, \vartheta_2) - u(\vartheta_1)u(\vartheta_2)\}, \tag{5}$$

which implies that

$$\nabla_{\vartheta_1}\xi = \alpha\{\vartheta_1 - u(\vartheta_1)\xi\}, \tag{6}$$

where α is a non-zero scalar and $\nabla_{\vartheta_1}\alpha = g(\vartheta_1, D\alpha) = \vartheta_1(\alpha) = \mu u(\vartheta_1)$ for some smooth fuction μ on M. Here D is used for gradient operator of g. From equation (5), it is obvious that the 1-form u is closed. Taking covariant derivative of equation (4) along ϑ_2 and using the fact $\vartheta_1(\alpha) = \mu u(\vartheta_1)$ and equation (5), we conclude that π is also closed. Hence, the unit torse-forming vector field ξ defined in (3) is a unit concircular vector field on M in Yano sense. The smooth function α on M is known as the potential function of the concircular vector field. Equation $\vartheta_1(\alpha) = \mu u(\vartheta_1)$ gives that $\xi(\alpha) = \mu \implies \xi(\xi(\alpha)) = \xi(\mu)$. Again $\vartheta_1(\alpha) = \mu u(\vartheta_1)$ infers that $D\alpha = \mu\xi$. The covariant derivative of $D\alpha = \mu\xi$ along ϑ_1 gives

$$\nabla_{\vartheta_1} D\alpha = \vartheta_1(\mu)\xi + \mu\alpha(\vartheta_1 - u(\vartheta_1)\xi). \tag{7}$$

Let us consider an orthonormal frame field on M and then contracting the above equation over ϑ_1, we lead to

$$\triangle\alpha = \xi(\xi(\alpha)) + \alpha(n-1)\xi(\alpha),$$

where $\triangle$ stands for the Laplace operator of g. A smooth function Ψ on M is said to be harmonic if and only if $\triangle\Psi = 0$. Suppose $\xi = \frac{\partial}{\partial t}$ on M, then the above equation takes the form

$$\triangle\alpha = \frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial t} + \frac{n-1}{2}\alpha^2\right). \tag{8}$$

Thus, we conclude the following:

Theorem 2.1. *If an n-dimensional Riemannian manifold M admits a unit concircular vector field ξ, then the potential function α of ξ satisfies the partial differential equation (8).*

From equation (8), we can also state:

Proposition 2.2. *The potential function α of a unit concircular vector field ξ on an n-dimensional Riemannian manifold is harmonic if and only if $\frac{\partial\alpha}{\partial t} + \frac{n-1}{2}\alpha^2 = constant$.*

Remark 2.3. *Equation (8) gives a clue to evaluate the potential function α of the concircular vector field ξ on M.*

Now, taking inner product of equation (7) with ϑ_2, we get

$$g(\nabla_{\vartheta_1} D\alpha, \vartheta_2) = Hess_g(\alpha)(\vartheta_1, \vartheta_2) = \vartheta_1(\mu)u(\vartheta_2) + \mu\alpha(g(\vartheta_1, \vartheta_2) - u(\vartheta_1)u(\vartheta_2)), \tag{9}$$

where $Hess_g$ is the Hessian operator of g and $Hess_g(\alpha)(\vartheta_1, \vartheta_2) = \vartheta_1\vartheta_2(\alpha) - d\alpha(\nabla_{\vartheta_1}\vartheta_2)$ for the exterior derivative d. We know that $Hess_g(\alpha)$ is symmetric, therefore from equation (9) we have

$$\vartheta_2(\mu)u(\vartheta_1) = \vartheta_1(\mu)u(\vartheta_2),$$

which gives $\vartheta_1(\mu) = \sigma u(\vartheta_1)$, where $\sigma = \xi(\mu)$. Thus, we state the following:

Lemma 2.4. *Let an n-dimensional Riemannian manifold admit a unit concircular vector field ξ. Then $D\mu = \sigma\xi$.*

Let the Riemannian manifold M admit a $(1, 1)$ tensor field ϕ such that

$$\alpha\phi\vartheta_1 = \nabla_{\vartheta_1}\xi, \quad \alpha \neq 0,$$

which gives

$$\phi\vartheta_1 = \vartheta_1 - u(\vartheta_1)\xi, \tag{10}$$

where equation (6) is used. Operating ϕ on either side of equation (10), and then following (10) and $u(\xi) = 1$, we obtain

$$\phi^2 = I - u \otimes \xi.$$

In view of equation (10), we have

$$g(\phi\vartheta_1, \phi\vartheta_2) = g(\vartheta_1, \vartheta_2) - u(\vartheta_1)u(\vartheta_2), \quad u(\vartheta_2) = g(\vartheta_2, \xi).$$

Remark that $g(\phi\vartheta_1, \phi\vartheta_2) = g(\phi\vartheta_1, \vartheta_2) = g(\vartheta_1, \phi\vartheta_2)$, $\forall\ \vartheta_1, \vartheta_2 \in \mathfrak{X}(M)$. Thus, we conclude that if M admits a unit concircular vector field ξ, a $(1, 1)$ tenosr field ϕ and a 1-form u, then we have

$$\phi^2 = I - u \otimes \xi, \quad u(\xi) = 1, \quad g(\phi\,\cdot, \phi\,\cdot) = g(\cdot\,, \cdot) - u \otimes u. \tag{11}$$

By considering all above facts, we give the following definition.

Definition 2.5. *Let the data (ϕ, ξ, u, g) satisfy (11) on an n-dimensional Riemannian manifold M. Then M equipped with (ϕ, ξ, u, g) is said to be a Riemannian concircular structure manifold (briefly, $(RCS)_n$-manifold), and the structure (ϕ, ξ, u, g) is said to be a Riemannian concircular structure on M.*

Now, we prove some basic results of $(RCS)_n$-manifolds in the following:

Proposition 2.6. *An n-dimensional $(RCS)_n$-manifold satisfies*
(i) $\phi\xi = 0$,
(ii) $u(\phi\vartheta_1) = 0$,
(iii) $rank\,(\phi) = n - 1$,
(iv) $(\nabla_{\vartheta_1}\phi)(\vartheta_2) = \alpha[2u(\vartheta_1)u(\vartheta_2)\xi - g(\vartheta_1, \vartheta_2)\xi - u(\vartheta_2)\vartheta_1]$, $\forall\ \vartheta_1, \vartheta_2 \in \mathfrak{X}(M)$.

Proof. Setting $\vartheta_1 = \xi$ in equation (10) and then following (11), we immediately get (*i*). From (11), we have $\phi^2\vartheta_1 = \vartheta_1 - u(\vartheta_1)\xi$. Again replacing ϑ_1 with $\phi\vartheta_1$ in (10), we find $\phi^2\vartheta_1 = \phi\vartheta_1 - u(\phi\vartheta_1)\xi$. The last two equations together with (10) give (*ii*). Since $\phi\xi = 0$ implies that $rank(\phi) < dimM$. If possible, we suppose that $\phi\vartheta_1 = 0 \implies \vartheta_1 = u(\vartheta_1)\xi$, since equation (10) is used. The Rank-Nullity Theorem (for a homogeous system of equations $A\vartheta_1 = 0$ in n unknowns, $rank(A) + nullity(A) = n$) prove (*iii*). The covariant derivative of (10) gives (*iv*). This completes the proof. □

Proposition 2.7. *Every $(RCS)_n$-manifold satisfies the following identities.*
(*i*) $R(\vartheta_1, \vartheta_2)\xi = (\alpha^2 + \mu)\{u(\vartheta_1)\vartheta_2 - u(\vartheta_2)\vartheta_1\}$,
(*ii*) $R(\xi, \vartheta_1)\vartheta_2 = (\alpha^2 + \mu)\{u(\vartheta_2)\vartheta_1 - g(\vartheta_1, \vartheta_2)\xi\}$,
(*iii*) $u(R(\vartheta_1, \vartheta_2)\vartheta_3) = (\alpha^2 + \mu)\{u(\vartheta_2)g(\vartheta_1, \vartheta_3) - u(\vartheta_1)g(\vartheta_2, \vartheta_3)\}$,
(*iv*) $S(\vartheta_1, \xi) = -(n-1)(\alpha^2 + \mu)u(\vartheta_1) \iff Q\xi = -(n-1)(\alpha^2 + \mu)\xi$
for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathfrak{X}(M)$ *and* $(\alpha^2 + \mu) \neq 0$.

Proof. The covariant derivative of equation (6) along the vector field ϑ_2 gives

$$\nabla_{\vartheta_2}\nabla_{\vartheta_1}\xi = \vartheta_2(\alpha)\{\vartheta_1 - u(\vartheta_1)\xi\} + \alpha\{\nabla_{\vartheta_2}\vartheta_1 - \alpha(g(\vartheta_1, \vartheta_2) - u(\vartheta_1)u(\vartheta_2))\xi - u(\nabla_{\vartheta_2}\vartheta_1)\xi - \alpha u(\vartheta_1)(\vartheta_2 - u(\vartheta_2)\xi)\},$$

where equations (5) and (6) are used. This equation together with curvature identity $R(\vartheta_1, \vartheta_2)\xi = [\nabla_{\vartheta_1}, \nabla_{\vartheta_2}]\xi - \nabla_{[\vartheta_1, \vartheta_2]}\xi$ gives

$$R(\vartheta_1, \vartheta_2)\xi = \vartheta_1(\alpha)\{\vartheta_2 - u(\vartheta_2)\xi\} - \vartheta_2(\alpha)\{\vartheta_1 - u(\vartheta_1)\xi\} + \alpha^2\{u(\vartheta_1)\vartheta_2 - u(\vartheta_2)\vartheta_1\}.$$

By hypothesis $\vartheta_1(\alpha) = \mu u(\vartheta_1)$, the above equation states assertion (*i*) of Proposition 2.7. The other assertions can be proved by simple straightforward calculations. □

Proposition 2.8. *In an $(RCS)_n$-manifold, we have*

1. $'R(\vartheta_1, \vartheta_2, \phi\vartheta_3, \vartheta_4) - 'R(\vartheta_1, \vartheta_2, \vartheta_3, \phi\vartheta_4) = (\alpha^2 + \mu)\{u(\vartheta_3)[u(\vartheta_2)g(\vartheta_1, \vartheta_4) - u(\vartheta_1)g(\vartheta_2, \vartheta_4)] + u(\vartheta_4)[u(\vartheta_2)g(\vartheta_1, \vartheta_3) - u(\vartheta_1)g(\vartheta_2, \vartheta_3)]\}$,
2. $'R(\vartheta_1, \vartheta_2, \phi\vartheta_3, \phi\vartheta_4) = 'R(\phi\vartheta_1, \phi\vartheta_2, \vartheta_3, \vartheta_4)$,
3. $'R(\phi\vartheta_1, \phi\vartheta_2, \phi\vartheta_3, \phi\vartheta_4) = 'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) - (\alpha^2 + \mu)\{u(\vartheta_4)[u(\vartheta_1)g(\vartheta_2, \vartheta_3) - u(\vartheta_2)g(\vartheta_1, \vartheta_3)] + u(\vartheta_3)[u(\vartheta_2)g(\vartheta_1, \vartheta_4) - u(\vartheta_1)g(\vartheta_2, \vartheta_4)]\}$,
4. $'R(\phi\vartheta_1, \vartheta_2, \vartheta_3, \phi\vartheta_4) - 'R(\vartheta_1, \phi\vartheta_2, \phi\vartheta_3, \vartheta_4) = (\alpha^2 + \mu)\{u(\vartheta_1)u(\vartheta_3)g(\vartheta_2, \vartheta_4) - u(\vartheta_2)u(\vartheta_3)g(\vartheta_1, \vartheta_4)\}$,
5. $S(\phi\vartheta_1, \phi\vartheta_2) = S(\vartheta_1, \vartheta_2) + (n-1)(\alpha^2 + \mu)u(\vartheta_1)u(\vartheta_2)$,

for all $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathfrak{X}(M)$. *Here* $'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = g(R(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_4)$.

Proof. From equation (10) and Proposition 2.7, we have

$$'R(\vartheta_1, \vartheta_2, \vartheta_3, \phi\vartheta_4) = 'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) - (\alpha^2 + \mu)u(\vartheta_4)\{u(\vartheta_2)g(\vartheta_1, \vartheta_3) - u(\vartheta_1)g(\vartheta_2, \vartheta_3)\}$$

and

$$'R(\vartheta_1, \vartheta_2, \phi\vartheta_3, \vartheta_4) = 'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + (\alpha^2 + \mu)u(\vartheta_3)\{u(\vartheta_2)g(\vartheta_1, \vartheta_4) - u(\vartheta_1)g(\vartheta_2, \vartheta_4)\},$$

where $'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = g(R(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_4)$. The last two equations deduce assertion (*1*) of Proposition 2.8. The assertion (2) of Proposition 2.8 can be obtained by straightforward calculations after considering equation (10) and Proposition 2.7. We have,

$$'R(\phi\vartheta_1, \phi\vartheta_2, \vartheta_3, \vartheta_4) = 'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + (\alpha^2 + \mu)\{u(\vartheta_4)[u(\vartheta_1)g(\vartheta_2, \vartheta_3) - u(\vartheta_2)g(\vartheta_1, \vartheta_3)] + u(\vartheta_3)[u(\vartheta_2)g(\vartheta_1, \vartheta_4) - u(\vartheta_1)g(\vartheta_2, \vartheta_4)]\}. \quad (12)$$

Replacing ϑ_1 and ϑ_2 with $\phi\vartheta_1$ and $\phi\vartheta_2$ in the assertion (*2*) of Proposition 2.8 and then following equations (10), (12) and Proposition 2.6, we obtain assertion (*3*). Next, equation (10) and Proposition 2.7 infer that

$$'R(\phi\vartheta_1, \vartheta_2, \vartheta_3, \phi\vartheta_4) = {}'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + (\alpha^2 + \mu)\{u(\vartheta_4)u(\vartheta_1)g(\vartheta_2, \vartheta_3) \\ -u(\vartheta_2)u(\vartheta_4)g(\vartheta_1, \vartheta_3) - u(\vartheta_3)u(\vartheta_1)g(\vartheta_2, \vartheta_4) + u(\vartheta_1)u(\vartheta_2)u(\vartheta_3)u(\vartheta_4)\}$$

and

$$'R(\vartheta_1, \phi\vartheta_2, \phi\vartheta_3, \vartheta_4) = {}'R(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) + (\alpha^2 + \mu)\{u(\vartheta_2)u(\vartheta_3)g(\vartheta_1, \vartheta_4) \\ -u(\vartheta_2)u(\vartheta_4)g(\vartheta_1, \vartheta_3) - u(\vartheta_3)u(\vartheta_1)g(\vartheta_2, \vartheta_4) + u(\vartheta_1)u(\vartheta_2)u(\vartheta_3)u(\vartheta_4)\}.$$

From the last two equations, we immediately get assertion (*4*). Assertion (*5*) is obvious from equation (10) and Proposition 2.7. Hence, the proof of Proposition 2.8 is completed. □

Theorem 2.9. *Every $(RCS)_n$-manifold is integrable.*

Proof. Let an n-dimensional Riemannian manifold M be an $(RCS)_n$-manifold. In 1951, Nijenhuis [25] found that when a $(1,1)$ tensor field ϕ is given in a manifold M, then the relation

$$N_\phi(\vartheta_1, \vartheta_2) = [\phi\vartheta_1, \phi\vartheta_2] - \phi[\phi\vartheta_1, \vartheta_2] - \phi[\vartheta_1, \phi\vartheta_2] + \phi^2[\vartheta_1, \vartheta_2] \tag{13}$$

holds for any vector fields ϑ_1 and ϑ_2 of M [17, 32]. Here N_ϕ denotes the Nijenhuis tensor of ϕ and $[\cdot,\cdot]$ represents the Lie bracket, and it is defined as:

$$[\vartheta_1, \vartheta_2]f = \vartheta_1\vartheta_2(f) - \vartheta_2\vartheta_1(f)$$

for some smooth function f on M. We have

$$[\phi\vartheta_1, \phi\vartheta_2] = \nabla_{\phi\vartheta_1}\phi\vartheta_2 - \nabla_{\phi\vartheta_2}\phi\vartheta_1 \\ = (\nabla_{\phi\vartheta_1}\phi)(\vartheta_2) + \phi(\nabla_{\phi\vartheta_1}\vartheta_2) - (\nabla_{\phi\vartheta_2}\phi)(\vartheta_1) - \phi(\nabla_{\phi\vartheta_2}\vartheta_1),$$
$$\phi[\phi\vartheta_1, \vartheta_2] = \phi(\nabla_{\phi\vartheta_1}\vartheta_2) - \phi(\nabla_{\vartheta_2}\phi)(\vartheta_1) - \phi^2(\nabla_{\vartheta_2}\vartheta_1),$$
$$\phi[\vartheta_1, \phi\vartheta_2] = -\phi(\nabla_{\phi\vartheta_2}\vartheta_1) + \phi(\nabla_{\vartheta_1}\phi)(\vartheta_2) + \phi^2(\nabla_{\vartheta_1}\vartheta_2).$$

In view of above equations, equation (13) consider the following form

$$N_\phi(\vartheta_1, \vartheta_2) = (\nabla_{\phi\vartheta_1}\phi)(\vartheta_2) - (\nabla_{\phi\vartheta_2}\phi)(\vartheta_1) + \phi\{(\nabla_{\vartheta_2}\phi)(\vartheta_1) - (\nabla_{\vartheta_1}\phi)(\vartheta_2)\}.$$

This equation together with equation (10), Proposition (2.6) and Proposition (2.7) declares that $N_\phi = 0$. Hence, the structure (ϕ, ξ, u, g) on M is integrable. This proves our Theorem 2.9. □

3. Semisymmetric $(RCS)_n$-manifolds

This section deals with the study of semisymmetric $(RCS)_n$-manifolds.

Theorem 3.1. *An n-dimensional Riemannian concircular structure manifold is semisymmetric if and only if it is concircularly flat.*

Proof. Let M be an n-dimensional $(RCS)_n$-manifold. We have

$$(R(\vartheta_1, \vartheta_2) \cdot R)(\vartheta_3, \vartheta_4)\vartheta_5 = R(\vartheta_1, \vartheta_2)R(\vartheta_3, \vartheta_4)\vartheta_5 - R(R(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_4)\vartheta_5 \\ -R(\vartheta_3, R(\vartheta_1, \vartheta_2)\vartheta_4)\vartheta_5 - R(\vartheta_3, \vartheta_4)R(\vartheta_1, \vartheta_2)\vartheta_5, \tag{14}$$

where $R(\vartheta_1, \vartheta_2)$ acts on R as a curvature derivation. If possible, we suppose that M is semisymmetric, that, $R \cdot R = 0$. Then the above equation becomes

$$R(\vartheta_1, \vartheta_2)R(\vartheta_3, \vartheta_4)\vartheta_5 - R(R(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_4)\vartheta_5 \\ -R(\vartheta_3, R(\vartheta_1, \vartheta_2)\vartheta_4)\vartheta_5 - R(\vartheta_3, \vartheta_4)R(\vartheta_1, \vartheta_2)\vartheta_5 = 0.$$

Setting $\vartheta_1 = \vartheta_5 = \xi$ in the above equation and then following Proposition 2.7, we find

$$R(\vartheta_3, \vartheta_4)\vartheta_2 = (\alpha^2 + \mu)\{g(\vartheta_2, \vartheta_3)\vartheta_4 - g(\vartheta_2, \vartheta_4)\vartheta_3\}, \tag{15}$$

which shows that M is a space form. The contraction of (15) over ϑ_3 gives

$$S(\vartheta_4, \vartheta_2) = -(n-1)(\alpha^2 + \mu)g(\vartheta_4, \vartheta_2), \tag{16}$$

which after contraction over ϑ_4 and ϑ_2 assumes the form

$$r = -n(n-1)(\alpha^2 + \mu). \tag{17}$$

In 1940, Yano [34] defined a transformation, named as concircular transformation, which preserve the geodesic circles. It is proved that the concircular curvature tensor C [5, 19, 24, 34, 36] defined on M as:

$$C(\vartheta_1, \vartheta_2)\vartheta_3 = R(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{r}{n(n-1)}\{g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\}, \ \ \forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in \mathfrak{X}(M) \tag{18}$$

is unaltered under the concircular transformation. M is said to be concircularly flat if and only if $C = 0$. From equations (15), (17) and (18), we notice that the manifold M under consideration is concircularly flat. Conversely, we suppose that M is concircularly flat, then from equation (18) we get

$$R(\vartheta_1, \vartheta_2)\vartheta_3 = \frac{r}{n(n-1)}\{g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\}. \tag{19}$$

From equations (14) and (19), we infer that $R \cdot R = 0$. That is, the concircularly flat manifold M is semisymmetric. This completes the proof. □

It is well-known that every semisymmetric Riemannian manifold is Ricci semisymmetric ($R(\vartheta_1, \vartheta_2) \cdot S = 0$), but the converse is not true in general. This fact together with Theorem 3.1 affirm the following:

Corollary 3.2. *An $(RCS)_n$-manifold is Ricci semisymmetric if and only if it is an Einstein manifold.*

From equation (6), we have

$$\mathcal{L}_\xi g = 2\alpha\{g - u \otimes u\},$$

which, in consequence of equation (16), gives

$$\frac{1}{2}\mathcal{L}_\xi g + \beta_1 S = \beta_2 g + \beta_3 u \otimes u,$$

where $\beta_1 = \frac{\alpha}{2(n-1)(\alpha^2+\mu)}$, $\beta_2 = \frac{\alpha}{2}$ and $\beta_3 = -\alpha$ are smooth functions on M and $\mathcal{L}_\xi$ stands for the Lie derivative operator of g along ξ.

Recently, Blaga and Chen [4] introduced the notion of generalized solitons on Riemannian manifolds. A Riemannian metric g of M is said to be a generalized soliton [4] if

$$\frac{1}{2}\mathcal{L}_\xi g + \beta \cdot S = \gamma \cdot g + \delta \cdot \eta \otimes \eta,$$

where β, γ, δ are smooth functions on M and η is the dual 1-form of ξ. Thus, the above two equations state the following:

Corollary 3.3. *Every semisymmetric $(RCS)_n$-manifold admits a generalized soliton.*

Remark 3.4. *Corollary 3.3 proves the existence of generalized solitons on Riemannian manifolds.*

In 1918, Hermann Weyl [31] introduced the notion of conformal curvature tensor $\check{C}$, which measures the curvature of a semi-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of body changes, but rather only how the shape of the body is distorted by tidal force. The mathematical expression of the conformal curvature tensor on a Riemannian manifold M is given by

$$\check{C}(\vartheta_1,\vartheta_2)\vartheta_3 = R(\vartheta_1,\vartheta_2)\vartheta_3 + \frac{r}{(n-1)(n-2)}\{g(\vartheta_2,\vartheta_3)\vartheta_1 - g(\vartheta_1,\vartheta_3)\vartheta_2\}$$
$$-\frac{1}{n-2}\{S(\vartheta_2,\vartheta_3)\vartheta_1 - S(\vartheta_1,\vartheta_3)\vartheta_2 + g(\vartheta_2,\vartheta_3)Q\vartheta_1 - g(\vartheta_1,\vartheta_3)Q\vartheta_2\}, \tag{20}$$

where $dimM = n$ and $\vartheta_1,\vartheta_2,\vartheta_3 \in \mathfrak{X}(M)$. We assume that the $(RCS)_n$-manifold M is semisymmetric, then we obtain equations (15)-(17). The straightforward calculations after considering equations (15)-(17) and (20) infer that M is conformally flat. Thus, we can declare our finding as:

Theorem 3.5. *Every semisymmetric $(RCS)_n$-manifold is conformally flat.*

4. Conformally flat $(RCS)_n$-manifolds

This section is concerned with the study of an n-dimensional conformally flat $(RCS)_n$-manifold with $n \geq 3$.

A Riemannian manifold M of dimension n is said to be conformally flat if and only if $\check{C} = 0$. In this section, we suppose that the $(RCS)_n$-manifold is conformally flat, then equation (20) reduces to

$$R(\vartheta_1,\vartheta_2)\vartheta_3 = \frac{1}{n-2}[S(\vartheta_2,\vartheta_3)\vartheta_1 - S(\vartheta_1,\vartheta_3)\vartheta_2 + g(\vartheta_2,\vartheta_3)Q\vartheta_1 - g(\vartheta_1,\vartheta_3)Q\vartheta_2]$$
$$-\frac{r}{(n-1)(n-2)}\{g(\vartheta_2,\vartheta_3)\vartheta_1 - g(\vartheta_1,\vartheta_3)\vartheta_2\}.$$

Putting $\vartheta_1 = \vartheta_3 = \xi$ in the above equation and then the forthcoming equation together with Proposition 2.7 gives

$$Q\vartheta_2 = \left(\frac{r}{n-1} + \alpha^2 + \mu\right)\vartheta_2 - \left(\frac{r}{n-1} + n(\alpha^2+\mu)\right)u(\vartheta_2)\xi, \tag{21}$$

which deduces that M is a quasi-Einstein manifold with associated scalars $a = \frac{r}{n-1} + \alpha^2 + \mu$ and $b = -\left(\frac{r}{n-1} + n(\alpha^2+\mu)\right)$. This implies that $a + b = -(n-1)(\alpha^2+\mu) \neq 0$. Thus, we state our results as:

Theorem 4.1. *Every conformally flat $(RCS)_n$-manifold is a quasi-Einstein with $a + b \neq 0$.*

It is well-known that every three-dimensional Riemannian manifold M is conformally flat. Thus, we can state the following:

Corollary 4.2. *A three-dimensional $(RCS)_3$-manifold is a quasi-Einstein manifold.*

Differentiating equation (21) covariantly with respect to ϑ_1 and then considering equations (4) and (5), we lead to

$$(\nabla_{\vartheta_1}Q)(\vartheta_2) = \vartheta_1(a)\vartheta_2 + \vartheta_1(b)u(\vartheta_2)\xi + b\alpha u(\vartheta_2)\vartheta_1 - 2b\alpha u(\vartheta_1)u(\vartheta_2)\xi + b\alpha g(\vartheta_1,\vartheta_2)\xi, \tag{22}$$

which gives $\vartheta_1(r) = n\vartheta_1(a) + \vartheta_1(b)$. Also, we have

$$\vartheta_1(a) = \frac{\vartheta_1(r)}{n-1} + (2\alpha\mu+\sigma)u(\vartheta_1), \quad \vartheta_1(b) = -\frac{\vartheta_1(r)}{n-1} - n(2\alpha\mu+\sigma)u(\vartheta_1), \tag{23}$$

where Lemma 2.4 and equation (21) have been used. Since the $(RCS)_n$-manifold M with $n \geq 4$ is conformally flat, then it will possesses a divergence free conformal curvature tensor. Form equation (20) we have

$$(div\check{C})(\vartheta_1, \vartheta_2)\vartheta_3 = \frac{n-3}{n-2}\{(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) - (\nabla_{\vartheta_2} S)(\vartheta_1, \vartheta_3)\}$$
$$-\frac{n-3}{2(n-1)(n-2)}\{dr(\vartheta_1)g(\vartheta_2, \vartheta_3) - dr(\vartheta_2)g(\vartheta_1, \vartheta_3)\} = 0. \quad (24)$$

In consequence of equations (22) and (23), equation (24) assumes the form

$$\frac{1}{2(n-1)}\{\vartheta_1(r)g(\vartheta_2, \vartheta_3) - \vartheta_2(r)g(\vartheta_1, \vartheta_3)\} - \frac{1}{n-1}\{\vartheta_1(r)u(\vartheta_2) - \vartheta_2(r)u(\vartheta_1)\}u(\vartheta_3)$$
$$+(2\alpha\mu + \sigma - b\alpha)\{u(\vartheta_1)g(\vartheta_2, \vartheta_3) - u(\vartheta_2)g(\vartheta_1, \vartheta_3)\} = 0.$$

Putting ξ in lieu of ϑ_3 in the above equation and then using (11), we obtain

$$\vartheta_1(r)u(\vartheta_2) = \vartheta_2(r)u(\vartheta_1) \implies \vartheta_1(r) = \xi(r)u(\vartheta_1) \iff Dr = \xi(r)\xi.$$

The covariant derivative of the above equation gives

$$\nabla_{\vartheta_1} Dr = \vartheta_1(\xi(r))\xi + \xi(r)\nabla_{\vartheta_1}\xi.$$

The contraction of the above equation along ϑ_1 infers that

$$\triangle r = \xi(\xi(r)) + \alpha(n-1)\xi(r).$$

Let us take $\xi = \frac{\partial}{\partial t}$, the above equation reduces to

$$\triangle r = \frac{\partial^2 r}{\partial t^2} + \alpha(n-1)\frac{\partial r}{\partial t}. \quad (25)$$

This states the following:

Theorem 4.3. *Every conformally flat $(RCS)_n$-manifold of dimension $n > 3$ satisfies the partial differential equation (25).*

Corollary 4.4. *The scalar curvature of an n-dimensional conformally flat $(RCS)_n$-manifold with $n > 3$ is harmonic if and only if $\frac{\partial^2 r}{\partial t^2} + \alpha(n-1)\frac{\partial r}{\partial t} = 0$.*

5. Existence of Riemannian concircular structure manifolds

In this section, we prove the existence of Riemannian concircular structure manifolds.
Let us consider an orthonormal frame field on M and then contracting equation (1), we lead to

$$r = na + b, \quad (26)$$

where r is the scalar curvature of M.

Suppose that $(QE)_n$-manifold M has a divergence free Riemann curvature tensor R and ξ is a unit vector such that $g(\vartheta_1, \xi) = u(\vartheta_1)$, $\forall\ \vartheta_1 \in \mathfrak{X}(M)$. Then from equation (2) we have

$$(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) - (\nabla_{\vartheta_2} S)(\vartheta_1, \vartheta_3) = 0. \quad (27)$$

The contraction of this equation over ϑ_2 and ϑ_3 shows that r = constant, and hence equation (26) gives $n\vartheta_1(a) = -\vartheta_1(b)$. Taking covariant derivative of equation (1) along ϑ_1, we get

$$(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) = \vartheta_1(a)g(\vartheta_2, \vartheta_3) + \vartheta_1(b)u(\vartheta_2)u(\vartheta_3)$$
$$+b\{(\nabla_{\vartheta_1} u)(\vartheta_2)u(\vartheta_3) + u(\vartheta_2)(\nabla_{\vartheta_1} u)(\vartheta_3)\}, \quad (28)$$

where $\vartheta_1(a) = g(\vartheta_1, Da)$, D denotes the gradient operator of g. Equations (27) and (28) give

$$\begin{aligned} & b\left[u(\vartheta_3)\{(\nabla_{\vartheta_1}u)(\vartheta_2) - (\nabla_{\vartheta_2}u)(\vartheta_1)\} + u(\vartheta_2)(\nabla_{\vartheta_1}u)(\vartheta_3) - u(\vartheta_1)(\nabla_{\vartheta_2}u)(\vartheta_3)\right] \\ & +\vartheta_1(a)g(\vartheta_2, \vartheta_3) - \vartheta_2(a)g(\vartheta_1, \vartheta_3) + u(\vartheta_3)\{\vartheta_1(b)u(\vartheta_2) - \vartheta_2(b)u(\vartheta_1)\} = 0. \end{aligned} \tag{29}$$

Changing ϑ_3 with ξ in the above equation and then considering the facts $n\vartheta_1(a) = -\vartheta_1(b)$, $u(\xi) = 1$ and $(\nabla_{\vartheta_1}u)(\xi) = 0$, we lead

$$(n-1)\vartheta_1(a)u(\vartheta_2) - (n-1)\vartheta_2(a)u(\vartheta_1) + b\{(\nabla_{\vartheta_2}u)(\vartheta_1) - (\nabla_{\vartheta_1}u)(\vartheta_2)\} = 0. \tag{30}$$

Again setting $\vartheta_2 = \xi$ in the above equation, we get

$$\begin{aligned} & (n-1)\{\vartheta_1(a) - \xi(a)u(\vartheta_1)\} + b(\nabla_\xi u)(\vartheta_1) = 0 \\ & \quad \Longleftrightarrow b\nabla_\xi \xi = -(n-1)\{Da - \xi(a)\xi\}. \end{aligned} \tag{31}$$

This infers that the vector field ξ is geodesic ($\nabla_\xi \xi = 0$) if and only if the gradient of scalar a (or b or both a and b) is pointwise collinear with the vector field ξ of M. Next, we consider an orthonormal frame field on M and then contracting equation (29) over ϑ_2 and ϑ_3, we assume the following form

$$\vartheta_1(a) - n\xi(a)u(\vartheta_1) + b\{(\nabla_\xi u)(\vartheta_1) + u(\vartheta_1)div\xi\} = 0, \tag{32}$$

since $(\nabla_{\vartheta_1}u)(\xi) = 0$ and the scalar curvature of M is constant. Putting $\vartheta_1 = \xi$ in the above equation, we infer

$$b\, div\xi = (n-1)\xi(a). \tag{33}$$

In view of (33), equation (32) reduces to

$$b(\nabla_\xi u)(\vartheta_1) = -\vartheta_1(a) + \xi(a)u(\vartheta_1). \tag{34}$$

From equations (31) and (34), we conclude that

$$\vartheta_1(a) = \xi(a)u(\vartheta_1), \;\; \nabla_\xi \xi = 0, \;\; (\nabla_\xi u)(\vartheta_1) = 0. \tag{35}$$

Using (35) in (30), we obtain

$$(\nabla_{\vartheta_1}u)(\vartheta_1) = 0, \tag{36}$$

since $b \neq 0$. This infers that the 1-form u is closed. In consequence of equations (35) and (36), equation (29) becomes

$$\xi(a)\{u(\vartheta_1)g(\vartheta_2, \vartheta_3) - u(\vartheta_2)g(\vartheta_1, \vartheta_3)\} + b\{u(\vartheta_2)(\nabla_{\vartheta_1}u)(\vartheta_3) - u(\vartheta_1)(\nabla_{\vartheta_2}u)(\vartheta_3)\} = 0.$$

Setting $\vartheta_1 = \xi$ in the above equation and recalling the equation $g(\xi, \xi) = 1$, we obtain

$$(\nabla_{\vartheta_2}u)(\vartheta_3) = \alpha\{g(\vartheta_2, \vartheta_3) - u(\vartheta_2)u(\vartheta_3)\}, \tag{37}$$

where $\alpha = -\frac{\xi(b)}{nb} \neq 0$. Equation (37) infers that the unit vector field ξ of M is a concircular vector field. Next we suppose that M admits a $(1,1)$ tensor field ϕ such that $\phi\vartheta_1 = \frac{1}{\alpha}\nabla_{\vartheta_1}\xi$, then we notice that the structure (ϕ, ξ, u, g) of M satisfies $\phi^2 = I - u \otimes \xi$, $g(\xi, \xi) = 1$ and $g(\phi\,\cdot, \phi\,\cdot) = g(\cdot\,, \cdot) - u \otimes u$. Thus, we can state the following:

Theorem 5.1. *An n-dimensional quasi-Einstein manifold M with divergence free Riemann curvature tensor is a Riemannian concircular structure manifold.*

From equation (35) we have $Da = \xi(a)\xi$, which gives

$$\nabla_{\vartheta_1}Da = \vartheta_1(\xi(a))\xi + \xi(a)\alpha(\vartheta_1 - u(\vartheta_1)\xi).$$

The contraction of above equation over ϑ_1 gives

$$\triangle a = \xi(\xi(a)) + (n-1)\alpha\xi(a). \tag{38}$$

This equation suggests us to select the values of a (or b or both a and b). Thus we state the following:

Proposition 5.2. *Let an n-dimensional quasi-Einstein manifold admit the divergence free Riemann curvature tensor. Then the associated scalar a of $(QE)_n$-manifold is governed by the partial differential equation (38).*

Let us suppose that the Riemannian manifold M be a $(QE)_n$-manifold. If the concircular curvature tensor C of M is divergence free, then $divC = 0$. Differentiating equation (18) covariantly along the vector field ϑ_4, we have

$$(\nabla_{\vartheta_4}C)(\vartheta_1,\vartheta_2)\vartheta_3 = (\nabla_{\vartheta_4}R)(\vartheta_1,\vartheta_2)\vartheta_3 - \frac{\vartheta_4(r)}{n(n-1)}\{g(\vartheta_2,\vartheta_3)\vartheta_1 - g(\vartheta_1,\vartheta_3)\vartheta_2\}. \tag{39}$$

Taking an orthonormal frame field on M and then contracting the above equation over ϑ_4, we find

$$(div\,C)(\vartheta_1,\vartheta_2)\vartheta_3 = (div\,R)(\vartheta_1,\vartheta_2)\vartheta_3 - \frac{1}{n(n-1)}\{\vartheta_1(r)g(\vartheta_2,\vartheta_3) - \vartheta_2(r)g(\vartheta_1,\vartheta_3)\}.$$

By the hypothesis that the concircular curvature tensor of M is divergence free, then from equations (2) and (39) we reach to

$$(\nabla_{\vartheta_1}S)(\vartheta_2,\vartheta_3) - (\nabla_{\vartheta_2}S)(\vartheta_1,\vartheta_3) = \frac{1}{n(n-1)}\{\vartheta_1(r)g(\vartheta_2,\vartheta_3) - \vartheta_2(r)g(\vartheta_1,\vartheta_3)\}.$$

Again, consider an orthonormal frame field on M and then contracting the above equation over ϑ_2 and ϑ_3, we obtain

$$\vartheta_1(r) = 0 \Longleftrightarrow r = \text{constant},$$

where identity $(divQ)(\vartheta_1) = \frac{1}{2}\vartheta_1(r)$ is used. The last two equations inform that the Ricci tensor S of M is of Codazzi type, that is, $(\nabla_{\vartheta_1}S)(\vartheta_2,\vartheta_3) = (\nabla_{\vartheta_2}S)(\vartheta_1,\vartheta_3)$. This result together with Theorem 5.1 state the following:

Corollary 5.3. *Let M be an n-dimensional quasi-Einstein manifold with $n \geq 3$. If the concircular curvature tensor of M is divergence free, then M is a Riemannian concircular structure manifold.*

Example 5.4. *Three-dimensional Riemannian concircular structure manifold.*

Let $M^3 = \{(x,y,z)|x,y,z \in \mathbb{R}$ and $z \neq 0\}$ be a differentiable manifold of dimension 3, where $\mathbb{R}$ is a real space. If $\theta_1 = e^{-f(z)}\frac{\partial}{\partial x}$, $\theta_2 = e^{-f(z)}\frac{\partial}{\partial y}$ and $\theta_3 = \frac{\partial}{\partial z}$ are the vector fields of M^3, then at each point of M^3 they form a basis of the tangent space of M^3. Here we consider that $f(z)$ is a non-vanishing smooth function of z and $f'(z) \neq 0$. Suppose that the associated metric g of M^3 is defined by $g = e^{2f(z)}\{dx\otimes dx + dy\otimes dy\} + dz\otimes dz$ and

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases},$$

where $i,j = 1,2,3$. Since the metric g is a Riemannian metric, thus M^3 equipped with g is a three-dimensional Riemannian manifold. Let u be a 1-form associated with the unit vector field $\xi = \theta_3$ such that $g(\vartheta_1,\theta_3) = u(\vartheta_1)$. Then we have $\nabla_{\theta_i}\theta_3 = f'(z)\{\theta_i - g(\theta_i,\xi)\xi\}$ for $i = 1,2,3$, where ∇ is the Levi-Civita connection of the metric g. This infers that (M^3,g) admits a concircular vector field ξ. Suppose that ϕ is a $(1,1)$ tensor field on M^3 such that $\phi(\theta_1) = \theta_1$, $\phi(\theta_2) = \theta_2$, $\phi(\theta_3) = 0$ and $f'(z)\phi\theta_i = \nabla_{\theta_i}\theta_3$, then by straightforward calculations we notice that the relations $\phi^2 = I - u\otimes\theta_3$, $g(\theta_3,\theta_3) = 1$ and $g(\phi\,\cdot,\phi\,\cdot) = g(\cdot,\cdot) - u\otimes u$ for all θ_i, $i = 1,2,3$ hold on M^3. Here I represents the identity transformation. Hence, we conclude that M^3 endowed with the structure (ϕ,θ_3,u,g) is a three-dimensional Riemannian concircular structure manifold.

Example 5.5. *Four-dimensional Riemannian concircular structure manifold.*

Let $\mathbb{R}$ be a real space and $M^4 = \{(x^1, x^2, x^3, x^4)|x^1, x^2, x^3, x^4 \in \mathbb{R}$ and $x^4 \neq 0\}$ is a differentiable manifold of dimension 4. Also, let $\Upsilon_1 = \partial_{x^1}$, $\Upsilon_2 = \partial_{x^2}$, $\Upsilon_3 = \partial_{x^3}$ and $\Upsilon_4 = \partial_{x^4} + \alpha x^1\partial_{x^1} + \alpha x^2\partial_{x^2} + \alpha x^3\partial_{x^3}$ be the vector fields of M^4, where α is a non-zero scalar and ∂_{x^i} denotes the partial derivative along x^i for $i = 1, 2, 3, 4$. Then $\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4\}$ forms a basis of the tangent space at each point of M^4. According to the fundamental theory of Riemannian geometry, there exists a Levi-Civita connection ∇ of the Riemannian metric g, where $g = g_{ij}dx^i \otimes dx^j = (1 + \alpha)\{dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3\} + dx^4 \otimes dx^4$ and $g_{ij} = \delta_{ij}$ for all $i, j = 1, 2, 3, 4$. The non-vanishing components of Lie bracket are $[\Upsilon_i, \Upsilon_4] = \alpha\Upsilon_i$, where $i = 1, 2, 3$. The Koszul's formula together with above informations infer that $\nabla_{\Upsilon_i}\Upsilon_4 = \alpha\Upsilon_i$, for $i = 1, 2, 3$ and $\nabla_{\Upsilon_4}\Upsilon_4 = 0$. This shows that Υ_4 satisfies the relation $\nabla_{\Upsilon_i}\Upsilon_4 = \alpha\{\Upsilon_i - g(\Upsilon_i, \Upsilon_4)\Upsilon_4\}$, $i \in \{1, 2, 3, 4\}$, and hence it is a concircular vector field on M^4. Let M^4 admit a $(1, 1)$ tensor field ϕ such that $\phi\Upsilon_i = \Upsilon_i$, $\phi\Upsilon_4 = 0$ and $\alpha\phi\Upsilon_i = \nabla_{\Upsilon_i}\Upsilon_4$ for $i = 1, 2, 3, 4$. Then we observe that the relations $\phi^2 = I - u \otimes \Upsilon_4$, $g(\Upsilon_4, \Upsilon_4) = 1$ and $g(\phi\,\cdot, \phi\,\cdot) = g(\cdot, \cdot) - u \otimes u$ for θ_i, $i = 1, 2, 3, 4$ hold on M^4. Here u denotes the 1-form associated with the unit vector field Υ_4, that is, $u(\Upsilon_i) = g(\Upsilon_i, \Upsilon_4)$, $1 \leq i \leq 4$. Thus, the manifold M^4 equipped with structure (ϕ, Υ_4, u, g) is a four-dimensional Riemannian concircular structure manifold.

Example 5.6. *Yano [33] defined a semi-symmetric metric connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold M as*

$$\tilde{\nabla}_{\vartheta_1}\vartheta_2 = \nabla_{\vartheta_1}\vartheta_2 + u(\vartheta_2)\vartheta_1 - g(\vartheta_1, \vartheta_2)\xi,$$

and studied its some properties. Here u, ∇, g and ξ denote the 1-form, Levi-Civita connection of the Riemannian metric g and the vector field such that $u(\cdot) = g(\cdot, \xi)$. Since then, the properties of semi-symmetric metric connection on different structures have been explored by several geometers. A linear connection $\tilde{\nabla}$ on M is said to be a semi-symmetric metric ξ-connection [8–10] if $\tilde{\nabla}_{\vartheta_1}\xi = 0 \implies \nabla_{\vartheta_1}\xi = \alpha\{\vartheta_1 - u(\vartheta_1)\xi\}$, where $\alpha = -1$. This shows that the Riemannian manifold M endowed with a semi-symmetric metric ξ-connection admits a concircular vector field ξ. Suppose $\alpha\phi = \nabla\xi$ and $g(\xi, \xi)$=1, then it is obvious that the structure (ϕ, ξ, u, g) is a Riemannian concircular structure on M. Hence, M equipped with (ϕ, ξ, u, g) is an $(RCS)_n$-manifold.

Example 5.7. *In [1], Bahadir and Chaubey introduced the notion of generalized symmetric metric connection in semi-Riemannian setting. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is said to be a generalized symmetric metric connection [1] if*

$$\bar{\nabla}_{\vartheta_1}\vartheta_2 = \nabla_{\vartheta_1}\vartheta_2 + \psi_1\{u(\vartheta_2)\vartheta_1 - g(\vartheta_1, \vartheta_2)\xi\} + \psi_2\{u(\vartheta_2)\phi\vartheta_1 - g(\phi\vartheta_1, \vartheta_2)\xi\},$$

where ψ_1 and ψ_2 are smooth functions on M. If we choose $\psi_2 = 0$ in the above equation, then the generalized symmetric metric connection reduces to the generalized semi-symmetric metric connection. Let us suppose that the generalized symmetric metric connection be a ξ-connection ($\bar{\nabla}\xi = 0$) and $\psi_2 = 0$. Then we get $\nabla_{\vartheta_1}\xi = \alpha\{\vartheta_1 - u(\vartheta_1)\xi\}$, where $\alpha = -\psi_1 \neq 0$. Let us assume that $\alpha\phi = \nabla\xi$ and $g(\xi, \xi) = 1$, then we notice that the structure (ϕ, ξ, u, g) satisfies equation (11) and hence it is a Riemannian concircular structure and the corresponding manifold M becomes $(RCS)_n$-manifold.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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