Filomat 36:19 (2022), 6679–6686
https://doi.org/10.2298/FIL2219679A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hypersoft Separation Axioms

Baravan A. Asaad[a,b], **Sagvan Y. Musa**[c]

[a]*Department of Mathematics, Faculty of Science, University of Zakho, Zakho 42002, Iraq*
[b]*Department of Computer Science, College of Science, Cihan University-Duhok, Duhok 42001, Iraq*
[c]*Department of Mathematics, Faculty of Education, University of Zakho, Zakho 42002, Iraq*

Abstract. In this manuscript, we continue to study the hypersoft topological space (for short, HSTS) by presenting hypersoft (HS) separation axioms, called HS T_i-spaces for $i = 0, 1, 2, 3, 4$. The notions of HS regular and HS normal spaces are explained in detail. We discuss the connections between them and present numerous examples to help clarify the interconnections between the different types of these spaces. We point out that HS T_i-axioms imply HS T_{i-1} for $i = 1, 2, 3$, and with the help of an example we show that HS T_4-space need not be HS T_3-space. We also clarify that the property that an HS space being HS T_i-spaces ($i = 0, 1, 2, 3$) is HS hereditary. Finally, we provide a diagram to illustrate the relationships between our proposed axioms.

1. Introduction

Most of our traditional tools for formal modeling, reasoning, and computation are clear, consistent, and precise. However, there are many complex problems in economics, engineering, the environment, social sciences, medicine, and other fields that require data that is not necessarily pure. Due to the many forms of uncertainty involved in these situations, we cannot use traditional approaches to resolve them. As a result, Molodtsov [12] pioneered soft set theory as a mathematical tool for dealing with uncertainty. Work on soft set theory and its applications has progressed rapidly in recent years (see, for example, [3, 6, 8–10, 19, 24]).

Shabir and Naz [21] introduced the concept of soft topological spaces in 2011 by defining soft sets over an initial universe set with a fixed set of parameters. Many authors have examined the notions of soft topology in the same way that they have been investigated in classical topology from the beginning of soft topology. The introduction of multiple varieties of soft axioms in terms of ordinary points [11, 21] and soft points [5, 23] is deriven by the various types of membership and non-membership relations in soft settings. The author of [4] corrected some reported conclusions about the soft separation axioms.

Smarandache [22] devised a novel approach to dealing with uncertainty. By modifying the function to a multiple decision function, the soft sets were generalized to HS sets. He also introduced fuzzy HS sets, intuitionistic fuzzy HS sets, neutrosophic HS sets, and plithogenic HS sets as extensions of the HS sets. Based on the HS sets and their extension, many researchers have developed various operators, properties, and applications [1, 2, 18, 20].

2020 *Mathematics Subject Classification*. Primary 54D10; Secondary 54D15
Keywords. HS separation axioms, HS topology, HS sets
Received: 19 January 2022; Revised: 08 October 2022; Accepted: 31 October 2022
Communicated by Ljubiša D.R. Kočinac
Email addresses: baravan.asaad@uoz.edu.krd (Baravan A. Asaad), sagvan.musa@uoz.edu.krd (Sagvan Y. Musa)

Recently, Musa and Asaad [13] applied HS sets to initiate the concept of HSTS. They defined basic HSTS concepts such as HS closure, HS interior operators, and HS connected spaces [14]. Musa and Asaad [15], investigated the concept of bipolar HS sets according to the role of bipolarity that was introduced by Dubois and Prade [7]. Some definitions, properties and operations on bipolar HS sets were studied in [15]. Topological structures across bipolar HS sets were first investigated by Musa and Asaad [16]. They defined the bipolar HSTS in a universal set along with a discussion of the connectivity of the bipolar HS [17].

The following is a list of the sections that make up the current paper: Section 2 is the preliminaries, which include definitions and properties of HS sets and hypersoft topology (HST). In section 3, we introduce the concept of HS separation axioms, namely HS T_i-spaces ($i = 0, 1, 2, 3, 4$). We show how they are related and describe some of their characteristics. In Section 4, we introduce the notions of HS regular and HS normal spaces. In Section 5, we draw conclusions and suggest some research topics for the future.

2. Preliminaries

The necessary concepts and results related to the HS sets and HST are presented in this section. Let $\mathfrak{R}$ be an initial universe, $2^{\mathfrak{R}}$ the power set of $\mathfrak{R}$, and $\sigma_i \cap \sigma_j = \phi$ for $i \neq j$. Let $\lambda_i, \gamma_i \subseteq \sigma_i$ for $i = 1, 2, ..., n$.

Definition 2.1. ([22]) A pair $(\pi, \sigma_1 \times \sigma_2 \times ... \times \sigma_n)$ is called an HS set over $\mathfrak{R}$, where π is a mapping given by $\pi : \sigma_1 \times \sigma_2 \times ... \times \sigma_n \to 2^{\mathfrak{R}}$.

We use the notations Σ, Λ, and Γ for $\sigma_1 \times \sigma_2 \times ... \times \sigma_n$, $\lambda_1 \times \lambda_2 \times ... \times \lambda_n$, and$\gamma_1 \times \gamma_2 \times ... \times \gamma_n$, respectively. Let $\Lambda, \Gamma \subseteq \Sigma$. Moreover, we identify HS set (π, Σ) as $(\pi, \Sigma) = \{(\ell, \pi(\ell)) : \ell \in \Sigma\}$.

Definition 2.2. ([18]) We called (π, Λ) is an HS subset of (θ, Γ) if $\Lambda \subseteq \Gamma$, and $\pi(\ell) \subseteq \theta(\ell)$ for all $\ell \in \Lambda$. We write $(\pi, \Lambda) \widetilde{\sqsubseteq} (\theta, \Gamma)$.

An HS set (π, Λ) is said to be an HS superset of (θ, Γ), if (θ, Γ) is an HS subset of (π, Λ). We write (π, Λ) $\widetilde{\sqsupseteq} (\theta, \Gamma)$.

Definition 2.3. ([18]) Two HS sets (π, Λ) and (θ, Γ) are said to be HS equal if (π, Λ) is an HS subset of (θ, Γ) and (θ, Γ) is an HS subset of (π, Λ).

Definition 2.4. ([18]) The complement of an HS set (π, Λ), denoted by $(\pi, \Lambda)^c$, is defined by (π^c, Λ) where $\pi^c : \Lambda \to 2^{\mathfrak{R}}$ is a mapping given by $\pi^c(\ell) = \mathfrak{R} \setminus \pi(\ell)$ for all $\ell \in \Lambda$.

Definition 2.5. ([20]) We called an HS set (π, Λ) a relative null HS set, denoted by $(\widetilde{\phi}, \Lambda)$, if $\pi(\ell) = \phi$ for all $\ell \in \Lambda$.

The null HS set over $\mathfrak{R}$ is denoted by $(\widetilde{\phi}, \Sigma)$.

Definition 2.6. ([20]) We called an HS set (π, Λ) a relative whole HS set, denoted by $(\widetilde{\mathfrak{R}}, \Lambda)$, if $\pi(\ell) = \mathfrak{R}$ for all $\ell \in \Lambda$.

The whole HS set over $\mathfrak{R}$ is denoted by $(\widetilde{\mathfrak{R}}, \Sigma)$.

Definition 2.7. ([18]) The difference of HS set (π, Λ) and HS set (θ, Γ) is an HS set (ω, Δ), denoted by (π, Λ) $\setminus (\theta, \Gamma) = (\omega, \Delta)$, where $\Delta = \Lambda \cap \Gamma$ and $\omega(\ell) = \pi(\ell) \setminus \theta(\ell)$ for all $\ell \in \Delta$.

Definition 2.8. ([20]) The union of HS set (π, Λ) and HS set (θ, Γ) is an HS set (ω, Δ), denoted by $(\pi, \Lambda) \widetilde{\sqcup}$ $(\theta, \Gamma) = (\omega, \Delta)$, where $\Delta = \Lambda \cap \Gamma$ and $\omega(\ell) = \pi(\ell) \cup \theta(\ell)$ for all $\ell \in \Delta$.

Definition 2.9. ([18]) The intersection of HS set (π, Λ) and HS set (θ, Γ) is an HS set (ω, Δ), denoted by (π, Λ) $\widetilde{\sqcap} (\theta, \Gamma) = (\omega, \Delta)$, where $\Delta = \Lambda \cap \Gamma$ and $\omega(\ell) = \pi(\ell) \cap \theta(\ell)$ for all $\ell \in \Delta$.

Definition 2.10. ([13]) Let (π, Σ) be an HS set over $\mathfrak{R}$ and $r \in \mathfrak{R}$. Then $r \in (\pi, \Sigma)$ if $r \in \pi(\ell)$ for all $\ell \in \Sigma$. Also for any $r \in \mathfrak{R}$, $r \notin (\pi, \Sigma)$, if $r \notin \pi(\ell)$ for some $\ell \in \Sigma$.

Definition 2.11. ([13]) Let $\Upsilon \subseteq \mathfrak{R}$. Then $(\widetilde{Y}, \Sigma)$ denotes the HS set over $\mathfrak{R}$ defined by $\widetilde{\Upsilon}(\ell) = \Upsilon$ for all $\ell \in \Sigma$.

Definition 2.12. ([13]) Let (π, Σ) be an HS set over $\mathfrak{R}$ and $\Upsilon \subseteq \mathfrak{R}$. Then the sub HS set of (π, Σ) over Υ denoted by (π_Υ, Σ) is defined as $\pi_\Upsilon(\ell) = \Upsilon \cap \pi(\ell)$ for all $\ell \in \Sigma$.

In other words, $(\pi_\Upsilon, \Sigma) = (\widetilde{\Upsilon}, \Sigma) \widetilde{\sqcap} (\pi, \Sigma)$.

Definition 2.13. ([13]) Let T_H be the collection of HS sets over $\mathfrak{R}$, then T_H is said to be an HST on $\mathfrak{R}$ if:
(1) $(\widetilde{\phi}, \Sigma)$, $(\widetilde{\mathfrak{R}}, \Sigma)$ belong to T_H;
(2) The intersection of any two HS sets in T_H belongs to T_H;
(3) The union of any number of HS sets in T_H belongs to T_H.

Then $(\mathfrak{R}, T_H, \Sigma)$ is called an HSTS. The members of T_H are said to be HS open sets and its complement is called an HS closed sets.

Proposition 2.14. ([13]) *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS. Then:*
(1) $(\widetilde{\phi}, \Sigma)$, $(\widetilde{\mathfrak{R}}, \Sigma)$ are HS closed set over $\mathfrak{R}$;
(2) The union of any two HS closed sets is an HS closed set over $\mathfrak{R}$;
(3) The intersection of any number of HS closed sets is an HS closed set over $\mathfrak{R}$.

Definition 2.15. ([13]) Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $\Upsilon \subseteq \mathfrak{R}$. Then $T_{H_\Upsilon} = \{(\pi_\Upsilon, \Sigma) \mid (\pi, \Sigma) \widetilde{\in} T_H\}$ is said to be the relative HST on Υ and $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is called an HS subspace of $(\mathfrak{R}, T_H, \Sigma)$.

Proposition 2.16. ([14]) *Let $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ be an HS subspace of HSTS $(\mathfrak{R}, T_H, \Sigma)$ and (π, Σ) be an HS set over $\mathfrak{R}$, then:*
(1) (π, Σ) is HS open in Υ if and only if $(\pi, \Sigma) = (\widetilde{\Upsilon}, \Sigma) \widetilde{\sqcap} (\theta, \Sigma)$ for some $(\theta, \Sigma) \in T_H$;
(2) (π, Σ) is HS closed in Υ if and only if $(\pi, \Sigma) = (\widetilde{\Upsilon}, \Sigma) \widetilde{\sqcap} (\theta, \Sigma)$ for some HS closed set (θ, Σ) in $\mathfrak{R}$.

Definition 2.17. ([14]) A property of an HSTS is said to be HS hereditary if every HS subspace of the space has that property.

Corollary 2.18. *Let (π, Σ) be an HS set over $\mathfrak{R}$ and $r \in \mathfrak{R}$. Then:*
(1) $r \in (\pi, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$;
(2) If $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$, then $r \notin (\pi, \Sigma)$.

Proof. Straightforward. □

Remark 2.19. The opposite of Corollary 2.18 (2) does not hold.

Example 2.20. Let $\mathfrak{R} = \{r_1, r_2\}$, $\sigma_1 = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, $\sigma_2 = \{\varepsilon_4\}$, and $\sigma_3 = \{\varepsilon_5\}$. Let $(\pi, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \mathfrak{R})\}$. Then, $r_2 \notin (\pi, \Sigma)$ but $(\pi_{r_2}, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) \neq (\widetilde{\phi}, \Sigma)$.

3. Hypersoft separation axioms

The definitions of HS T_i-spaces $(i = 0, 1, 2)$ are given in this section. The essential characteristics of these spaces are discussed, as well as the relationships between them.

Definition 3.1. An HSTS $(\mathfrak{R}, T_H, \Sigma)$ is said to be:
(1) An HS T_0-space if for every $r \neq s \in \mathfrak{R}$, there is an HS open set (π, Σ) with $r \in (\pi, \Sigma)$, $s \notin (\pi, \Sigma)$ or $s \in (\pi, \Sigma)$, $r \notin (\pi, \Sigma)$;
(2) An HS T_1-space if for every $r \neq s \in \mathfrak{R}$, there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \notin (\pi_1, \Sigma)$ and $s \in (\pi_2, \Sigma)$, $r \notin (\pi_2, \Sigma)$;
(3) An HS T_2-space if for every $r \neq s \in \mathfrak{R}$, there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Next, we examine some results related to the HS T_0-space.

Proposition 3.2. *Let $(\Re, T_H, \Sigma)$ be an HSTS and $r \neq s \in \Re$. If there is an HS open set (π, Σ) with $r \in (\pi, \Sigma)$, $s \in (\pi, \Sigma)^c$ or $s \in (\pi, \Sigma)$, $r \in (\pi, \Sigma)^c$, then $(\Re, T_H, \Sigma)$ is an HS T_0-space.*

Proof. Let $r \neq s \in \Re$ and (π, Σ) be an HS open set with $r \in (\pi, \Sigma)$, $s \in (\pi, \Sigma)^c$. Since $s \in (\pi, \Sigma)^c$ then $s \in \pi^c(\ell)$ for all $\ell \in \Sigma$. This means $s \notin \pi(\ell)$ for all $\ell \in \Sigma$. Therefore $s \notin (\pi, \Sigma)$. Similarly, we may verify $s \in (\pi, \Sigma)$ and $r \notin (\pi, \Sigma)$. Hence, $(\Re, T_H, \Sigma)$ is an HS T_0-space. □

Proposition 3.3. *Let $(\Re, T_H, \Sigma)$ be an HSTS and $\Upsilon \subseteq \Re$. If $(\Re, T_H, \Sigma)$ is an HS T_0-space, then $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_0-space.*

Proof. Let $r \neq s \in \Upsilon$. Since $(\Re, T_H, \Sigma)$ is an HS T_0-space, then there is an HS open set (π, Σ) with $r \in (\pi, \Sigma)$, $s \notin (\pi, \Sigma)$ or $s \in (\pi, \Sigma)$, $r \notin (\pi, \Sigma)$. Say, $r \in (\pi, \Sigma)$ and $s \notin (\pi, \Sigma)$. As, $r \in (\pi, \Sigma)$ then $r \in \pi(\ell)$ for all $\ell \in \Sigma$. Since $r \in \Upsilon$, then $r \in \Upsilon \cap \pi(\ell) = \pi_\Upsilon(\ell)$ for all $\ell \in \Sigma$. Hence, $r \in (\pi_\Upsilon, \Sigma)$. Consider $s \notin (\pi, \Sigma)$. Then, $s \notin \pi(\ell)$ for some $\ell \in \Sigma$. This implies $s \notin \Upsilon \cap \pi(\ell) = \pi_\Upsilon(\ell)$ for some $\ell \in \Sigma$. Hence, $s \notin (\pi_\Upsilon, \Sigma)$. Similarly, we may verify $s \in (\pi_\Upsilon, \Sigma)$ and $r \notin (\pi_\Upsilon, \Sigma)$. Hence, $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_0-space. □

In the following result, we present a complete description of an HS T_1-space and then establish various characteristics of this space.

Proposition 3.4. *If (π_r, Σ) is an HS closed set of $(\Re, T_H, \Sigma)$ for each $r \in \Re$, then $(\Re, T_H, \Sigma)$ is an HS T_1-space.*

Proof. Let (π_r, Σ) is an HS closed set of $(\Re, T_H, \Sigma)$ for each $r \in \Re$. Then, $(\pi_r, \Sigma)^c$ is an HS open set in T_H. For $r \neq s \in \Re$, $(\pi_r, \Sigma)^c$ is an HS open set with $s \in (\pi_r, \Sigma)^c$ and $r \notin (\pi_r, \Sigma)^c$. Similarly, $(\pi_s, \Sigma)^c \widetilde{\in} T_H$ with $r \in (\pi_s, \Sigma)^c$ and $s \notin (\pi_s, \Sigma)^c$. Thus, $(\Re, T_H, \Sigma)$ is an HS T_1-space. □

Remark 3.5. The next example illustrates the converse of Proposition 3.4 is not true.

Example 3.6. Let $\Re = \{r_1, r_2\}$, $\sigma_1 = \{\varepsilon_1, \varepsilon_2\}$, $\sigma_2 = \{\varepsilon_3\}$, and $\sigma_3 = \{\varepsilon_4\}$. Let $T_H = \{(\widetilde{\phi}, \Sigma), (\widetilde{\Re}, \Sigma), (\pi_1, \Sigma), (\pi_2, \Sigma), (\pi_3, \Sigma)\}$ be an HST defined on $\Re$, where
$(\pi_1, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_2\})\}$;
$(\pi_2, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \Re)\}$;
$(\pi_3, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \Re), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_2\})\}$.
Then, $(\Re, T_H, \Sigma)$ is an HS T_1-space. We note that for (π_{r_1}, Σ), (π_{r_2}, Σ) over $\Re$, where
$(\pi_{r_1}, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_1\})\}$;
$(\pi_{r_2}, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_2\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_2\})\}$.
The HS complement $(\pi_{r_1}, \Sigma)^c$, $(\pi_{r_2}, \Sigma)^c$ over $\Re$ are defined by
$(\pi_{r_1}, \Sigma)^c = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_2\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_2\})\}$;
$(\pi_{r_2}, \Sigma)^c = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \{r_1\})\}$.
Neither $(\pi_{r_1}, \Sigma)^c$ nor $(\pi_{r_2}, \Sigma)^c$ belong to T_H. Thus, (π_{r_1}, Σ) and (π_{r_2}, Σ) are not HS closed sets of $(\Re, T_H, \Sigma)$.

Proposition 3.7. *Let $(\Re, T_H, \Sigma)$ be an HSTS and $r \in \Re$. If $\Re$ is an HS T_1-space, then for each HS open set (π, Σ) such that $r \in (\pi, \Sigma)$:*
(1) $(\pi_r, \Sigma) \widetilde{\sqsubseteq} [\widetilde{\sqcap}\, (\pi, \Sigma)]$;
(2) $s \notin \widetilde{\sqcap}\, (\pi, \Sigma)$ for all $s \neq r$.

Proof. (1) Since $r \in \widetilde{\sqcap}\, (\pi, \Sigma)$, then by Corollary 2.18, $(\pi_r, \Sigma) \widetilde{\sqsubseteq} [\widetilde{\sqcap}\, (\pi, \Sigma)]$.

(2) Let $r \neq s \in \Re$, then there are HS open sets (θ, Σ) with $r \in (\theta, \Sigma)$ and $s \notin (\theta, \Sigma)$. So, $s \notin \theta(\ell)$ for some $\ell \in \Sigma$, and hence $s \notin \cap_{\ell \in \Sigma} \pi(\ell)$. Thus, $s \notin \widetilde{\sqcap}\, (\pi, \Sigma)$. □

Remark 3.8. In Proposition 3.7 (1), the equality does not hold.

Example 3.9. Let $\Re = \{r_1, r_2\}$, $\sigma_1 = \{\varepsilon_1, \varepsilon_2\}$, $\sigma_2 = \{\varepsilon_3, \varepsilon_4\}$, and $\sigma_3 = \{\varepsilon_5\}$. Let $T_H = \{(\widetilde{\phi}, \Sigma), (\widetilde{\Re}, \Sigma), (\pi_1, \Sigma), (\pi_2, \Sigma), (\pi_3, \Sigma), (\pi_4, \Sigma), (\pi_5, \Sigma)\}$ be an HST defined on $\Re$, where
$(\pi_1, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_5), \Re), ((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\})\}$;
$(\pi_2, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_5), \{r_2\}), ((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_5), \{r_2\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \Re)\}$;

$(\pi_3, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_5), \mathfrak{R}), ((\varepsilon_1, \varepsilon_4, \varepsilon_5), \mathfrak{R}), ((\varepsilon_2, \varepsilon_3, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\})\}$;
$(\pi_4, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_5), \{r_2\}), ((\varepsilon_1, \varepsilon_4, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_3, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\})\}$;
$(\pi_5, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_5), \{r_2\}), ((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\})\}$.

Then, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space. But for all HS open sets $r_1 \in (\pi_1, \Sigma)$ and $r_1 \in (\pi_3, \Sigma)$ we have $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_3, \Sigma) = (\pi_1, \Sigma) \neq (\pi_{r_1}, \Sigma)$.

Proposition 3.10. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \neq s \in \mathfrak{R}$. If there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_1, \Sigma)^c$ and $s \in (\pi_2, \Sigma)$, $r \in (\pi_2, \Sigma)^c$, then $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space.*

Proof. Similar to the proof of Proposition 3.2. □

Proposition 3.11. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $\Upsilon \subseteq \mathfrak{R}$. If $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space, then $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_1-space.*

Proof. Similar to the proof of Proposition 3.3. □

In the following results, we characterize an HS T_2-space and investigate some of its properties.

Proposition 3.12. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \in \mathfrak{R}$. If $\mathfrak{R}$ is an HS T_2-space, then for each HS open set (π, Σ) with $r \in (\pi, \Sigma)$, $(\pi_r, \Sigma) = \widetilde{\sqcap} (\pi, \Sigma)$.*

Proof. Let $r \neq s \in \mathfrak{R}$ and $s \in \cap\pi(\ell)$ for some $\ell \in \Sigma$. Since $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. Therefore $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_{2_s}, \Sigma) = (\widetilde{\phi}, \Sigma)$ and hence $\pi_1(\ell) \cap \pi_{2_s}(\ell) = \phi$. This contradicts $s \in \cap\pi(\ell)$ for some $\ell \in \Sigma$. This is the complete of the proof. □

Proposition 3.13. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \neq s \in \mathfrak{R}$. If there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_1, \Sigma)^c$ and $s \in (\pi_2, \Sigma)$, $r \in (\pi_2, \Sigma)^c$, then $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space.*

Proof. Let $r \neq s \in \mathfrak{R}$ and (π_1, Σ), (π_2, Σ) be two HS open sets with $r \in (\pi_1, \Sigma)$, $s \in (\pi_1, \Sigma)^c$ and $s \in (\pi_2, \Sigma)$, $r \in (\pi_2, \Sigma)^c$. Then, $r \in \pi_1(\ell)$, $s \in \pi_1^c(\ell)$ and $s \in \pi_2(\ell)$, $r \in \pi_2^c(\ell)$ for all $\ell \in \Sigma$. This means $r \in \pi_1(\ell)$, $s \notin \pi_1(\ell)$ and $s \in \pi_2(\ell)$, $r \notin \pi_2(\ell)$ with $\pi_1(\ell) \cap \pi_2(\ell) = \phi$ for all $\ell \in \Sigma$. Then, we have $r \in (\pi_1, \Sigma)$ and $s \in (\pi_2, \Sigma)$ with $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. Thus, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space. □

Proposition 3.14. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $\Upsilon \subseteq \mathfrak{R}$. If $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space, then $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_2-space.*

Proof. Let $r \neq s \in \Upsilon$. Since $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. This means $r \in \pi_1(\ell)$ and $s \in \pi_2(\ell)$ with $\pi_1(\ell) \cap \pi_2(\ell) = \phi$ for all $\ell \in \Sigma$. Since $r, s \in \Upsilon$, then $r \in \Upsilon \cap \pi_1(\ell) = \pi_{1_\Upsilon}(\ell)$ and $s \in \Upsilon \cap \pi_2(\ell) = \pi_{2_\Upsilon}(\ell)$ with $\pi_{1_\Upsilon}(\ell) \cap \pi_{2_\Upsilon}(\ell) = \phi$ for all $\ell \in \Sigma$. Then, $r \in (\pi_{1_\Upsilon}, \Sigma)$ and $s \in (\pi_{2_\Upsilon}, \Sigma)$ with $(\pi_{1_\Upsilon}, \Sigma) \widetilde{\sqcap} (\pi_{2_\Upsilon}, \Sigma) = (\widetilde{\phi}, \Sigma)$. Hence $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_2-space. □

Proposition 3.15. *Every HS T_i-space is HS T_{i-1}-space, for $i = 1, 2$.*

Proof. Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \neq s \in \mathfrak{R}$. For the case $i = 1$, let $(\mathfrak{R}, T_H, \Sigma)$ be an HS T_1-space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \notin (\pi_1, \Sigma)$ and $s \in (\pi_2, \Sigma)$, $r \notin (\pi_2, \Sigma)$. This implies $r \in (\pi_1, \Sigma)$, $s \notin (\pi_1, \Sigma)$ or $s \in (\pi_2, \Sigma)$, $r \notin (\pi_2, \Sigma)$. Thus, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_0-space. Now, for the case $i = 2$, let $(\mathfrak{R}, T_H, \Sigma)$ be an HS T_2-space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. This means $r \in \pi_1(\ell)$, $s \in \pi_2(\ell)$ and $\pi_1(\ell) \cap \pi_2(\ell) = \phi$ for all $\ell \in \Sigma$. Then, we have $r \in \pi_1(\ell)$, $s \notin \pi_1(\ell)$ and $s \in \pi_2(\ell)$, $r \notin \pi_2(\ell)$ for all $\ell \in \Sigma$. Thus, $r \in (\pi_1, \Sigma)$, $s \notin (\pi_1, \Sigma)$ and $s \in (\pi_2, \Sigma)$, $r \notin (\pi_2, \Sigma)$. Therefore, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space. □

Remark 3.16. The next example shows the opposite of Proposition 3.15 is incorrect.

Example 3.17. Let $(\mathfrak{R}, T_H, \Sigma)$ be the same as in Example 3.6. Then, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space. But for r_1, $r_2 \in \mathfrak{R}$ there do not exist any two HS open sets (π_1, Σ) and (π_2, Σ) with $r_1 \in (\pi_1, \Sigma)$, $r_2 \in (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. Hence it is not an HS T_2-space. Now let $T_H = \{(\widetilde{\phi}, \Sigma), (\widetilde{\mathfrak{R}}, \Sigma), (\pi, \Sigma)\}$ where $(\pi, \Sigma) = \{((\varepsilon_1, \varepsilon_3, \varepsilon_4), \{r_1\}), ((\varepsilon_2, \varepsilon_3, \varepsilon_4), \mathfrak{R})\}$. Then, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_0-space . But since for $r_1, r_2 \in \mathfrak{R}$ there do not exist HS open sets (π_1, Σ) and (π_2, Σ) with $r_1 \in (\pi_1, \Sigma)$, $r_2 \notin (\pi_1, \Sigma)$ and $r_2 \in (\pi_2, \Sigma)$, $r_1 \notin (\pi_2, \Sigma)$. Then, it is not an HS T_1-space

4. Hypersoft regular and hypersoft normal spaces

In this section, we study and characterize the regular and normal spaces of HS in detail.

Definition 4.1. An HSTS $(\mathfrak{R}, T_H, \Sigma)$ is said to be HS regular if for every HS closed set (ω, Σ) with $r \notin (\omega, \Sigma)$, there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $(\omega, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Corollary 4.2. *Let (ω, Σ) be an HS closed set of an HSTS $(\mathfrak{R}, T_H, \Sigma)$ such that $r \notin (\omega, \Sigma)$. If $(\mathfrak{R}, T_H, \Sigma)$ is an HS regular space, then there is HS open set (π, Σ) with $r \in (\pi, \Sigma)$ and $(\pi, \Sigma) \widetilde{\sqcap} (\omega, \Sigma) = (\widetilde{\phi}, \Sigma)$.*

Proposition 4.3. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \in \mathfrak{R}$. If $\mathfrak{R}$ is an HS regular space, then:*

(1) For an HS closed set (ω, Σ), $r \notin (\omega, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqcap} (\omega, \Sigma) = (\widetilde{\phi}, \Sigma)$;

(2) For an HS open set (π, Σ), $r \notin (\pi, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Proof. (1) Let $r \notin (\omega, \Sigma)$. Then, there is an HS open set (π, Σ) with $r \in (\pi, \Sigma)$ and $(\pi, \Sigma) \widetilde{\sqcap} (\omega, \Sigma) = (\widetilde{\phi}, \Sigma)$ by Corollary 4.2. Since $r \in (\pi, \Sigma)$, then by Corollary 2.18 (1), $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$. Hence, $(\pi_r, \Sigma) \widetilde{\sqcap} (\omega, \Sigma) = (\widetilde{\phi}, \Sigma)$. The converse is obtained by Corollary 2.18 (2).

(2) Let $r \notin (\pi, \Sigma)$. Then, we have two cases: (a) for all $\ell \in \Sigma$, $r \notin \pi(\ell)$ and (b) for some $\ell, \beta \in \Sigma$, $r \notin \pi(\ell)$ and $r \in \pi(\beta)$. In case (a) we have $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$. In case (b) for some $\ell, \beta \in \Sigma$, $r \in \pi^c(\ell)$ and $r \notin \pi^c(\beta)$. Hence, $(\pi, \Sigma)^c$ is an HS closed set with $r \notin (\pi, \Sigma)^c$, by (1), $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma)^c = (\widetilde{\phi}, \Sigma)$. So $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$ but this contradicts $r \notin \pi(\ell)$ for some $\ell \in \Sigma$. Thus, $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$. The converse is obvious. □

Proposition 4.4. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \in \mathfrak{R}$. Then, these are equivalent:*

1) $(\mathfrak{R}, T_H, \Sigma)$ is an HS regular space;

(2) For each HS closed set (ω, Σ) with $(\pi_r, \Sigma) \widetilde{\sqcap} (\omega, \Sigma) = (\widetilde{\phi}, \Sigma)$, there are HS open sets (π_1, Σ) and (π_2, Σ) with $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi_1, \Sigma)$, $(\omega, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Proof. Follows from Proposition 4.3 (1) and Corollary 2.18 (1). □

Proposition 4.5. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $r \in \mathfrak{R}$. If $\mathfrak{R}$ is an HS regular space, then:*

(1) For an HS open set (π, Σ), $r \in (\pi, \Sigma)$ if and only if $r \in \pi(\ell)$ for some $\ell \in \Sigma$;

(2) For an HS open set (π, Σ), $(\pi, \Sigma) = \widetilde{\sqcup}\{(\pi_r, \Sigma) : r \in \pi(\ell)$ for some $\ell \in \Sigma\}$.

Proof. (1) Let $r \in \pi(\ell)$ for some $\ell \in \Sigma$, and $r \notin (\pi, \Sigma)$. Then, $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$ by Proposition 4.3 (2). But this contradicts our assumption and hence $r \in (\pi, \Sigma)$. The converse is obvious.

(2) Follows from (1) and $r \in (\pi, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$. □

Proposition 4.6. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS . If $(\mathfrak{R}, T_H, \Sigma)$ is an HS regular space, then the these are equivalent:*

(1) $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space;

(2) For $r \neq s \in \mathfrak{R}$, there are HS open sets (π_1, Σ) and (π_2, Σ) with $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi_1, \Sigma)$ and $(\pi_s, \Sigma) \widetilde{\sqcap} (\pi_1, \Sigma) = (\widetilde{\phi}, \Sigma)$; and $(\pi_s, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ and $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Proof. $r \in (\pi, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$ and, by Proposition 4.3 (2), $r \notin (\pi, \Sigma)$ if and only if $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$. Therefore the above statements are equivalent. □

Definition 4.7. An HSTS $(\mathfrak{R}, T_H, \Sigma)$ is said to be HS T_3-space if it is HS regular and HS T_1-space.

Proposition 4.8. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS and $\Upsilon \subseteq \mathfrak{R}$. If $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_3-space, then $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_3-space.*

Proof. Since $(\mathfrak{R}, T_H, \Sigma)$ be an HS T_3-space, then it is HS T_1-space. By Proposition 3.11, $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_1-space. Let $r \in \Upsilon$ and let (ω, Σ) be an HS closed set in Υ with $r \notin (\omega, \Sigma)$. Then, $r \notin \omega(\ell)$ for some $\ell \in \Sigma$. Since (ω, Σ) be an HS closed set in Υ, then there is an HS closed set (η, Σ) in $\mathfrak{R}$ with $\omega(\ell) = \eta(\ell) \cap \Upsilon$. Since $r \notin \omega(\ell)$ for some $\ell \in \Sigma$, then $r \notin \eta(\ell) \cap \Upsilon = \omega(\ell)$ and hence $r \notin (\eta, \Sigma)$. As $(\mathfrak{R}, T_H, \Sigma)$ is HS regular space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$ and $(\eta, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ with $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. Now, if we take $(\pi_{1_\Upsilon}, \Sigma)$ and $(\pi_{2_\Upsilon}, \Sigma)$ as two HS open sets in Υ, then $\pi_{1_\Upsilon}(\ell) = \pi_1(\ell) \cap \Upsilon$ and $\pi_{2_\Upsilon}(\ell) = \pi_2(\ell) \cap \Upsilon$. This means $r \in (\pi_{1_\Upsilon}, \Sigma)$ and $(\omega, \Sigma) \widetilde{\sqsubseteq} (\pi_{1_\Upsilon}, \Sigma)$ with $(\pi_{1_\Upsilon}, \Sigma) \widetilde{\sqcap} (\pi_{2_\Upsilon}, \Sigma) = (\widetilde{\phi}, \Sigma)$. Thus, $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS regular space and hence $(\Upsilon, T_{H_\Upsilon}, \Sigma)$ is an HS T_3-space. □

Proposition 4.9. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS . If $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_3-space, then (π_r, Σ) is an HS closed for each $r \in \mathfrak{R}$.*

Proof. For each $s \in \mathfrak{R} \setminus \{r\}$, since $(\mathfrak{R}, T_H, \Sigma)$ is an HS regular and HS T_1-space, then by Proposition 4.6, there is an HS open set (π, Σ) with $(\pi_s, \Sigma) \widetilde{\sqsubseteq} (\pi, \Sigma)$ and $(\pi_r, \Sigma) \widetilde{\sqcap} (\pi, \Sigma) = (\widetilde{\phi}, \Sigma)$. So, $\widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}(\pi, \Sigma) \widetilde{\sqsubseteq} (\pi_r, \Sigma)^c$. Now, for each $s \in \mathfrak{R} \setminus \{r\}$ and for each $\ell \in \Sigma$, $\pi_r{}^c(\ell) = \mathfrak{R} \setminus \{r\} = \widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}\{s\} = \widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}\{s(\ell)\} \widetilde{\sqsubseteq} \widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}\{\pi(\ell)\}$. This means $(\pi_r, \Sigma)^c \widetilde{\sqsubseteq} \widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}(\pi, \Sigma)$ and hence $(\pi_r, \Sigma)^c = \widetilde{\sqcup}_{s\in\mathfrak{R}\setminus\{r\}}(\pi, \Sigma)$. Since (π, Σ) is an HS open set for each $s \in \mathfrak{R} \setminus \{r\}$. Hence, (π_r, Σ) is an HS closed. □

Proposition 4.10. *An HS T_3-space is HS T_2-space.*

Proof. Let $(\mathfrak{R}, T_H, \Sigma)$ be an HS T_3-space. For $r \neq s \in \mathfrak{R}$, (π_s, Σ) is an HS closed set and $r \notin (\pi_s, \Sigma)$ by Proposition 4.9. Since $(\mathfrak{R}, T_H, \Sigma)$ is HS regular space, then there are HS open sets (π_1, Σ) and (π_2, Σ) with $r \in (\pi_1, \Sigma)$, $s \in (\pi_s, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ and $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$. Hence, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space. □

Remark 4.11. The opposite of Proposition 4.10 is not true in general.

Example 4.12. Let $\mathfrak{R} = \{r_1, r_2\}$, $\sigma_1 = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, $\sigma_2 = \{\varepsilon_4\}$, and $\sigma_3 = \{\varepsilon_5\}$. Let $T_H = \{(\widetilde{\phi}, \Sigma), (\widetilde{\mathfrak{R}}, \Sigma), (\pi_1, \Sigma), (\pi_2, \Sigma), (\pi_3, \Sigma), (\pi_4, \Sigma), (\pi_5, \Sigma), (\pi_6, \Sigma), (\pi_7, \Sigma)\}$ be an HST defined on $\mathfrak{R}$, where
$(\pi_1, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \{r_1\})\}$;
$(\pi_2, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \{r_2\})\}$;
$(\pi_3, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \{r_1\})\}$;
$(\pi_4, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \{r_2\})\}$;
$(\pi_5, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_1\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \mathfrak{R}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \mathfrak{R})\}$;
$(\pi_6, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \{r_2\}), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \mathfrak{R}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \mathfrak{R})\}$;
$(\pi_7, \Sigma) = \{((\varepsilon_1, \varepsilon_4, \varepsilon_5), \phi), ((\varepsilon_2, \varepsilon_4, \varepsilon_5), \mathfrak{R}), ((\varepsilon_3, \varepsilon_4, \varepsilon_5), \mathfrak{R})\}$.

Then, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_2-space. But it is not HS T_3-space since for $r_1 \notin (\pi_3, \Sigma)^c$, there do not exist HS open sets (π, Σ) and (θ, Σ) with $r_1 \in (\pi, \Sigma)$, $(\pi_3, \Sigma)^c \widetilde{\sqsubseteq} (\theta, \Sigma)$ and $(\pi, \Sigma) \widetilde{\sqcap} (\theta, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Definition 4.13. An HSTS $(\mathfrak{R}, T_H, \Sigma)$ is said to be HS normal if for every HS closed sets (ω_1, Σ) and (ω_2, Σ) with $(\omega_1, \Sigma) \widetilde{\sqcap} (\omega_2, \Sigma) = (\widetilde{\phi}, \Sigma)$, there are HS open sets (π_1, Σ) and (π_2, Σ) such that $(\omega_1, \Sigma) \widetilde{\sqsubseteq} (\pi_1, \Sigma)$ and $(\omega_2, \Sigma) \widetilde{\sqsubseteq} (\pi_2, \Sigma)$ with $(\pi_1, \Sigma) \widetilde{\sqcap} (\pi_2, \Sigma) = (\widetilde{\phi}, \Sigma)$.

Proposition 4.14. *Let $(\mathfrak{R}, T_H, \Sigma)$ be an HSTS. If $(\mathfrak{R}, T_H, \Sigma)$ is an HS normal space and if (π_r, Σ) is an HS closed set for each $r \in \mathfrak{R}$, then $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_3-space.*

Proof. Since (π_r, Σ) is an HS closed set for each $r \in \mathfrak{R}$, then by Proposition 3.4, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_1-space. It is also HS regular space by Proposition 4.4 and Definition 4.13. Hence, $(\mathfrak{R}, T_H, \Sigma)$ is an HS T_3-space. □

Definition 4.15. An HSTS $(\mathfrak{R}, T_H, \Sigma)$ is said to be HS T_4-space if it is HS normal and HS T_1-space.

Remark 4.16. An HS T_4-space need not be HS T_3-space.

Example 4.17. Suppose that $(\mathfrak{R}, T_H, \Sigma)$ is the same as in Example 4.12. Then, it is obvious that $(\mathfrak{R}, T_H, \Sigma)$ is HS T_4-space but is not HS T_3-space.

Now, we summarize the relationships between the HS T_i-spaces for $i = 0, 1, 2, 3, 4$.

$$\text{HS } T_4\text{-space} \nrightarrow \text{HS } T_3\text{-space} \overset{\nleftarrow}{\rightarrow} \text{HS } T_2\text{-space} \overset{\nleftarrow}{\rightarrow} \text{HS } T_1\text{-space} \overset{\nleftarrow}{\rightarrow} \text{HS } T_0\text{-space}$$

5. Conclusions

In this article, we continued to investigate the topological structures of HS sets by introducing HS separation axioms, namely HS T_i-spaces ($i = 0, 1, 2, 3, 4$). We showed that every HS T_i-space is HS T_{i-1} for $i = 1, 2, 3$, but HS T_4-space need not be HS T_3-space using an example as reference. We also showed that every HS subspace of HS T_i-spaces is HS T_i-spaces for $i = 0, 1, 2, 3$. Finally, since we indicated the relation between the HS sets and the ordinary points, some other relations between them can be defined. This lead to the study of many other types of HS separation axioms.

Acknowledgement

We thank the editor and referees for valuable comments and suggestions for improving the paper.

References

[1] M. Abbas, G. Murtaza, F. Smarandache, Basic operations on hypersoft sets and hypersoft point, Neutrosophic Sets Syst. 35 (2020) 407–421.
[2] M. Abdel-Basset, W. Ding, R. Mohamed, N. Metawa, An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries, Risk Manag. 22 (2020) 192–218.
[3] M. Ali, F. Feng, X. Liu, W. Min, M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[4] T. M. Al-shami, Comments on some results related to soft separation axioms, Afr. Mat. 31 (2020) 1105–1119.
[5] S. Bayramov, C.G. Aras, A new approach to separability and compactnees in soft topological spaces, TWMS J. Pure Appl. Math. 9 (2018) 82–93.
[6] N. Çağman, S. Enginoglu, Soft set theory and uni-int decision making, Eur. J. Oper. Res 207 (2010) 848–855.
[7] D. Dubois, H. Prade, An introduction to representations of information and preference. Int. J. Intell. Syst. 23 (2008) 866–877.
[8] X. Ge, S. Yang, Investigations on some operations of soft set, World Acad. Eng. Technol 75 (2011) 1113–1116.
[9] F. Li, Notes on soft set operations, ARPN J. Syst. Software (2011) 205–208.
[10] P. K. Maji, R. Biswas, R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[11] M. Matejdes, On some operations on soft topological spaces, Filomat 5 (2021) 1693–1705.
[12] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.
[13] S.Y. Musa, B.A. Asaad, Hypersoft topological spaces, Neutrosophic Sets Syst. 49 (2022) 397–415.
[14] S.Y. Musa, B.A. Asaad, Connectedness on hypersoft topological spaces, Neutrosophic Sets Syst. 51 (2022) 666–680.
[15] S.Y. Musa, B.A. Asaad, Bipolar hypersoft sets, Mathematics 9 (2021) 1826.
[16] S.Y. Musa, B.A. Asaad, Topological structures via bipolar hypersoft sets, J. Math. 2022 (2022), Article ID 2896053.
[17] S.Y. Musa, B.A. Asaad, Connectedness on bipolar hypersoft topological spaces, J. Intell. Fuzzy Syst. 43 (2022) 4095–4105.
[18] M. Saeed, M. Ahsan, M. Siddique, M. Ahmad, A study of the fundamentals of hypersoft set theory, Int. J. Sci. Eng. Res. 11 (2020).
[19] M. Saeed, M. Hussain, A.A. Mughal, A study of soft sets with soft members and soft elements: A new appraoch, Punjab Univ. J. Math 52 (2020).
[20] M. Saeed, A. Rahman, M. Ahsan, F. Smarandache, An inclusive study on fundamentals of hypersoft set. In: Theory and Application of Hypersoft Set, 2021 ed., Pons Publishing House: Brussels, Belgium, 2021, pp. 1–23.
[21] M. Shabir, M. Naz, On Soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[22] F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic Sets Syst. 22 (2018) 168–170.
[23] O. Tantawy, S. A. El-Sheikh, S. Hamde, Separation axioms on soft topological spaces, Ann. Fuzzy Math. Inform. 11 (2016) 511–525.
[24] M. Zhou, S. Li, M. Akram, Categorical properties of soft sets, Sci. World J. (2014) Article ID 783056.