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Generalized Wintgen Inequality for Submanifolds in Standard Warped Product Manifolds

Aliya Naaz Siddiqui[a], Kamran Ahmad[a], Ali Hussain Alkhaldi[b], Lamia Saeed Alqahtani[c]

[a]*Division of Mathematics, School of Basic & Applied Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India*
[b]*Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia*
[c]*Department of Mathematics, Faculty of Science, King Abdulaziz University, 21589 Jeddah, Saudi Arabia*

Abstract. Roth (Bull. Aust. Math. Soc. 95, 495-499 (2017)) gave a result for DDVV inequality submanifolds of warped product manifolds. Then Murathan et al. (J. Geom. (2018) 109:30) obtained the Wintgen-like inequality for statistical submanifolds of statistical warped product manifolds. Recently, Gorunus et al. (Inter. Elec. J. Geom. 12(1), 43-56 (2019)) established the generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu statistical manifolds. Thus, in the present manuscript, we find the generalized Wintgen inequality for a Legendrian submanifold in the standard warped product manifolds (β-Kenmotsu manifold).

1. Introduction

The concept of warped products plays crucial roles in mathematical physics, especially in general relativity. For example, different models of space-time in general relativity can be expressed in terms of warped geometry such as the Robertson-Walker space-time, the Friedman cosmological models and the standard static space-time; and on the other hand, the Einstein field equations and modified field equations have so many exact solutions as the warped products. In this reason, the study of warped products is the most inventive topic in differential geometry. R.L. Bishop and O'Neill give the idea of warped product manifolds in order to construct examples of Riemannian manifolds of negative curvature in [2].

Let (N_1, g^{N_1}) and (N_2, g^{N_2}) be two Riemannian manifolds and $f > 0$ be a smooth function on N_1. Consider the product $f_1 : N_1 \times N_2 \longrightarrow N_1$ and $f_2 : N_1 \times N_2 \longrightarrow N_2$. Then the warped product $\overline{N} = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$\overline{g}(E, F) = g^{N_1}(f_{1*}E, f_{1*}F) + f^2(u)g^{N_2}(f_{2*}E, f_{2*}F), \tag{1}$$

for any $E, F \in \Gamma(T_{(u,v)}\overline{N})$, $u \in N_1$ and $v \in N_2$, where $*$ is the symbol for the tangent maps, and we have $\overline{g} =<,>= g^{N_1} + f^2 g^{N_2}$. The function f is called the warping function of the warped product.

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Communicated by Mića Stanković
Corresponding author: Aliya Naaz Siddiqui
Email addresses: aliya.siddiqui@galgotiasuniversity.edu.in (Aliya Naaz Siddiqui), kamran.ahmad@galgotiasuniversity.edu.in (Kamran Ahmad), ahalkhaldi@kku.edu.sa (Ali Hussain Alkhaldi), lalqahtani@kau.edu.sa (Lamia Saeed Alqahtani)

Let $\chi(N_1)$ and $\chi(N_2)$ be the set of all vector fields on $N_1 \times N_2$ which is the horizontal lift of a vector field on N_1 and the vector lift of a vector field on N_2, respectively. Thus, it can be seen that $f_{1*}(\chi(N_1)) = \Gamma(TN_1)$ and $f_{2*}(\chi(N_2)) = \Gamma(TN_2)$.

Lemma 1.1. *[2] Let $\overline{N} = N_1 \times_f N_2$ be a warped product manifold. For any $X, Y \in \chi(N_1)$ and $U, V \in \chi(N_2)$, we have*

1. $\overline{\nabla}_X Y \in \chi(N_1)$,
2. $\overline{\nabla}_U X = \overline{\nabla}_X U = (X \ln f)U$,
3. $\overline{\nabla}_U V = \nabla^{N_2}_U V - < U, V > \overline{\nabla} \ln f$,

where $\overline{\nabla}$ and ∇^{N_2} denotes the Levi-Civita connections on $\overline{N}$ and N_2, and $\overline{\nabla} \ln f$ is the gradient of $\ln f$.

For the survey on warped product submanifolds and their geometric obstructions in different structures, see [5].

P. Wintgen [21] proposed a nice relationship between the Gauss curvature (intrinsic), the normal curvature (extrinsic) and squared mean curvature (extrinsic) of any surface in a 4-dimensional Euclidean space and also discussed the necessary and sufficient conditions for which the equality case holds. Later or, I.V. Guadalupe and L. Rodriguez extended Wintgen's inequality to surfaces of arbitrary codimension in real space forms $\mathbb{R}^{2+m}(c)$, $m \geq 2$. Then, B.-Y. Chen extended this inequality to surfaces in a 4-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ with a neutral metric.

De Smet, et al. [6] have conjectured generalized Wintgen inequality for any submanifold in real space forms $\mathbb{R}^m(c)$:

$$\rho_{nor} \leq \|\mathcal{H}\|^2 - \rho^{\perp} + c,$$

where ρ_{nor} denotes the normalized scalar curvature of the submanifold M, called an intrinsic invariant of M, $\rho^{\perp}$ denotes the normalized normal scalar curvature of M and $\mathcal{H}$ is the mean curvature vector of M. Here $\rho^{\perp}$ and $\mathcal{H}$ are called extrinsic invariants of M. This conjecture is known as DDVV conjecture or also known as the generalized Wintgen inequality. Many authors have expanded the foregoing inequality to the situation of ambient spaces other than real space forms in many geometrical features. The generalized Wintgen inequality for submanifolds in Riemannian space forms was proved by Z. Lu in [11] and J. Ge and Z.Z. Tang in [7], independently. Afterwards, I. Mihai refined a generalized Wintgen inequality for Lagrangian submanifolds in complex space forms [12] and as well as Legendrian submanifolds in Sasakian space forms [15]. In addition, A.N. Siddiqui et al. [19] examined the generalized Wintgen inequality for totally real submanifolds in LCS-manifolds. Recently, H. Alodan et al. [1] intensively investigated the generalized Wintgen inequality for quaternionic CR-submanifolds of quaternionic Kähler manifolds of constant quaternionic sectional curvature. In fact, many remarkable articles were published in the recent years and several inequalities of this type have been obtained for other classes of submanifolds in several ambient spaces, for details see [4, 18, 20].

In this paper, first we are going to prove that the standard warped product manifold $\overline{N} = I \times_f N$ is a β-Kenmotsu manifold, if (N, g^N, ∇^N, J) is a Kähler manifold, where $\beta = \frac{f'(z)}{f(z)}$. Then the Riemannian curvature tensor of $\overline{N}$ can also be obtained by taking help from [10] and [8]. Now, the main purpose of this paper is to extend the classical DDVV inequality proved by Roth in [17] to a Legendrian submanifold in $\overline{N} = I \times_f N(c)$, where $N(c)$ is a complex space form, a new study to the developments in Wintgen inequality and Wintgen ideal submanifolds done mainly in the last 15 years.

2. Preliminaries

In this section, we give some basic definitions and notations for the next sections.

Let $\overline{N}$ be a complex m-dimensional Kähler manifold, that is, $\overline{N}$ is endowed with an almost complex structure J and with a J-Hermitian metric g, we have

$$J^2 = -I, \quad g(JE, JF) = g(E, F), \quad \overline{\nabla} J = 0,$$

where $\overline{\nabla}$ is the Levi-Civita connection of g and the covariant derivative of the complex structure J is defined as [22]

$$(\overline{\nabla}_E J)F = \overline{\nabla}_E JF - J\overline{\nabla}_E F,$$

for any $E, F \in \Gamma(T\overline{N})$. It follows that J is integrable. Here $\Gamma(T\overline{N})$ denotes the Lie algebra of vector fields and on $\overline{N}$.

If the ambient manifold $\overline{N}$ is of constant holomorphic sectional curvature c, then $\overline{N}$ is called a complex space form and is denoted by $\overline{N}(c)$. Thus, the Riemannian curvature tensor $\overline{R}$ of $\overline{N}(c)$ is given as [22]

$$\begin{aligned}\overline{R}(E, F, G, H) \quad = \quad & \frac{c}{4}\Big[g(F, G)g(E, H) - g(E, G)g(F, H) + g(JF, G)g(JE, H) - g(JE, G)g(JF, H) \\ & +2g(E, JF)g(JG, H)\Big],\end{aligned} \tag{2}$$

for any $E, F, G, H \in \Gamma(T\overline{N})$.

A complete simply-connected complex space form $\overline{N}(c)$ is holomorphically isometric to the complex Euclidean m-space $\mathbb{C}^m$, the complex projective m-space $\mathbb{CP}^m(4c)$, or the complex hyperbolic m-space $\mathbb{CH}^m(4c)$, according to $c = 0$, $c > 0$ or $c < 0$, respectively.

It is well known from literature that a $(2m + 1)$-dimensional manifold $\overline{N}$ endowed with almost contact structure (ϕ, ξ, η, g) is called an almost contact metric manifold when satisfies the following properties [3, 22]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{3}$$

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F), \quad \text{and} \quad \eta(E) = g(E, \xi), \tag{4}$$

for any $E, F \in \Gamma(T\overline{N})$, where ϕ, g, ξ and η are called $(1, 1)$-tensor fields, a structure vector field and dual 1-form, respectively.

An almost contact metric manifold $(\overline{N}, g, \phi, \eta, \xi)$ is said to be β-Kenmotsu manifold if the following conditions hold [9]:

$$(\overline{\nabla}_E \phi)F = -\beta\{\eta(F)\phi E + g(E, \phi F)\xi\}, \quad \overline{\nabla}_E \xi = -\beta\{-E + \eta(E)\xi\},$$

for any $E, F \in \Gamma(T\overline{N})$. Here β is a differentiable function on $\overline{N}$.

Remark 2.1. *A β-Kenmotsu manifold is called Kenmotsu manifold, if $\beta = 1$ and it is called homothetic Kenmotsu manifold, if β = constant.*

A differentiable map

$$\psi : M \longrightarrow \overline{N}$$

of a differentiable manifold M into another differentiable manifold $\overline{N}$ is called an immersion if the differential

$$d\psi : TM \longrightarrow T\overline{N}$$

is injective. Now, let M be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold $\overline{N}$ of dimension m. The Riemannian metric on M and $\overline{N}$ is denoted by the same symbol g. Let $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ denote the Lie algebra of vector fields and set of all normal vector fields on M respectively. The operator of covariant differentiation with respect to the Levi-Civita connection on M and $\overline{N}$ is denoted by ∇ and $\overline{\nabla}$, respectively. The Gauss and Weingarten formulae are respectively given as [22]

$$\overline{\nabla}_E F = \nabla_E F + h(E,F), \tag{5}$$

and

$$\overline{\nabla}_E \mathcal{N} = -A_{\mathcal{N}}(E) + \nabla^{\perp}_E \mathcal{N}, \tag{6}$$

for any $E, F \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^{\perp}M)$. Here h is the second fundamental form, A is the shape operator of M and $\nabla^{\perp}$ is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T^{\perp}M$ of M.

The second fundamental form and the shape operator are related by the following equation [22]:

$$g(h(E,F),\mathcal{N}) = g(A_{\mathcal{N}}(E),F), \tag{7}$$

for any $E, F \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^{\perp}M)$.

Denote by $\overline{R}$ and R the curvature tensor of $\overline{N}$ and M, respectively. Then the equation of Gauss is given by [22]

$$\begin{aligned}\overline{R}(E,F,G,H) &= R(E,F,G,H) - g(A_{h(F,G)}E,H) \\ &\quad + g(A_{h(E,G)}F,H),\end{aligned} \tag{8}$$

for any $E, F, G, H \in \Gamma(TM)$. Here

$$\overline{R}(E,F,G,H) = g(\overline{R}(E,F)G,H)$$

and

$$R(E,F,G,H) = g(R(E,F)G,H).$$

The Ricci equation is given by [22]

$$\overline{R}(E,F,\mathcal{N},\mathcal{N}') = R^{\perp}(E,F,\mathcal{N},\mathcal{N}') - g([A_{\mathcal{N}},A_{\mathcal{N}'}]E,F), \tag{9}$$

for any $\mathcal{N}, \mathcal{N}' \in \Gamma(T^{\perp}M)$. Here $R^{\perp}$ is the normal curvature tensor with respect to $\nabla^{\perp}$ and $[A_{\mathcal{N}}, A_{\mathcal{N}'}] = A_{\mathcal{N}}A_{\mathcal{N}'} - A_{\mathcal{N}'}A_{\mathcal{N}}$.

Let $x \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, \ldots, e_m\}$ be an orthonormal basis of $T_x^{\perp}M$. The mean curvature vector $\mathcal{H}$ of the submanifold M at x is given by the following relation [22]:

$$\mathcal{H}(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

Also, we set

$$h^a_{ij} = g(h(e_i,e_j),e_a), \quad i,j \in \{1,\ldots,n\}, \quad a \in \{n+1,\ldots,m\},$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).$$

The mean curvature vector is said to be parallel in the normal bundle, if

$$\nabla^{\perp}\mathcal{H} = 0$$

holds identically.

Let M be an n-dimensional submanifold of an $(2m+1)$-dimensional almost contact metric manifold $(\overline{N}, g, \phi, \eta, \xi)$. If $\phi(T_xM) \subset T_x^{\perp}M$, then M is called C-totally real. But if $n = m$, then a C-totally real submanifold becomes Legendrian submanifold. Thus, it is easy to say that Legendrian submanifold is a C-totally real submanifold with the smallest possible codimension (see [22]).

3. Standard Warped Products

In the following, we consider an open interval I in $\mathbb{R}$ with metric

$$g^I = dz^2$$

and a $2m$-dimensional almost Hermitian manifold N. We deal with the standard warped product manifold $\overline{N} = I \times_f N$ with warping function $f > 0$, endowed with the Riemannian metric

$$\overline{g} =<,>= g^I + f^2 g^N.$$

Theorem 3.1. *Let $(\overline{N} = I \times_f N, \overline{\nabla}, <,>)$ be the standard warped product manifold. Then $\overline{N}$ is a β-Kenmotsu manifold, if (N, g^N, ∇^N, J) is a Kähler manifold, where $\beta = \frac{f'(z)}{f(z)}$ and ∇^N denotes the Levi-Civita connection on N.*

Proof. First we define an almost contact metric structure on $\overline{N}$. So, we denote the structure vector field on $\overline{N}$ by

$$\xi = \frac{\partial}{\partial z},$$

for $z \in I$. Also, an arbitrary vector field on $\overline{N}$ by

$$U = E - \eta(E)\xi \ \in \chi(N), \quad dz = \eta.$$

With the help of tensor field J, a new tensor field ϕ of type $(1,1)$ is defined on $\overline{N}$ as

$$\phi E = JU,$$

for any $E \in \Gamma(T\overline{N})$. So, we can get the following:

$$\begin{aligned}
\phi\xi &= \phi(\xi + 0) = J0 = 0, \\
\eta(\phi E) &= <\phi E, \xi>= f^2 g^N(JU, \xi) = 0, \\
\phi^2 E &= -E + \eta(E)\xi, \\
<\phi E, F> &= -<X, \phi F>, \\
<\phi E, \phi F> &= <E, F> -\eta(E)\eta(F),
\end{aligned}$$

for any $E, F \in \Gamma(T\overline{N})$. Hence, $(\overline{N}, <,>, \phi, \xi, \eta)$ is an almost contact metric manifold.

Moreover, by using Lemma 1.1, it is easy to prove that

$$\begin{aligned}(\overline{\nabla}_E\phi)F &= (\nabla^N_E J)F + \Big(-\frac{f'(z)}{f(z)}\Big)\Big\{<E,\phi F>\xi + \eta(F)\phi E\Big\}\\ &= \Big(-\frac{f'(z)}{f(z)}\Big)\Big\{<E,\phi F>\xi + \eta(F)\phi E\Big\}\\ &= -\beta\Big\{<E,\phi F>\xi + \eta(F)\phi E\Big\}\end{aligned}$$

and

$$\overline{\nabla}_E\xi = \Big(-\frac{f'(z)}{f(z)}\Big)(-E+\eta(E)\xi) = -\beta(-E+\eta(E)\xi).$$

□

We recall methods as in [10] and by using equation (2) to derive the Riemannian curvature tensor $\overline{S}$ of β-Kenmotsu manifold $(\overline{N} = I\times_f N(c), <,>, \phi, \xi, \eta)$, where $N(c)$ is a complex space form.

$$\begin{aligned}\overline{S}(E,F,G,H) &= \Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2\Big)(<F,G><E,H> - <E,G><F,H>)\\ &\quad -\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)(<F,G><E,\partial><\partial,H>\\ &\quad - <E,G><F,\partial><\partial,H> + <F,\partial><G,\partial><E,H>\\ &\quad - <E,\partial><G,\partial><F,H>) + \Big(\frac{c}{4f^2}\Big)(<E,\phi G><\phi F,H>\\ &\quad - <F,\phi G><\phi E,H> + 2<E,\phi F><\phi G,H>,\end{aligned} \tag{10}$$

for any $E,F,G,H \in \Gamma(T\overline{N})$. Here $f' = \frac{df}{dz}$, $f'' = \frac{d^2f}{dz^2}$ and the unit tangent vector field on I by $\partial = \frac{\partial}{\partial z}$, for $z \in I$.

By using (10) and (8), we can rewrite the Riemannian curvature tensor R of M as

$$\begin{aligned}R(E,F,G,H) &= \Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2\Big)(<F,G><E,H> - <E,G><F,H>)\\ &\quad -\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)(<F,G><E,\partial><\partial,H>\\ &\quad - <E,G><F,\partial><\partial,H> + <F,\partial><G,\partial><E,H>\\ &\quad - <E,\partial><G,\partial><F,H>) + \Big(\frac{c}{4f^2}\Big)(<E,\phi G><\phi F,H>\\ &\quad - <F,\phi G><\phi E,H> + 2<E,\phi F><\phi G,H>\\ &\quad + g(A_{h(E,G)}F,H) - g(A_{h(F,G)}E,H),\end{aligned} \tag{11}$$

for any $E,F,G,H \in \Gamma(TM)$.

4. Generalized Wintgen Inequality for Legendrian Submanifolds

In this section, we obtain the generalized Wintgen inequality for an n-dimensional Legendrian submanifold in a $(2n+1)$-dimensional standard warped product manifold $(\overline{N} = I\times_f N(c), \overline{\nabla}, <,>)$, where $N(c)$ is a complex space form. Before going to proceed further, first we give the following propositions and lemmas which are useful in obtaining the said inequality.

Suppose a local orthonormal tangent frame be $\{e_1, \dots, e_n\}$ of T_xM and a local orthonormal normal frame be $\{e_{n+1} = \phi e_1, \dots, e_2n, e_{2n+1} = \xi\}$ of $T_x^{\perp}N$ in $\overline{N}$, $x \in N$. The normalized scalar curvature ρ_{nor} of M is given by

$$\begin{aligned}\rho_{nor} &= \frac{2}{n(n-1)} \sum_{1\le i<j\le n} g(R(e_i, e_j)e_j, e_i) \\ &= \frac{1}{n(n-1)} \sum_{i\neq j} g(R(e_i, e_j)e_j, e_i). \end{aligned} \tag{12}$$

We denote the tangent part of $\partial \in \chi(I)$ by $T \in \Gamma(TM)$, given by

$$T = \sum_{i=1}^{n} < \partial, e_i > e_i.$$

On using (11) and (12), we have

Proposition 4.1. *Let M be an n-dimensional Legendrian submanifold in the standard warped product manifold $\overline{N} = I \times_f N(c)$, then the normalized scalar curvature of M is the following:*

$$\rho_{nor} = \frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 - \frac{2}{n}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)\|T\|^2 + \frac{1}{n(n-1)}[n^2\|\mathcal{H}\|^2 - \|h\|^2].$$

If we fix $\tau = h - \mathcal{H}g$ the traceless part of the second fundamental form, then we get

$$\|\tau\|^2 = \|h\|^2 - n\|\mathcal{H}\|^2.$$

Thus, Proposition 4.1 reduces to

Corollary 4.2. *Let M be an n-dimensional Legendrian submanifold in the standard warped product manifold $\overline{N} = I \times_f N(c)$, then the normalized scalar curvature of M is the following:*

$$\rho_{nor} = \frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 - \frac{2}{n}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)\|T\|^2 - \frac{\|\tau\|^2}{n(n-1)} + \|\mathcal{H}\|^2.$$

The normalized normal scalar curvature of M is given by

$$\rho^{\perp} = \frac{2}{n(n-1)}\Big\{ \sum_{1\le a<b\le n} \sum_{1\le i<j\le n} < R^{\perp}(e_i, e_j)e_{n+a}, e_{n+b} >^2 \Big\}^{1/2}, \tag{13}$$

where $R^{\perp}$ is the normal connection of M.

Proposition 4.3. *Let M be an n-dimensional Legendrian submanifold in the standard warped product manifold $\overline{N} = I \times_f N(c)$, then we have*

$$\rho^{\perp} \le \frac{\sqrt{2}}{2n(n-1)}\Big\{\frac{c^2}{16f^4}n^2(n-1)^2 + \sum_{a,b=1}^{n} \|[A_a, A_b]\|^2\Big\}^{1/2}.$$

Proof. With the help of (9) and (13), we directly compute the following:

$$\begin{aligned}\rho^{\perp} &= \frac{1}{n(n-1)}\Big\{ \sum_{1\le a<b\le n} \sum_{1\le i<j\le n} \Big[g([A_{e_{n+a}}, A_{e_{n+b}}]e_i, e_j) + \frac{c}{4f^2}(-\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})\Big]^2\Big\}^{1/2} \\ &\le \frac{\sqrt{2}}{n(n-1)}\Big\{ \sum_{1\le a<b\le n} \sum_{1\le i<j\le n} \Big[g([A_{e_{n+a}}, A_{e_{n+b}}]e_i, e_j)^2 + \frac{c^2}{16f^4}(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})^2\Big]\Big\}^{1/2} \\ &\le \frac{\sqrt{2}}{n(n-1)}\Big\{\frac{c^2}{64f^4}n^2(n-1)^2 + \frac{1}{4}\sum_{a,b=1}^{n}\sum_{i,j=1}^{n} g([A_a, A_b]e_i, e_j)^2\Big\}^{1/2},\end{aligned}$$

where we have used the algebraic inequality $(P+Q)^2 \leq 2(P^2+Q^2)$, for $P, Q \in \mathbb{R}$.

On simplifying the last inequality, we get the desired inequality. □

For any $a \in \{1, \dots, n\}$, the operator $B_a : T_xM \to T_xM$ is defined as

$$< B_a E, F > = < \tau(E,F), \mathcal{N}_a > . \tag{14}$$

Clearly, we have the relations

$$B_a = A_a - < \mathcal{H}, \mathcal{N}_a > Id$$

and

$$[A_a, A_b] = [B_a, B_b], \tag{15}$$

for $b \in \{1, \dots, n\}$. Thus, we have

Corollary 4.4. *Let M be an n-dimensional Legendrian submanifold in the standard warped product manifold $\overline{N} = I \times_f N(c)$, then we have*

$$\rho^{\perp} \leq \frac{\sqrt{2}}{2n(n-1)} \Big\{ \frac{c^2}{16 f^4} n^2 (n-1)^2 + \sum_{a,b=1}^{n} \| [B_a, B_b] \|^2 \Big\}^{1/2}.$$

The following result, derived by Lu for the symmetric and trace-free operators in [11], is the most important ingredient in the proof of our desired DDVV inequality:

Theorem 4.5. *Let M be an n-dimensional Riemannian submanifold of $n+m$-dimensional Riemannian space form $\overline{N}(c)$. For every set $\{B_1, \dots, B_m\}$ of symmetric $(m \times m)$-matrices with trace zero the following inequality holds:*

$$\sum_{r,s=1}^{m} \|B_r, B_s\|^2 \leq \Big(\sum_{r=1}^{m} \|B_r\|^2 \Big)^2.$$

By using Corollaries 4.2, 4.4 and Theorem 4.5, we derive the main inequality of this section:

Theorem 4.6. *Let M be an n-dimensional Legendrian submanifold in a $(2n+1)$-dimensional standard warped product manifold $\overline{N} = I \times_f N(c)$, then we have*

$$\sqrt{2}\rho^{\perp} + \rho_{nor} \leq \frac{|c| + c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 - \frac{2}{n}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)\|T\|^2 + \|\mathcal{H}\|^2.$$

Proof. From Corollary 4.4, Theorem 4.5 and relation (14), it follows that

$$\begin{aligned} \rho^{\perp} &\leq \frac{1}{\sqrt{2}n(n-1)} \Big\{ \frac{c^2}{16 f^4} n^2 (n-1)^2 + \sum_{a,b=1}^{n} \| [B_a, B_b] \|^2 \Big\}^{1/2} \\ &\leq \frac{\sqrt{2}|c|}{8f^2} + \frac{1}{\sqrt{2}n(n-1)} \sum_{a,b=1}^{n} \|B_a\|^2 \\ &\leq \frac{\sqrt{2}|c|}{8f^2} + \frac{1}{\sqrt{2}n(n-1)} \|\tau\|^2. \end{aligned} \tag{16}$$

On combining (16) with the following relation:

$$\rho_{nor} = \frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 - \frac{2}{n}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)\|T\|^2 - \frac{\|\tau\|^2}{n(n-1)} + \|\mathcal{H}\|^2,$$

we arrive at

$$\rho^{\perp} + \frac{1}{\sqrt{2}}\rho_{nor} \quad \leq \quad \frac{\sqrt{2}|c|}{8f^2} + \frac{1}{\sqrt{2}}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2\Big) + \frac{1}{\sqrt{2}}\|\mathcal{H}\|^2 - \frac{2}{\sqrt{2}n}\Big(\frac{c}{4f^2} - \Big(\frac{f'}{f}\Big)^2 + \frac{f''}{f}\Big)\|T\|^2.$$

Thus, we get the desired inequality. □

Warped product plays a crucial role not only for Kenmotsu manifold, but for a cosymplectic manifold as well. A Kenmotsu manifold can be locally expressed as the warped product of a Kähler manifold and a line. In this reason, we set $I = \mathbb{R}$ and $f(z) = e^z$, so $\beta = \frac{f'}{f} = 1$ and we have the following consequence:

Corollary 4.7. *Let M be an n-dimensional Legendrian submanifold in Kenmotsu manifold $\overline{N} = \mathbb{R} \times_{e^z} \mathbb{C}$, then we have*

$$\|\mathcal{H}\|^2 \geq \sqrt{2}\rho^{\perp} + \rho_{nor} + 1.$$

When $f = constant(\neq 0)$, then $\beta = 0$. Thus, we have

Corollary 4.8. *Let M be an n-dimensional Legendrian submanifold in a $(2n+1)$-dimensional cosymplectic manifold $\overline{N} = \mathbb{R} \times N(c)$, then we have*

$$\sqrt{2}\rho^{\perp} + \rho_{nor} \leq \frac{|c| + c}{4} - \Big(\frac{c}{2n}\Big)\|T\|^2 + \|\mathcal{H}\|^2.$$

5. Conclusion

It is well-known that intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds. The most fascinating problem in the general theory of Riemannian submanifolds is

Problem A. "To establish some simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold."

Obviously, such simple and nice relationships can be provided by certain types of inequalities. In the present paper, we have mainly considered an isometric immersion of Riemannian manifold into the standard warped product manifold $\overline{N} = I \times_f N(c)$, where $N(c)$ is a complex space form and constant holomorphic sectional curvature c, with arbitrary codimension. A solution to the **Problem A** is obtained by using the fundamental equations for Riemannian submanifolds to establish a sharp relationship between the normalized normal scalar curvature (defined from the normal curvature tensor), normalized scalar curvature and mean curvature. The results proved here motivate further studies to modify or find similar relationships for many kinds of invariants (of similar nature) for Riemannian submanifolds in different kinds of warped product manifold, for example $\overline{N} = I \times_f N(c)$, where $N(c)$ is the simply connected real space form with constant curvature c (see [17]) and other ambient spaces (see [4]).

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