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On Pointwise Quasi Bi-Slant Submanifolds

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Abstract. In this paper, we introduce a new class of submanifolds which are called *pointwise quasi bi-slant submanifolds* in almost Hermitian manifolds which extends quasi bi-slant, bi-slant, hemi-slant, semi-slant and slant submanifolds in a very natural way. Several basic results in this respect are proved in this paper. Moreover, we obtain some conditions of the distributions which is involved in the definition of the new submanifolds. We also get some results for totally geodesic and mixed totally geodesic conditions for pointwise quasi bi-slant submanifolds. Finally, we illustrate some examples in order to guarantee the new kind of submanifolds.

1. Introduction

One of the interesting and active research topic is the theory of submanifolds in differential geometry. The theory has many interesting applications such as economic modelling, mechanics, image processing and computer design. In the theory of submanifolds, the notion of slant submanifolds was introduced by B. Y. Chen as a natural generalizations of holomorphic immersions and totally real immersions. Most of the studies and examples related to slant submanifolds can be found in [5].

In the course of time, this interesting notion has been studied broadly by several geometers ([6], [9], [11], [12], [22], [25], [26]). As a generalization of slant submanifolds, there are several kinds of submanifolds: semi-slant submanifolds ([2], [14], [23]), generic submanifolds ([20], [21]), hemi-slant submanifolds ([13], [28]), bi-slant submanifolds ([3], [4], [27]), quasi hemi-slant submanifolds ([18], [19]), quasi bi-slant submanifolds ([1], [17]). In 2012, B. Y. Chen and O. J. Garay [7] studied pointwise slant submanifolds in almost Hermitian manifolds which was first proposed by F. Etayo [10] under the notion of quasi slant submanifold.

In 2013, B. Şahin [24] defined the notion of pointwise semi-slant submanifolds. In 2014, K. S. Park ([15], [16]) defined the notion of pointwise almost h-slant submanifolds and pointwise almost h-semi-slant submanifolds in an almost quaternionic Hermitian manifold. The author obtained some geometrically important properties of these manifolds.

On the other hand, in Akyol and Beyendi [1] initiated the study of quasi bi-slant submanifolds of an almost contact metric manifold by generalizing slant, semi-slant, hemi-slant and bi-slant submanifolds. (See also: [17]).

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Taking into account of the above studies, we introduce the notion of pointwise quasi-bi-slant submanifolds, in which the tangent bundle consists of one invariant and two slant distributions which have slant functions instead of slant angle, of almost Hermitian manifolds as a generalization of quasi bi-slant, bi-slant, hemi-slant, semi-slant and slant submanifolds in the present paper.

The summary of the present paper is as follows: In Sect. 2, the basic notions, important definitions and some properties of both almost Hermitian manifolds and the geometry of submanifolds are given. In Sect. 3, we define the notion of pointwise quasi-bi-slant submanifolds, giving a non-trivial example and obtain some basic results for the next sections. In Sect. 4, we deals with main theorems related to the geometry of distributions in this section. Finally, we mention another non-trivial example for this new class of submanifolds.

2. Preliminaries

In this section, we give the definition of a Kaehler manifold and some background on submanifolds theory.

Let $\widetilde{M}$ be a smooth manifold of dimension $2m$. Then, $\widetilde{M}$ is said to be an almost Hermitian manifold if it admits a tensor field J of type $(1,1)$ and a Riemannian metric g on $\widetilde{M}$ satisfying

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \tag{1}$$

for any vector fields X, Y on $T\widetilde{M}$, where I denotes the identity transformation. The fundamental 2 -form Ω on $\widetilde{M}$ is defined by $\Omega(X, Y) = g(X, JY)$, $\forall X, Y \in \Gamma(T\widetilde{M})$, with $\Gamma(T\widetilde{M})$ being the section of tangent bundle $T\widetilde{M}$ of $\widetilde{M}$. An almost Hermitian manifold $\widetilde{M}$ is called a Kaehler manifold [29] if

$$(\widetilde{\nabla}_X J)Y = 0, \tag{2}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$ with respect to g.

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let $\mathcal{A}$ and h denote the shape operator and second fundamental form, respectively, of immersion of M into $\widetilde{M}$. The Gauss and Weingarten formulas of M into $\widetilde{M}$ are given by [6]

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3}$$

and

$$\widetilde{\nabla}_X V = -\mathcal{A}_V X + \nabla^{\perp}_X V, \tag{4}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, $\mathcal{A}_V$ and h are related by

$$g(h(X, Y), V) = g(\mathcal{A}_V X, Y), \tag{5}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now, we have the following definition from [7]:

Definition 2.1. *A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is called pointwise slant if, at each point $p \in M$, the Wirtinger angle $\theta(X)$ is independent of the choice of nonzero vector $X \in T^*_pM$, where T^*_pM is the tangent space of nonzero vectors. In this case, θ is called slant function of M (cf. [7]).*

Finally, a submanifold M is called (i) $(\mathfrak{D}_1, \mathfrak{D}_2)$-mixed totally geodesic if $h(Z, W) = 0$, for any $Z \in \Gamma(\mathfrak{D}_1)$ and $W \in \Gamma(\mathfrak{D}_2)$ (ii) $\mathfrak{D}$-totally geodesic if it is $(\mathfrak{D}, \mathfrak{D})$-mixed totally geodesic.

3. Pointwise quasi bi-slant submanifolds

In this section, we define a new class of submanifolds which can be considered as a generalization of quasi bi-slant, bi-slant, hemi-slant, semi-slant, slant etc. submanifolds. We also give some non-trivial examples for illustrating the notion.

First, we have the following definition:

Definition 3.1. *Let M be an isometrically immersed submanifold in a Kaehler manifold $\widetilde{M}$. Then we say that M is a pointwise quasi bi-slant submanifold if it is furnished with three orthogonal distributions $(\mathfrak{D}, \mathfrak{D}_{\theta_1}, \mathfrak{D}_{\theta_2})$ satisfying the conditions:*

(i) $TM = \mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}$,

(ii) The distribution $\mathfrak{D}$ is invariant, i.e. $J\mathfrak{D} = \mathfrak{D}$,

(iii) $J\mathfrak{D}_{\theta_1} \perp \mathfrak{D}_{\theta_2}$ *and* $J\mathfrak{D}_{\theta_2} \perp \mathfrak{D}_{\theta_1}$,

(iv) The distributions $\mathfrak{D}_{\theta_1}, \mathfrak{D}_{\theta_2}$ are pointwise slant distributions with slant function θ_1, θ_2, respectively.

The pair θ_1, θ_2 of slant functions is called the bi-slant functions. A pointwise quasi bi-slant submanifold M is called proper if its bi-slant function satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, and both θ_1, θ_2 are not constant on M.

If we represent by d_1, d_2 and d_3 the dimensions of $\mathfrak{D}$, $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$, respectively, then from our generalized definition of pointwise quasi bi-slant submanifold M, we can obtain the following known subclasses as remarks:

Remark 3.2. *M reduces to a pointwise bi-slant submanifold [8] for $d_1 = 0$.*

Remark 3.3. *M is a quasi bi-slant submanifold [1] for $d_1 \neq 0$ and $0 < d_2, d_3 < \frac{\pi}{2}$ where d_2 and d_3 are different slant angles.*

Remark 3.4. *M is called a pointwise-slant submanifold [7] for $d_1 = d_2 = 0$.*

Remark 3.5. *M is a bi-slant submanifold [3] for $d_1 = 0$ and $0 < d_2, d_3 < \frac{\pi}{2}$ where d_2 and d_3 are different slant angles.*

Remark 3.6. *M is simply to a slant submanifold [5] for $d_1 = d_2 = 0$ and $0 < d_3 < \frac{\pi}{2}$.*

Now, we present a non-trivial example of proper pointwise quasi bi-slant submanifold in $\mathbb{R}^{12}$.

Example 3.7. *For $t, s \neq 0, 1$; $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$, consider a submanifold $\mathcal{M}$ of a Kaehler manifold $\mathbb{R}^{12}$ given by the equations*

$$x_1 = s\cos\theta_2, y_1 = t\cos\theta_1, x_2 = s\sin\theta_2, y_2 = t\sin\theta_1, x_3 = s\cos\theta_1, y_3 = t\cos\theta_2$$
$$x_4 = s\sin\theta_1, y_4 = t\sin\theta_2, x_5 = \frac{1}{\sqrt{3}}\theta_1 + \frac{1}{\sqrt{2}}\theta_2, y_5 = \frac{1}{\sqrt{2}}\theta_1 + \frac{1}{\sqrt{3}}\theta_2, x_6 = u, y_6 = v.$$

Then the tangent bundle of $\mathcal{M}$ is spanned by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ where

$$e_1 = \cos\theta_1\frac{\partial}{\partial y_1} + \sin\theta_1\frac{\partial}{\partial y_2} + \cos\theta_2\frac{\partial}{\partial y_3} + \sin\theta_2\frac{\partial}{\partial y_4},$$
$$e_2 = \cos\theta_2\frac{\partial}{\partial x_1} + \sin\theta_2\frac{\partial}{\partial x_2} + \cos\theta_1\frac{\partial}{\partial x_3} + \sin\theta_1\frac{\partial}{\partial x_4},$$
$$e_3 = -t\sin\theta_1\frac{\partial}{\partial y_1} + t\cos\theta_1\frac{\partial}{\partial y_2} - s\sin\theta_1\frac{\partial}{\partial x_3} + s\cos\theta_1\frac{\partial}{\partial x_4} + \frac{1}{\sqrt{3}}\frac{\partial}{\partial x_5} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_5},$$
$$e_4 = -s\sin\theta_2\frac{\partial}{\partial x_1} + s\cos\theta_2\frac{\partial}{\partial x_2} - t\sin\theta_2\frac{\partial}{\partial y_3} + t\cos\theta_2\frac{\partial}{\partial y_4} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5} + \frac{1}{\sqrt{3}}\frac{\partial}{\partial y_5},$$
$$e_5 = \frac{\partial}{\partial x_6}, \quad e_6 = \frac{\partial}{\partial y_6}.$$

Then $\mathfrak{D} = \{e_5, e_6\}$ is holomorphic distribution and $\mathfrak{D}_{\theta_1} = span\{e_1, e_2\}$, $\mathfrak{D}_{\theta_2} = span\{e_3, e_4\}$ are pointwise bi-slant distributions with slant functions $\cos^{-1}[\cos(\theta_1 - \theta_2)]$ and $\cos^{-1}(\frac{1}{6(t^2+s^2)+5})$. Hence $\mathcal{M}$ is a proper 6-dimensional pointwise quasi bi-slant submanifold in $\mathbb{R}^{12}$.

Let M be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then, for any $X \in \Gamma(TM)$, we have

$$X = \mathcal{P}X + \mathcal{Q}X + \mathcal{R}X, \tag{6}$$

where $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ denotes the projections on the distributions $\mathfrak{D}$, $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$, respectively.

$$JX = \mathcal{T}X + \mathcal{F}X, \tag{7}$$

where $\mathcal{T}X$ and $\mathcal{F}X$ are tangential and normal components on M. By using (6) and (7), we get immediately

$$JX = \mathcal{T}\mathcal{P}X + \mathcal{T}\mathcal{Q}X + \mathcal{F}\mathcal{Q}X + \mathcal{T}\mathcal{R}X + \mathcal{F}\mathcal{R}X, \tag{8}$$

here since $J\mathfrak{D} = \mathfrak{D}$, we have $\mathcal{F}\mathcal{P}X = 0$. Thus we get

$$J(TM) = \mathfrak{D} \oplus \mathcal{T}\mathfrak{D}_{\theta_1} \oplus \mathcal{T}\mathfrak{D}_{\theta_2} \tag{9}$$

and

$$T^{\perp}M = \mathcal{F}\mathfrak{D}_{\theta_1} \oplus \mathcal{F}\mathfrak{D}_{\theta_2} \oplus \mu, \tag{10}$$

where μ is the orthogonal complement of $\mathcal{F}\mathfrak{D}_{\theta_1} \oplus \mathcal{F}\mathfrak{D}_{\theta_2}$ in $T^{\perp}M$ and $J\mu = \mu$. Also, for any $Z \in T^{\perp}M$, we have

$$JZ = \mathcal{B}Z + \mathcal{C}Z, \tag{11}$$

where $\mathcal{B}Z \in \Gamma(TM)$ and $\mathcal{C}Z \in \Gamma(T^{\perp}M)$.

Taking into account of the condition (iii) in Definition (3.1), (7) and (11), we obtain the followings:
$T\mathfrak{D}_{\theta_1} \subset \mathfrak{D}_{\theta_1}$, $T\mathfrak{D}_{\theta_2} \subset \mathfrak{D}_{\theta_2}$, $\mathcal{B}\mathcal{F}\mathfrak{D}_{\theta_1} = \mathfrak{D}_{\theta_1}$, $\mathcal{B}\mathcal{F}\mathfrak{D}_{\theta_2} = \mathfrak{D}_{\theta_2}$.

With the help of (7) and (11), we obtain the following Lemma:

Lemma 3.8. *Let M be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then, we have*

$$\textbf{(a)}\ \mathcal{B}\mathcal{F}U_1 = -\sin^2\theta_1 U_1, \quad \textbf{(b)}\ \mathcal{B}\mathcal{F}U_2 = -\sin^2\theta_2 U_2,$$

$$\textbf{(c)}\ \mathcal{T}^2U_1 + \mathcal{B}\mathcal{F}U_1 = -U_1, \quad \textbf{(d)}\ \mathcal{T}^2U_2 + \mathcal{B}\mathcal{F}U_2 = -U_2,$$

$$\textbf{(e)}\ \mathcal{F}\mathcal{T}U_1 + \mathcal{C}\mathcal{F}U_1 = 0, \quad \textbf{(f)}\ \mathcal{F}\mathcal{T}U_2 + \mathcal{C}\mathcal{F}U_2 = 0,$$

for any $U_1 \in \mathfrak{D}_{\theta_1}$ and $U_2 \in \mathfrak{D}_{\theta_2}$.

By using (2), Definition (3.1), (7) and (11), we obtain the following Lemma:

Lemma 3.9. *Let M be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then, we have*

(i) $\mathcal{T}_i^2 U_i = -\cos^2\theta_i U_i$,
(ii) $g(\mathcal{T}_i U_i, \mathcal{T}_i V_i) = (\cos^2\theta_i) g(U_i, V_i)$,
(iii) $g(\mathcal{F}_i U_i, \mathcal{F}_i V_i) = (\sin^2\theta_i) g(U_i, V_i)$

for any $i = 1, 2$, $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $U_2, V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$.

Proof. The proof follows using similar steps as in Proposition 2.8 of [7]. □

Lemma 3.10. *Let M be a pointwise quasi bi-slant submanifold of a Kaehler manifold $\tilde{M}$ with pointwise slant distributions $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$ with distinct slant functions θ_1 and θ_2, respectively. Then*

(i) For any $X, Y \in \mathfrak{D}_{\theta_1}$ and $Z \in \mathfrak{D}_{\theta_2}$, we have

$$\left(\sin^2\theta_1 - \sin^2\theta_2\right) g(\nabla_X Y, Z) = g\left(A_{FT_2Z}Y - A_{FZ}T_1Y, X\right) + g\left(A_{FT_1Y}Z - A_{FY}T_2Z, X\right)$$

(ii) For $Z, W \in \mathfrak{D}_{\theta_2}$ and $X \in \mathfrak{D}_{\theta_1}$, we have

$$\left(\sin^2\theta_2 - \sin^2\theta_1\right) g(\nabla_Z W, X) = g\left(A_{FT_2W}X - A_{FW}T_1X, Z\right) + g\left(A_{FT_1X}W - A_{FX}T_2W, Z\right).$$

4. Main Results

Theorem 4.1. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, the invariant distribution $\mathfrak{D}$ defines a totally geodesic foliation on M if and only if*

$$g(A_{\mathcal{F}Q_1Z}JU - \nabla_{JU}\mathcal{T}Q_1Z, V) = g(\mathcal{T}\nabla_{JU}V + \mathcal{B}h(JU, V), Q_2Z)$$

and

$$\nabla_{JU}\mathcal{B}W - A_{CW}JU \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}),$$

for any $U, V \in \mathfrak{D}$, $Z = Q_1Z + Q_2Z \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^\perp$.

Proof. From equations (3) and (7), we have

$$g(\widetilde{\nabla}_{JU}JV, Z) = -g(\widetilde{\nabla}_{JU}V, \mathcal{T}Q_1Z) - g(\widetilde{\nabla}_{JU}V, \mathcal{F}Q_1Z) + g(J(\nabla_{JU}V + h(JU, V)), Q_2Z),$$

for any $U, V \in \mathfrak{D}, Z = Q_1Z + Q_2Z \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2})$. Further, taking into account of (4), the above equation becomes

$$g(\widetilde{\nabla}_{JU}JV, Z) = g(\nabla_{JU}\mathcal{T}Q_1Z - A_{\mathcal{F}Q_1Z}JU, V) + g(\mathcal{T}\nabla_{JU}V + \mathcal{B}h(JU, V), Q_2Z). \tag{12}$$

Now, for any $W \in \Gamma(TM)^\perp$ and $U, V \in \Gamma(\mathfrak{D})$, we have

$$\begin{aligned} g(\widetilde{\nabla}_{JU}JV, W) &= -g(\widetilde{\nabla}_{JU}V, \mathcal{B}W) - g(\widetilde{\nabla}_{JU}V, CW) \\ &= g(V, \nabla_{JU}\mathcal{B}W + h(JU, \mathcal{B}W)) + g(-A_{CW}JU + \nabla^\perp_{JU}CW, V) \\ &= g(\nabla_{JU}\mathcal{B}W - A_{CW}JU, V). \end{aligned} \tag{13}$$

The proof comes from (12) and (13). □

As a direct consequence of the above theorem, we can give the following result:

Corollary 4.2. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, we have*

$$J\nabla_{JU}V + \mathcal{B}h(JU, V) \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}),$$

$$\nabla_{JU}\mathcal{T}Q_1Z - A_{\mathcal{F}Q_1Z}JU \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}),$$

$$\nabla_{JU}\mathcal{B}W - A_{CW}JU \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}),$$

for any $U, V \in \Gamma(\mathfrak{D})$, $Z = Q_1Z + Q_2Z \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^\perp$.

Theorem 4.3. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, the slant distribution $\mathfrak{D}_{\theta_1}$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}\sin^2\theta_1 g([U_1,Z],V_1) - \sin 2\theta_1 Z(\theta_1) g(U_1,V_1) &= g(A_{\mathcal{F}\mathcal{T}U_1}Z,V_1) + g(A_{\mathcal{F}U_1}Z,\mathcal{T}V_1)\\ &\quad - g(\nabla^{\perp}_Z\mathcal{F}U_1,\mathcal{F}V_1)\end{aligned}$$

and

$$\nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1 + \nabla^{\perp}_{U_1}\mathcal{C}\mathcal{F}V_1 + h(U_1,\mathcal{B}\mathcal{F}V_1) = 0,$$

where $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1}), Z \in \Gamma(\mathfrak{D}\oplus\mathfrak{D}_{\theta_2})$.

Proof. For any $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1}), Z \in \Gamma(\mathfrak{D}\oplus\mathfrak{D}_{\theta_2})$ we have

$$\begin{aligned}g(\widetilde{\nabla}_{U_1}V_1,Z) &= U_1 g(V_1,Z) - g(V_1,\widetilde{\nabla}_{U_1}Z)\\ &= -g([U_1,Z],V_1) + g(\widetilde{\nabla}_Z\mathcal{T}^2U_1,V_1) - g(\widetilde{\nabla}_Z\mathcal{F}\mathcal{T}U_1,V_1) - g(\widetilde{\nabla}_Z\mathcal{F}U_1,JV_1).\end{aligned}$$

Furthermore, from Lemma 3.9 and using the property of slant function, we deduce

$$\begin{aligned}g(\widetilde{\nabla}_{U_1}V_1,Z) &= -g([U_1,Z],V_1) + \sin 2\theta_1 Z(\theta_1) g(U_1,V_1) - \cos^2\theta_1 g(\widetilde{\nabla}_Z U_1,V_1)\\ &\quad + g(\widetilde{\nabla}_Z\mathcal{F}\mathcal{T}U_1,V_1) - g(\widetilde{\nabla}_Z\mathcal{F}U_1,JV_1)\\ &= -g([U_1,Z],V_1) + \sin 2\theta_1 Z(\theta_1) g(U_1,V_1) + \cos^2\theta_1 g(\widetilde{\nabla}_{U_1}V_1,Z)\\ &\quad + \cos^2\theta_1 g([U_1,Z],V_1) - g(\widetilde{\nabla}_Z\mathcal{F}\mathcal{T}U_1,V_1) - g(\widetilde{\nabla}_Z\mathcal{F}U_1,JV_1).\end{aligned}$$

From the above equation, we obtain

$$\begin{aligned}\sin^2\theta_1 g(\widetilde{\nabla}_{U_1}V_1,Z) &= -\sin^2\theta_1 g([U_1,Z],V_1) + \sin 2\theta_1 Z(\theta_1) g(U_1,V_1)\\ &\quad g(A_{\mathcal{F}\mathcal{T}U_1}Z,V_1) + g(A_{\mathcal{F}U_1}Z,\mathcal{T}V_1) - g(\nabla^{\perp}_Z\mathcal{F}U_1,\mathcal{F}V_1).\end{aligned} \tag{14}$$

Now, for any $W \in \Gamma(TM)^{\perp}$ and $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$, we have

$$\begin{aligned}g(\widetilde{\nabla}_{U_1}V_1,W) &= -g(\widetilde{\nabla}_{U_1}\mathcal{J}\mathcal{T}U_1,W) - g(\widetilde{\nabla}_{U_1}\mathcal{J}\mathcal{F}V_1,W)\\ &= -g(\widetilde{\nabla}_{U_1}\mathcal{T}^2V_1,W) - g(\widetilde{\nabla}_{U_1}\mathcal{F}\mathcal{T}V_1,W) - g(\widetilde{\nabla}_{U_1}\mathcal{B}\mathcal{F}V_1,W)\\ &\quad - g(\widetilde{\nabla}_{U_1}\mathcal{C}\mathcal{F}V_1,W).\end{aligned}$$

By using the property of the slant function, we get

$$\begin{aligned}g(\widetilde{\nabla}_{U_1}V_1,W) =& -(\sin 2\theta_1)U_1(\theta_1)g(V_1,W) + \cos^2\theta_1 g(\widetilde{\nabla}_{U_1}V_1,W)\\ & - g(\nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1,W) - g(h(U_1,\mathcal{B}\mathcal{F}V_1),W) - g(\nabla^{\perp}_{U_1}\mathcal{C}\mathcal{F}V_1,W)\end{aligned}$$

which gives

$$\sin^2\theta_1 g(\widetilde{\nabla}_{U_1}V_1,W) = -g(\nabla^{\perp}_{U_1}\mathcal{F}\mathcal{T}V_1 + \nabla^{\perp}_{U_1}\mathcal{C}\mathcal{F}V_1 + h(U_1,\mathcal{B}\mathcal{F}V_1),W). \tag{15}$$

The proof comes from (14) and (15). □

For the geometry of $\mathfrak{D}_{\theta_2}$, we have the following theorem:

Theorem 4.4. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, the slant distribution $\mathfrak{D}_{\theta_2}$ defines a totally geodesic foliation on M if and only if*

$$\sin^2\theta_2 g([U_2,Z],V_2) - \sin 2\theta_2 Z(\theta_2) g(U_2,V_2) = g(\nabla_Z\mathcal{B}\mathcal{F}U_2 - A_{\mathcal{F}\mathcal{T}U_2}Z - A_{\mathcal{C}\mathcal{F}U_2}Z, V_2)$$

and

$$\nabla^{\perp}_{U_2}\mathcal{F}\mathcal{T}V_2 - \mathcal{F}A_{\mathcal{F}V_2}U_2 + \nabla^{\perp}_{U_2}\mathcal{F}V_2 = 0$$

for any $W \in \Gamma(TM)^{\perp}$ and $U_2, V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$.

Theorem 4.5. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D}$ is totally geodesic if and only if*

$$\nabla_U \mathcal{B}W - A_{CW}U \in \Gamma(\mathcal{F}\mathcal{D}_1 \oplus \mathcal{F}\mathcal{D}_2)$$

and

$$h(U, \mathcal{B}W) + \nabla^{\perp}_U CW \in \Gamma(\mu)$$

for any $U, V \in \Gamma(\mathfrak{D})$ and $W \in \Gamma(TM)^{\perp}$.

Proof. For any $U, V \in \Gamma(\mathfrak{D})$ and $W \in \Gamma(TM)^{\perp}$, from (3), (4), (7) and (11), we have

$$\begin{aligned} g(h(U,V), W) &= g(\widetilde{\nabla}_U V, W) \\ &= g(J\widetilde{\nabla}_U JW, V) \\ &= g(\mathcal{T}(\nabla_U \mathcal{B}W - A_{CW}U) + \mathcal{B}(h(U, \mathcal{B}W) + \nabla^{\perp}_U CW), V) \end{aligned}$$

which achieves the proof. □

Theorem 4.6. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D}_{\theta_1}$ is totally geodesic if and only if*

$$g(\widetilde{\nabla}^{\perp}_{U_1} W, \mathcal{F}\mathcal{T}V_1) = g(A_{\mathcal{F}V_1}, \mathcal{B}W) + g(\nabla^{\perp}_{U_1} CW, \mathcal{F}V_1)$$

where $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $W \in \Gamma(\mathcal{T}M)^{\perp}$.

Proof. By virtue of (7), (11), we get

$$\begin{aligned} g(h(U_1, V_2), W) &= -g(\widetilde{\nabla}_{U_1} JW, JV_1) \\ &= g(\widetilde{\nabla}_{U_1} W, \mathcal{T}^2 V_1) + g(\widetilde{\nabla}_{U_1} W, \mathcal{F}\mathcal{T}V_1) \\ &- g(\widetilde{\nabla}_{U_1} \mathcal{B}W, \mathcal{F}V_1) - g(\widetilde{\nabla}_{U_1} CW, \mathcal{F}V_1) \end{aligned}$$

for any $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $W \in \Gamma(\mathcal{T}M)^{\perp}$. Furthermore, taking into account of Lemma 3.9 and using the property of slant function, we obtain

$$\begin{aligned} g(h(U_1, V_2), W) &= -\cos^2\theta_1 g(\widetilde{\nabla}_{U_1} W, V_1) + g(-A_W U_1 + \nabla^{\perp}_{U_1} W, \mathcal{F}\mathcal{T}V_1) \\ &+ g(-A_{\mathcal{F}V_1} U_1, \mathcal{B}W) - g(\nabla^{\perp}_{U_1} CW, \mathcal{F}V_1) \end{aligned}$$

which gives

$$g(h(U_1, V_1), W) = \csc^2\theta_1 \{g(\nabla^{\perp}_{U_1} W, \mathcal{F}\mathcal{T}V_1) - g(A_{\mathcal{F}V_1} U_1, \mathcal{B}W) - g(\nabla^{\perp}_{U_1} CW, \mathcal{F}V_1)\},$$

which completes the proof. □

For the slant distribution $\mathfrak{D}_{\theta_2}$, we have the following theorem:

Theorem 4.7. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D}_{\theta_2}$ is totally geodesic if and only if*

$$g(h(U_2, \mathcal{B}W) + \nabla^{\perp}_{U_2} CW, \mathcal{F}V_2) = g(\nabla^{\perp}_{U_2} W, \mathcal{F}\mathcal{T}V_2)$$

where $U_2, V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $W \in \Gamma(\mathcal{T}M)^{\perp}$.

Theorem 4.8. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D} - \mathfrak{D}_{\theta_1}$ mixed totally geodesic if and only if*

$$\mathcal{F}\nabla_V \mathcal{T}U = -Ch(V, \mathcal{T}U)$$

for any $U \in \Gamma(\mathfrak{D})$, $V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $W \in \Gamma(TM)^{\perp}$.

Proof. For any $U \in \Gamma(\mathfrak{D})$, $V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $W \in \Gamma(TM)^{\perp}$, we have

$$g(h(U, V_1), W) = g(\widetilde{\nabla}_U V_1, W) = g(\widetilde{\nabla}_U JV_1, JW).$$

On the other hand, by using (3), (7) and (11), we get

$$\begin{aligned} g(h(U, V_1), W) &= g(\widetilde{\nabla}_{V_1} JU, JW) \\ &= -g(J\widetilde{\nabla}_{V_1} \mathcal{T}U, W) \\ &= -g(\mathcal{F}\nabla_{V_1}\mathcal{T}U + Ch(V_1, \mathcal{T}U), W) \end{aligned}$$

which gives the proof. □

Theorem 4.9. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D} - \mathfrak{D}_{\theta_2}$ mixed totally geodesic if and only if*

$$g(h(U, \mathcal{T}V_2) + \nabla_U^{\perp}\mathcal{F}V_2, CW) = g(A_{\mathcal{F}V_2}U - \nabla_U\mathcal{T}V_2, \mathcal{B}W)$$

where $U \in \Gamma(\mathfrak{D})$, $V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^{\perp}$.

Proof. For any $U \in \Gamma(\mathfrak{D})$, $V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^{\perp}$, we obtain

$$g(h(U, V_2), W) = g(\widetilde{\nabla}_U JV_2, JW).$$

By virtue of (3), (4) and (7), we deduce

$$\begin{aligned} g(h(U, V_2), W) &= g(\widetilde{\nabla}_U JV_2, JW) \\ &= g(\nabla_U\mathcal{T}V_2 + h(U, \mathcal{T}V_2), JW) + g(-A_{\mathcal{F}V_2}U + \nabla_U^{\perp}\mathcal{F}V_2, JW). \end{aligned}$$

Taking into account of (11) in the above equation, we have

$$g(h(U, V_2), W) = g(\nabla_U\mathcal{T}V_2 - A_{\mathcal{F}V_2}U, \mathcal{B}W) + g(h(U, \mathcal{T}V_2) + \nabla_U^{\perp}\mathcal{F}V_2, CW)$$

which proves the assertion. □

Theorem 4.10. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. Then, $\mathfrak{D}_{\theta_1} - \mathfrak{D}_{\theta_2}$ mixed totally geodesic if and only if*

$$\mathcal{F}(A_{\mathcal{F}V_2}U_1 - \nabla_{U_1}\mathcal{T}V_2) = C(h(U_1, \mathcal{T}V_2) + \nabla_{U_1}^{\perp}\mathcal{F}V_2)$$

where $U_1 \in \Gamma(\mathfrak{D}_{\theta_1})$, $V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^{\perp}$.

Proof. For any $U_1 \in \Gamma(\mathfrak{D}_{\theta_1})$, $V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $W \in \Gamma(TM)^{\perp}$, we obtain

$$g(\widetilde{\nabla}_{U_1} V_2, W) = g(\widetilde{\nabla}_{U_1} JV_2, JW).$$

By using (3), (4), (7) and (11), we have

$$\begin{aligned} g(h(U_1, V_2), W &= -g(J\widetilde{\nabla}_{U_1} JV_2, W) \\ &= -g(J(\widetilde{\nabla}_{U_1}\mathcal{T}V_2 + \widetilde{\nabla}_{U_1}\mathcal{F}V_2), W) \\ &= -g(F\nabla_{U_1}\mathcal{T}V_2 + Ch(U_1, \mathcal{T}V_2) - \mathcal{F}A_{\mathcal{F}V_2}U_1 + C\nabla_{U_1}^{\perp}\mathcal{F}V_2, W) \end{aligned}$$

which has desired the assertion. □

Theorem 4.11. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. The invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$\mathcal{T}(\nabla_U \mathcal{T}V - \nabla_V \mathcal{T}U) + \mathcal{B}(h(U, \mathcal{T}V - h(U, \mathcal{T}V)$$

has no component in $\Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2})$.

Proof. For any $U, V \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2})$, by using (1), (2), (3) and (7), we have

$$\begin{aligned} g([U, V], Z) &= g(J(\widetilde{\nabla}_V JU) - J(\widetilde{\nabla}_U JV), Z) \\ &= g(J(\nabla_V \mathcal{T}U + h(V, \mathcal{T}U)) - J(\nabla_U \mathcal{T}V + h(U, \mathcal{T}V)), Z) \end{aligned} \tag{16}$$

here we have used that $\mathcal{F}U = 0$ for any $U \in \Gamma(D)$. Further, by using (7), (11) in the above equation, we get

$$g([U, V], Z) = g(\mathcal{T}(\nabla_V \mathcal{T}U - \nabla_U \mathcal{T}V) + \mathcal{B}(h(V, \mathcal{T}U) - h(U, \mathcal{T}V), Z)$$

which achieves the proof. □

Theorem 4.12. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. The slant distribution $\mathfrak{D}_{\theta_1}$ is integrable if and only if*

$$\begin{aligned} &\sin^2\theta_1 g([U_1, Z], V_1) - \cos^2\theta_1 g(\nabla_{U_1} V_1, Z) - \sin 2\theta_1 Z(\theta_1) g(U_1, V_1) \\ &\qquad = g(T(A_{\mathcal{F}U_1} V_1 - \nabla_{V_1} \mathcal{T}U_1) - \mathcal{B}(h(V_1, \mathcal{T}U_1) + \nabla^{\perp}_{V_1} \mathcal{F}U_1), Z) \\ &\qquad + g(\nabla_Z \mathcal{B}\mathcal{F}U_1 - A_{C\mathcal{F}U_1} Z, V_1) \end{aligned}$$

where $U_1, V_1 \in \Gamma(\mathfrak{D}_{\theta_1})$, $Z \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_2})$, $Z = PZ + RZ$.

Proof. For any $U_1, V_1 \in \Gamma(\mathfrak{D}_1)$, $Z \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_2})$, $Z = \mathcal{P}Z + \mathcal{R}Z$, by using (3), (4) and (7), we get

$$\begin{aligned} g([U_1, V_1], Z) &= g(\widetilde{\nabla}_{U_1} V_1, Z) - g(\widetilde{\nabla}_{V_1} U_1, Z) \\ &= -g(\widetilde{\nabla}_Z JU_1, JV_1) - g([U_1, Z], V_1) - g(\widetilde{\nabla}_{V_1} JU_1, JZ) \\ &= g(\widetilde{\nabla}_Z \mathcal{T}^2 U_1, V_1) + g(\widetilde{\nabla}_Z \mathcal{F}\mathcal{T}U_1, V_1) - g(\widetilde{\nabla}_Z \mathcal{F}U_1, JV_1) - g([U_1, Z], V_1) \\ &- g(J(\nabla_{V_1} \mathcal{T}U_1 + h(V_1, \mathcal{T}U_1) - J(-A_{\mathcal{F}U_1} V_1 + \nabla^{\perp}_{V_1} \mathcal{F}U_1), Z). \end{aligned} \tag{17}$$

On the other hand, taking into account of Lemma 3.9 and using the property of slant function, equation (17)

$$\begin{aligned} g([U_1, V_1], Z) = &\sin 2\theta_1 Z(\theta_1) g(U_1, V_1) + \cos^2\theta_1 \{g(\widetilde{\nabla}_{U_1} V_1, Z) + g([U_1, Z], V_1)\} \\ &- g(A_{\mathcal{F}\mathcal{T}U_1} Z, V_1) + g(\nabla_Z \mathcal{B}\mathcal{F}U_1, V_1) - g([U_1, Z], V_1) \\ &+ g(T(A_{\mathcal{F}U_1} V_1 - \nabla_{V_1} \mathcal{T}U_1) - \mathcal{B}(h(V_1, \mathcal{T}U_1) + \nabla^{\perp}_{V_1} \mathcal{F}U_1), Z), \end{aligned}$$

which gives the assertion. □

For the slant distribution $\mathfrak{D}_{\theta_2}$, we have the following theorem:

Theorem 4.13. *Let M be a pointwise quasi bi-slant submanifolds of a Kaehler manifold $\widetilde{M}$. The slant distribution $\mathfrak{D}_{\theta_2}$ is integrable if and only if*

$$\begin{aligned} &\sin^2\theta_2 g([U_2, Z], V_2) - \cos^2\theta_2 g(\nabla_{U_2} V_2, Z) - \sin 2\theta_2 Z(\theta_2) g(U_2, V_2) \\ &= g(\nabla_Z \mathcal{B}\mathcal{F}U_2 - A_{\mathcal{F}\mathcal{T}U_2} Z, V_2) + g(\nabla_{V_2} U_2, \mathcal{T}\mathcal{P}Z + \mathcal{T}\mathcal{Q}Z) + g(h(V_2, U_2), \mathcal{F}\mathcal{Q}Z), \end{aligned}$$

where $U_2, V_2 \in \Gamma(\mathfrak{D}_{\theta_2})$, $Z \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_1})$, $Z = PZ + QZ$.

Finally, we mention the following example:

Example 4.14. *For $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$, consider a submanifold M of a Kaehler manifold $\widetilde{M}$ defined by immersion ψ as follows:*

$$\psi(\theta_1, \theta_2, u, v, s, t) = (u\cos\theta_2, v\cos\theta_2, u\cos\theta_1, v\cos\theta_1, u\sin\theta_2, v\sin\theta_2, u\sin\theta_1, v\sin\theta_1, \theta_1, \theta_2, u\theta_1 + v\theta_2, 0, s, t).$$

We can easily see that the tangent bundle of M is spanned by the tangent vectors

$$\begin{aligned}
e_1 &= -u\sin\theta_1\frac{\partial}{\partial x_2} - v\sin\theta_1\frac{\partial}{\partial y_2} + u\cos\theta_1\frac{\partial}{\partial x_4} + v\cos\theta_1\frac{\partial}{\partial y_4} + \frac{\partial}{\partial x_5} + u\frac{\partial}{\partial x_6},\\
e_2 &= -u\sin\theta_2\frac{\partial}{\partial x_1} - v\sin\theta_2\frac{\partial}{\partial y_1} + u\cos\theta_2\frac{\partial}{\partial x_3} + v\cos\theta_2\frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_5} + v\frac{\partial}{\partial x_6},\\
e_3 &= \cos\theta_2\frac{\partial}{\partial x_1} + \cos\theta_1\frac{\partial}{\partial x_2} + \sin\theta_2\frac{\partial}{\partial x_3} + \sin\theta_1\frac{\partial}{\partial x_4} + \theta_1\frac{\partial}{\partial x_6},\\
e_4 &= \cos\theta_2\frac{\partial}{\partial y_1} + \cos\theta_1\frac{\partial}{\partial y_2} + \sin\theta_2\frac{\partial}{\partial y_3} + \sin\theta_1\frac{\partial}{\partial y_4} + \theta_2\frac{\partial}{\partial x_6},\\
e_5 &= \frac{\partial}{\partial x_7}, \qquad e_6 = \frac{\partial}{\partial y_7}.
\end{aligned}$$

We define Kaehler structure J of $\mathbb{R}^{14}$ by,

$$J(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i},\ J(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j},\ \ 1 \le i, j \le 6.$$

We have,

$$\begin{aligned}
Je_1 &= -v\sin\theta_1\frac{\partial}{\partial x_2} + u\sin\theta_1\frac{\partial}{\partial y_2} + v\cos\theta_1\frac{\partial}{\partial x_4}, -u\cos\theta_1\frac{\partial}{\partial y_4} - \frac{\partial}{\partial y_5} - u\frac{\partial}{\partial y_6}\\
Je_2 &= -v\sin\theta_2\frac{\partial}{\partial x_1} + u\sin\theta_2\frac{\partial}{\partial y_1} + v\cos\theta_2\frac{\partial}{\partial x_3} - u\cos\theta_2\frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_5} - v\frac{\partial}{\partial y_6},\\
Je_3 &= -\cos\theta_2\frac{\partial}{\partial y_1} - \cos\theta_1\frac{\partial}{\partial y_2} - \sin\theta_2\frac{\partial}{\partial y_3} - \sin\theta_1\frac{\partial}{\partial y_4} - \theta_1\frac{\partial}{\partial y_6},\\
Je_4 &= \cos\theta_2\frac{\partial}{\partial x_1} + \cos\theta_1\frac{\partial}{\partial x_2} + \sin\theta_2\frac{\partial}{\partial x_3} + \sin\theta_1\frac{\partial}{\partial x_4} - \theta_2\frac{\partial}{\partial y_6},\\
Je_5 &= -\frac{\partial}{\partial y_7}, \qquad Je_6 = \frac{\partial}{\partial x_7}.
\end{aligned}$$

Then $\mathfrak{D} = span\{e_5, e_6\}$ is holomorphic distribution and $\mathfrak{D}_{\theta_1} = span\{e_1, e_2\}$, $\mathfrak{D}_{\theta_2} = span\{e_3, e_4\}$ are pointwise bi-slant with slant functions $\cos^{-1}\left(\frac{-1}{\sqrt{(2u^2+v^2+1)(2v^2+u^2+1)}}\right)$ and $\cos^{-1}\left(\frac{-2}{\sqrt{(\theta_1^2+2)(\theta_2^2+2)}}\right)$. Thus ψ defines a proper 6-dimensional pointwise quasi bi-slant submanifold M in $\widetilde{M}$.

References

[1] M. A. Akyol and S. Beyendi, A note on quasi bi-slant submanifolds of cosymplectic manifolds, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 69(2), (2020), 1508-1521.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Semi-slant submanifolds of a Sasakian manifold, Geom. Dedic. 78(2), (1999), 183–199.

[3] A. Carriazo, New developments in slant submanifolds theory, Narasa Publishing Hause New Delhi, India, 2002.

[4] A. Carriazo, Bi-slant immersions. In: Proceeding of the ICRAMS 2000, Kharagpur, pp. (2000), 88-97.

[5] B. Y. Chen, Geometry of slant submanifolds, Katholieke Universiteit Leuven, Leuven, Belgium, View at Zentralblatt Math., 1990.

[6] B. Y. Chen, Slant immersions, Bull. Austral. Math. Soc., 41 (1990), 135-147.

[7] B. Y. Chen and O. J. Garay, Pointwise slant submanifolds in almost Hermitian manifolds, Turk. J. Math. 2012, 36, 630–640.

[8] B. Y. Chen and S. Uddin, Warped product pointwise bi-slant submanifolds of Kaehler manifolds, Publ. Math: Debrecen, 92(1-2): (2018), 183-199.
[9] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, Glasgow Math. J., 42 (2000), 125-138.
[10] F. Etayo, On quasi-slant submanifolds of an almost Hermitian manifold, Publ. Math. Debrecen. 53, (1998), 217-223.
[11] A. Lotta, Slant submanifolds in contact geometry, Bulletin Mathematical Society Roumanie, 39 (1996), 183-198.
[12] K. Matsumoto, I. Mihai Y. Tazawa, Ricci tensor of slant submanifolds in complex space forms, Kodai Math.J. 26, (2003), 85–94.
[13] M. A. Lone, M. S. Lone and M. H. Shahid, Hemi-slant submanifolds of cosymplectic manifolds, Cogent Mathematics 3(1), (2016), 120–143.
[14] N. Papaghuic, Semi-slant submanifolds of a Kaehlarian manifold, An. St. Univ. Al. I. Cuza. Univ. Iasi, 40 (2009), 55-61.
[15] K. S. Park, Pointwise slant and pointwise semi-slant submanifolds in almost contact metric manifolds, Mathematics 2020, 8(6), 985; https://doi.org/10.3390/math8060985.
[16] K. S. Park, On the pointwise slant submanifolds, In Hermitian-Grassmannian Submanifolds; Suh, Y., Ohnita, Y., Zhou, J., Kim, B., Lee, H., Eds.; Springer Proceedings in Mathematics and Statistics, 203; Springer: Singapore, 2017.
[17] S. Y. Perktaş, A. M. Blaga, and E. Kılıç, Almost bi-slant submanifolds of an almost contact metric manifold, J. Geom., 112, 2 (2021), 1–23. https://doi.org/10.1007/s00022-020-00564-1.
[18] R. Prasad, S. K. Verma and S. Kumar, Quasi hemi-slant submanifolds of Sasakian manifolds, Journal of Mathematical and Computational Science, 10(2), (2020), 418-435.
[19] R. Prasad, S. K. Verma, S. Kumar and S. K. Chaubey, Quasi hemi-slant submanifolds of cosymplectic manifolds, Korean J. Math. 28(2), (2020), 257-273.
[20] G. S. Ronsse, Generic and skew CR-submanifolds of a Kaehler manifold, Bull. Inst. Math. Acad. Sin. 18(2), (1990), 127-141.
[21] C. Sayar, H. M. Taştan, F. Özdemir and M. M. Tripathi, Generic submersions from Kaehlerian manifolds, Bull. Malays. Math. Sci. Soc. 43, (2020), 809-831.
[22] B. Şahin, Slant submanifolds of quaternion Kaehler manifolds, Commum. Korean Math. Soc., 22, (2007), 123-135.
[23] B. Şahin, Non-existence of warped product semi-slant submanifolds of Kaehler manifolds, Geom. Dedicata, 117, (2006), 195–202.
[24] B. Şahin, Warped product pointwise semi-slant submanifolds of Kaehler manifolds, Port. Math., 70, (2013), 252–268.
[25] B. Şahin, Slant submanifolds of an almost product Riemannian manifold, Journal of the Korean Mathematical Society, 43(4), (2006), 717-732.
[26] B. Şahin and S. Keleş, Slant submanifolds of Kaehler product manifolds, Turkish Journal of Mathematics, 31 (2007), 65–77.
[27] S. Uddin, B. Y. Chen and F. R. Al-Solamy, Warped product bi-slant immersions in Kaehler manifolds, Mediterr. J. Math. 14(2), (2017), 14–95.
[28] H. M. Taştan and F. Özdemir, The geometry of hemi-slant submanifolds of a locally product Riemannian manifold, Turkish Journal of Mathematics, 39 (2015), 268-284.
[29] K. Yano and M. Kon, Structures on manifolds, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.