Filomat 36:19 (2022), 6655–6667
https://doi.org/10.2298/FIL2219655M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Zeros of One Class of Quaternionic Polynomials

Gradimir V. Milovanović[a], Abdullah Mir[b]

[a] *Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia & University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia*
[b] *Department of Mathematics, University of Kashmir, Srinagar, 190006, India*

Abstract. The goal of this paper is to study the properties of zeros of some special quaternionic polynomials with restricted coefficients, namely coefficients whose real and imaginary components satisfy suitable inequalities. We extend the well-known Eneström-Kakeya theorem and its various generalizations from complex to the quaternionic setting. The main tools used to derive the bounds for the zeros of these polynomials are the maximum modulus theorem and the structure of the zero sets established in the newly developed theory of regular functions and polynomials of a quaternionic variable.

1. Introduction

The task of determining the regions, mostly circular or annular, containing all the zeros of a polynomial on using various methods of the geometric function theory is a problem of interest both in mathematics and in the application areas such as physical systems. Due to this fact, various authors have studied extensively problems involving polynomials and their properties in general, and location of their zeros (cf. [1, 5, 12, 13, 16]). We can find a large body of research concerning an upper bound for the moduli of all the zeros of a polynomial when its coefficients are restricted with special conditions. One of the classical results concerning the zeros and their regional location of a restricted coefficient polynomial is known in the literature as Eneström-Kakeya theorem [12]. This classical result is particularly important in the study of stability of numerical methods for differential equations and, subsequently, many related results giving extensions and generalizations to this theorem appeared in the literature. We refer the reader to the comprehensive books of Marden [12] and Milovanović et al. [13] and the references therein for a survey of extensions and refinements of this well known result.

Theorem A (Eneström-Kakeya Theorem). *If $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, where z is a complex variable, with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(z)$ lie in $|z| \leq 1$.

2020 *Mathematics Subject Classification*. Primary 30E10; Secondary 30G35, 16K20
Keywords. Holomorphic functions, Quaternionic polynomials; Eneström-Kakeya theorem; Zero-sets of a regular function.
Received: 07 May 2022; Accepted: 12 August 2022
Communicated by Vladimir Rakočević
Research of the first author was partly supported by the Serbian Academy of Sciences and Arts (Project Φ-96).
Email addresses: gvm@mi.sanu.ac.rs (Gradimir V. Milovanović), drabmir@yahoo.com (Abdullah Mir)

The above result is of great importance in the geometric function theory because of its numerous applications in many areas of scientific disciplines and its extensions, generalizations and improvements in several directions. In 1967, Joyal, Labelle and Rahman [10] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative by proving the following result.

Theorem B. *If $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, where z is a complex variable, with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $T(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

The extension of Theorem A to complex coefficients was established by Govil and Rahman [9] in the form of the following result.

Theorem C. *If $T(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg a_\nu - \beta| \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \nu \leq n,$$

and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $T(z)$ lie in

$$|z| \leq \cos\theta + \sin\theta + 2\frac{\sin\theta}{|a_n|}\sum_{\nu=0}^{n-1} |a_\nu|.$$

As remarked before, the location of zeros of complex polynomials has been vastly studied with more focus on polynomials with restricted coefficients. The Eneström-Kakeya theorem and its various extensions as mentioned above are the classic and significant examples of this kind. The richness of the theory of holomorphic functions of one complex variable, along with motivations from many areas of scientific disciplines, aroused a natural interest in the theory of quaternion-valued functions of a quaternionic variable. Provided such a richness of the complex setting, a natural question is to ask what kind of results in the quaternionic setting can be obtained. The goal of this paper is to present extensions to the quaternionic setting of some classical results of Eneström-Kakeya type as discussed above.

2. Preliminary knowledge

In order to introduce the framework in which we will work, let us introduce some preliminaries on quaternions and regular functions of a quaternionic variable which will be useful in the sequel. Quaternions are essentially a generalization of complex numbers to four dimensions (one real and three imaginary parts) and were first studied and developed by Sir Rowan William Hamilton in 1843. This number system of quaternions is denoted by $\mathbb{H}$ in honor of Hamilton. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [17] for the basic features of quaternionic functions. In the recent study (for example, see [3], [4]–[11]), a new theory of regularity for functions, particularly for polynomials of a quaternionic variable was developed, and is extremely useful in replicating many important properties of holomorphic functions. One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except when the function vanishes identically). Given a regular function of a quaternionic variable, all its restrictions to complex lines are holomorphic and hence either have a discrete zero set or vanishes identically. In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato [6] gave a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the power series expansion of the function. It is important to mention that this result was proved for the polynomial case since long time ago, and can be found in the classical book of Lam [11]. Before we proceed further, we need to introduce some preliminaries on quaternions and quaternionic polynomials. The set of quaternions denoted by $\mathbb{H}$ is a non commutative division ring. It consists of elements of the form $q = \alpha + \beta i + \gamma j + \delta k$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = \alpha$ and the imaginary part $\mathrm{Im}(q) = \beta i + \gamma j + \delta k$. The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$. For $r > 0$, we define the ball $B(0, r) = \left\{q \in \mathbb{H} : |q| < r\right\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centred at the origin, i.e.,

$$\mathbb{B} = \left\{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\right\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \left\{q = \beta i + \gamma j + \delta k : \beta^2 + \gamma^2 + \delta^2 = 1\right\}.$$

The angle between two quaternions $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$ is given by

$$\measuredangle(q_1, q_2) = \cos^{-1}\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + \delta_1\delta_2}{|q_1||q_2|}\right).$$

We represent the indeterminate for a quaternionic polynomial as q. A quaternionic polynomial T of degree n in the variable q, namely a polynomial with coefficients on the right and indeterminate on the left is given by $T(q) =: \sum_{\nu=0}^{n} q^\nu a_\nu$, $a_\nu \in \mathbb{H}$, $\nu = 0, 1, 2, \ldots, n$. It is known that these are the only polynomials in the quaternion-valued functions of a quaternionic variable to satisfy the regularity conditions, and therefore, their behavior resembles very closely to that of holomorphic functions of a complex variable. In the theory of polynomials of this kind over skew-fields, one defines a different product (we use the symbol $*$ to denote such a product) which guarantees that the product of regular functions is regular. For polynomials, for example, this product is defined as follows:

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) = \sum_{i=0}^{n} q^i a_i$ and $T_2(q) = \sum_{j=0}^{m} q^j b_j$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{i=0,1,\ldots,n \\ j=0,1,\ldots,m}} q^{i+j} a_i b_j.$$

If T_1 has real coefficients, the so called $*$ multiplication coincides with the usual point wise multiplication. Notice that the $*$ product is associative and not, in general, commutative. The absence of commutativity leads to a behavior of polynomials rather unlike their behavior in the real or complex setting. For example, a real or complex polynomial of degree n can have at most n (real or complex) zeros counted with multiplicity. In the quaternionic setting, the second degree polynomial $q^2 + 1$ has an infinite number of zeros, namely; any $q \in \mathbb{S}$. The following result which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors is from [8] (see also [11]).

Theorem D. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.*

Since the regular functions of a quaternionic variable have been introduced and intensively studied in the past decade. In this context, Gentili and Struppa [7] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem E (Maximum Modulus Theorem). *Let $B = B(0,r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is a constant on B.*

Very recently, Carney et al. [2] extended the Eneström-Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials by making use of Theorems D and E. Firstly, they established the following quaternionic analogue of Theorem A.

Theorem F. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable, with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \leq 1.$$

In the same paper, Carney et al. [2] also established the following quaternionic analogue of Theorem C.

Theorem G. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ be a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non zero quaternion and suppose $\measuredangle(a_\nu, b) \leq \theta \leq \pi/2$ for some θ and for $\nu = 0, 1, 2, \ldots, n$. Assume that*

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|.$$

Then all the zeros of $T(q)$ lie in

$$|q| \leq \cos\theta + \sin\theta + 2\frac{\sin\theta}{|a_n|}\sum_{\nu=0}^{n-1}|a_\nu|.$$

They also proved the following result similar to Theorem B but instead of polynomials with monotone increasing real coefficients, it considers quaternionic polynomials with monotone increasing real and imaginary parts and thus giving the quaternionic analogue of Theorem B.

Theorem H. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$, is a polynomial of degree n, where q is a quaternionic variable, with quaternionic coefficients, where $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\begin{aligned}
&\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,\\
&\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0,\\
&\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_1 \geq \gamma_0,\\
&\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_1 \geq \delta_0,
\end{aligned}$$

then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)}{|a_n|}.$$

In the meantime, Tripathi [18] besides proving some other results also established the following generalization of Theorem F.

Theorem I. *Let $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ be a polynomial of degree n, where q is a quaternionic variable, with quaternionic coefficients, where $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\begin{aligned}
&\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_\ell,\\
&\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_\ell,\\
&\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_\ell,\\
&\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_\ell,
\end{aligned}$$

for $0 \le \ell \le n$. *Then all the zeros of* $T(q)$ *lie in*

$$\left|q\right| \le \frac{1}{|a_n|}\Big[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_\ell) + (\beta_n - \beta_\ell) + (\gamma_n - \gamma_\ell) + (\delta_n - \delta_\ell) + M_\ell\Big],$$

where

$$M_\ell = \sum_{\nu=1}^{\ell} \Big[|\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|\Big].$$

In the past few years, a series of papers related to regular functions of a quaternionic variable has been published and significant advances have been achieved. In ([8]–[6]) the structure of the zeros of polynomials was used and a topological proof of the Fundamental Theorem of Algebra was established. We point out that the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ was already proved by Niven (for reference, see [14], [15]), by using different techniques. This lead to the complete identification of the zeros of polynomials in terms of their factorization. Thus it became an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of quaternionic variable. In the literature, we could not find much except the above mentioned papers about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. The main purpose of this paper is to determine the regions containing all the zeros of a regular polynomial of quaternionic variable when the real and imaginary parts of its coefficients are restricted. We shall make use of a recently established maximum modulus theorem (Theorem E) and the structure of the zero sets of regular functions (Theorem D) of a quaternionic variable. The obtained results also produce various generalizations of Theorems F, G, H and I.

3. Main results

In this section, we state our main results and their proofs are given in the next section. We begin with the following generalizations of Theorems F, H and I.

Theorem 3.1. *If* $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ *is a polynomial of degree* n*, where* q *is a quaternionic variable, with quaternionic coefficients, where* $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ *for* $\nu = 0, 1, 2, \ldots, n$*, satisfying*

$$\begin{aligned}
&R\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_\ell,\\
&R\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_\ell,\\
&R\gamma_n \ge \gamma_{n-1} \ge \cdots \ge \gamma_\ell,\\
&R\delta_n \ge \delta_{n-1} \ge \cdots \ge \delta_\ell
\end{aligned}$$

for some $R \ge 1$ *and* $0 \le \ell \le n-1$. *Then all the zeros of* $T(q)$ *lie in*

$$|q + R - 1| \le \frac{1}{|a_n|}\Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big],$$

where

$$M_\ell = \sum_{\nu=1}^{\ell} \Big[|\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|\Big].$$

Remark 3.1. Taking $R = 1$ in Theorem 3.1, we get Theorem I.

By assuming $\alpha_\nu \le \alpha_{\nu-1}$, $\beta_\nu \le \beta_{\nu-1}$, $\gamma_\nu \le \gamma_{\nu-1}$, $\delta_\nu \le \delta_{\nu-1}$ for $1 \le \nu \le \ell$, we get the following result from which we get the quaternionic analogue of Theorem B.

Corollary 3.1. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable, with quaternionic coefficients $a_\nu = \alpha_\nu + \beta_\nu i + \gamma_\nu j + \delta_\nu k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\begin{aligned}
&R\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_{\ell+1} \geq \alpha_\ell \leq \alpha_{\ell-1} \leq \cdots \leq \alpha_1 \leq \alpha_0,\\
&R\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_{\ell+1} \geq \beta_\ell \leq \beta_{\ell-1} \leq \cdots \leq \beta_1 \leq \beta_0,\\
&R\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_{\ell+1} \geq \gamma_\ell \leq \gamma_{\ell-1} \leq \cdots \leq \gamma_1 \leq \gamma_0,\\
&R\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_{\ell+1} \geq \delta_\ell \leq \delta_{\ell-1} \leq \cdots \leq \delta_1 \leq \delta_0,
\end{aligned}$$

for some $R \geq 1$, then all the zeros of $T(q)$ lie in

$$\begin{aligned}
\left|q + R - 1\right| \leq \frac{1}{|a_n|}\Big[&(R\alpha_n + |\alpha_0| + \alpha_0 - 2\alpha_\ell) + (R\beta_n + |\beta_0| + \beta_0 - 2\beta_\ell)\\
&+ (R\gamma_n + |\gamma_0| + \gamma_0 - 2\gamma_\ell) + (R\delta_n + |\delta_0| + \delta_0 - 2\delta_\ell)\Big].
\end{aligned}$$

Remark 3.2. If we take $R = 1$ and $l = 0$ in Corollary 3.1, we recover Theorem H.

Taking $\beta_\nu = \gamma_\nu = \delta_\nu = 0$ for $\nu = 0, 1, 2, \ldots, n$, in Corollary 3.1, we get the following result. As a consequence, we obtain the quaternionic analogue of Theorem B from this result.

Corollary 3.2. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable, with real coefficients satisfying*

$$Ra_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_{\ell+1} \geq a_\ell \leq a_{\ell-1} \leq \cdots \leq a_1 \leq a_0,$$

for some $R \geq 1$ and $0 \leq \ell \leq n-1$, then all the zeros of $T(q)$ lie in

$$\left|q + R - 1\right| \leq \frac{1}{|a_n|}\Big[Ra_n + |a_0| + a_0 - 2a_\ell\Big].$$

Remark 3.3. *If we take $R = 1$ and $l = 0$ in Corollary 3.2, we get the quaternionic analogue of Theorem B. If in addition to $R = 1$ and $\ell = 0$, we suppose $a_0 \geq 0$, we get Theorem F.*

The next corollary is obtained by taking $R = a_{n-1}/a_n \geq 1$ in Corollary 3.2.

Corollary 3.3. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable, with real coefficients satisfying*

$$a_n \leq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_{\ell+1} \geq a_\ell \leq a_{\ell-1} \leq \cdots \leq a_1 \leq a_0,$$

where $0 \leq \ell \leq n-1$, then all the zeros of $T(q)$ lie in

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{a_{n-1} + |a_0| + a_0 - 2a_\ell}{|a_n|}.$$

If in Corollary 3.3, we take $\ell = 0$ and assume $a_0 > 0$, we get the following result.

Corollary 3.4. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n, where q is a quaternionic variable, with real coefficients satisfying*

$$a_n \leq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0 > 0,$$

then all the zeros of $T(q)$ lie in

$$\left|q + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{a_{n-1}}{a_n}.$$

Next, we establish the following generalization of Theorem G.

Theorem 3.2. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\measuredangle(a_\nu, b) \le \theta \le \pi/2$ for some θ and for $\nu = \ell, \ell+1, \ell+2, \dots, n$. Assume that for some $R \ge 1$,*

$$R|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_\ell|,$$

for $0 \le \ell \le n-1$, then all the zeros of $T(q)$ lie in

$$|q + R - 1| \le \frac{1}{|a_n|}\left[R|a_n|(\cos\theta + \sin\theta) + (|a_0| - |a_\ell|) + 2\sin\theta \sum_{\nu=\ell}^{n-1} |a_\nu| + N_\ell\right],$$

where

$$N_\ell = \sum_{\nu=1}^{\ell} |a_\nu - a_{\nu-1}|.$$

Remark 3.4. If we take $R = 1$ and $\ell = 0$ in Theorem 3.2, we recover Theorem G.

By assuming $|a_\nu| \le |a_{\nu-1}|$ for $1 \le \nu \le \ell$, we get the following result from Theorem 3.2.

Corollary 3.5. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$ is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\measuredangle(a_\nu, b) \le \theta \le \pi/2$ for some θ and for $\nu = 0, 1, 2, \dots, n$. Assume that for some $k \ge 1$,*

$$R|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_\ell| \le |a_{\ell-1}| \le \cdots \le |a_1| \le |a_0|,$$

then all the zeros of $T(q)$ lie in

$$|q + R - 1| \le \frac{1}{|a_n|}\left[R|a_n|(\cos\theta + \sin\theta) + (|a_0| - |a_\ell|)(1 + \cos\theta) + (|a_0| + |a_\ell) \sin\theta + 2\sin\theta \sum_{\nu=\ell}^{n-1} |a_\nu|\right].$$

If in Corollary 3.5, we take $\ell = 0$, we get the following generalization of Theorem G.

Corollary 3.6. *If $T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu$, is a polynomial of degree n with quaternionic coefficients and quaternionic variable. Let b be a non-zero quaternion and suppose $\measuredangle(a_\nu, b) \le \theta \le \pi/2$ for some θ and for $\nu = 0, 1, 2, \dots, n$. Assume that for some $R \ge 1$,*

$$R|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_0|,$$

then all the zeros of $T(q)$ lie in

$$|q + R - 1| \le R(\cos\theta + \sin\theta) + 2\frac{\sin\theta}{|a_n|} \sum_{\nu=0}^{n-1} |a_\nu|.$$

Remark 3.5. For $R = 1$, Corollary 3.6 reduces to Theorem G.

Finally, we shall try to relax the hypothesis of Theorem F by assuming that the real parts of the quaternionic coefficients of the polynomial $T(q) = a_0 + qa_1 + q^2a_2 + \cdots + q^na_n$ are alternately monotonic. In this connection, we prove the following result.

Theorem 3.3. *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ is a polynomial of degree n, where q is a quaternionic variable, with quaternionic coefficients, where $a_{\nu} = \alpha_{\nu} + \beta_{\nu} i + \gamma_{\nu} j + \delta_{\nu} k$ for $\nu = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_3 \geq \alpha_1 > 0 \quad \text{and} \quad \alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_2 \geq \alpha_0 > 0$$

if n is odd, or

$$\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_2 \geq \alpha_0 > 0 \quad \text{and} \quad \alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_3 \geq \alpha_1 > 0$$

if n is even, then all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} \right| \leq \frac{1}{|a_n|} \left[\alpha_n + \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| + 2 \sum_{\nu=0}^{n-2} \Big(|\beta_{\nu}| + |\gamma_{\nu}| + |\delta_{\nu}| \Big) \right].$$

If we take $\beta_{\nu} = \gamma_{\nu} = \delta_{\nu} = 0$ for $\nu = 0, 1, 2, \ldots, n$, in Theorem 3.3, we get the following result.

Corollary 3.7. *If $T(q) = \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$ is a polynomial of degree n, where q is a quaternionic variable, with real coefficients satisfying*

$$a_n \geq a_{n-2} \geq \cdots \geq a_3 \geq a_1 > 0 \quad \text{and} \quad a_{n-1} \geq a_{n-3} \geq \cdots \geq a_2 \geq a_0 > 0$$

if n is odd, or

$$a_n \geq a_{n-2} \geq \cdots \geq a_2 \geq a_0 > 0 \quad \text{and} \quad a_{n-1} \geq a_{n-3} \geq \cdots \geq a_3 \geq a_1 > 0$$

if n is even, then all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

4. Proofs of the main results

We need the following lemma due to Carney et al. [2] for the proof of Theorem 3.2.

Lemma 4.1. *Let $q_1, q_2 \in \mathbb{H}$, where $q_1 = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k$ and $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, $\measuredangle(q_1, q_2) = 2\theta' \leq 2\theta$ and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

Proof of Theorem 3.1. Consider the polynomial

$$\begin{aligned} T(q) * (1 - q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1} a_n \\ &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(R a_n - a_{n-1}) - q^n R a_n + q^n a_n - q^{n+1} a_n \\ &= \phi(q) - q^n R a_n + q^n a_n - q^{n+1} a_n, \end{aligned}$$

where

$$\phi(q) = a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(R a_n - a_{n-1}).$$

By Theorem D, $T(q) * (1 - q) = 0$ if and only if either $T(q) = 0$ or $T(q) \neq 0$ implies $T(q)^{-1} q T(q) - 1 = 0$, that is $T(q)^{-1} q T(q) = 1$. Thus, if $T(q) \neq 0$, this implies $q = 1$, so the only zero of $T(q) * (1 - q)$ are $q = 1$ and the zeros of $T(q)$.

For $|q| = 1$, we have

$$\begin{aligned}
|\phi(q)| &\leq \left| a_0 + \sum_{\nu=1}^{\ell} q^\nu (a_\nu - a_{\nu-1}) + \sum_{\nu=\ell+1}^{n-1} q^\nu (a_\nu - a_{\nu-1}) + q^n (Ra_n - a_{n-1}) \right| \\
&\leq |a_0| + \sum_{\nu=1}^{\ell} |q^\nu||a_\nu - a_{\nu-1}| + \sum_{\nu=\ell+1}^{n-1} |q^\nu||a_\nu - a_{\nu-1}| + |q^n||Ra_n - a_{n-1}| \\
&= |\alpha_0 + \beta_0 i + \gamma_0 j + \delta_0 k| + \sum_{\nu=1}^{\ell} |(\alpha_\nu - \alpha_{\nu-1}) + (\beta_\nu - \beta_{\nu-1})i + (\gamma_\nu - \gamma_{\nu-1})j + (\delta_\nu - \delta_{\nu-1})k| \\
&\quad + \sum_{\nu=\ell+1}^{n-1} |(\alpha_\nu - \alpha_{\nu-1}) + (\beta_\nu - \beta_{\nu-1})i + (\gamma_\nu - \gamma_{\nu-1})j + (\delta_\nu - \delta_{\nu-1})k| \\
&\quad + |(R\alpha_n - \alpha_{n-1}) + (R\beta_n - \beta_{n-1})i + (R\gamma_n - \gamma_{n-1})j + (R\delta_n - \delta_{n-1})k| \\
&\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{\nu=1}^{\ell} (|\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|) \\
&\quad + \sum_{\nu=\ell}^{n-1} (|\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}|) \\
&\quad + |R\alpha_n - \alpha_{n-1}| + |R\beta_n - \beta_{n-1}| + |R\gamma_n - \gamma_{n-1}| + |R\delta_n - \delta_{n-1}|,
\end{aligned}$$

i.e.,

$$|\phi(q)| = (R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell,$$

where

$$M_\ell = \sum_{\nu=1}^{\ell} \Big[|\alpha_\nu - \alpha_{\nu-1}| + |\beta_\nu - \beta_{\nu-1}| + |\gamma_\nu - \gamma_{\nu-1}| + |\delta_\nu - \delta_{\nu-1}| \Big].$$

Note that, we have

$$\max_{|q|=1} \left| q^n * \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| q^n \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} |\phi(q)|,$$

it is clear that $q^n * \phi(1/q)$ has the same bound on $|q| = 1$ as ϕ, that is

$$\left| q^n * \phi\left(\frac{1}{q}\right) \right| \leq (R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell$$

for $|q| = 1$.

Since $q^n * \phi(1/q)$ is a polynomial and so is regular in $|q| \leq 1$, it follows by the Maximum Modulus Theorem (Theorem E), that

$$\left| q^n * \phi\left(\frac{1}{q}\right) \right| = \left| q^n \phi\left(\frac{1}{q}\right) \right| \leq (R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell)$$

for $|q| \leq 1$.

Hence

$$\left| \phi\left(\frac{1}{q}\right) \right| \leq \frac{1}{|q|^n} \Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell \Big]$$

for $|q| \leq 1$.

Replacing q by $1/q$, we see that for $|q| \geq 1$,

$$|\phi(q)| \leq \Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big]|q|^n. \tag{1}$$

align For $|q| \geq 1$, using (1), we get

$$\begin{aligned}|T(q) * (1-q)| &= |\phi(q) - q^n Ra_n + q^n a_n - q^{n+1}a_n| \\ &\geq |q|^n|a_n||q + R - 1| - |\phi(q)| \\ &\geq |q|^n\Big\{|a_n||q + R - 1| - \Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) \\ &\qquad + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big]\Big\}.\end{aligned}$$

Hence, if

$$|q + R - 1| > \frac{1}{|a_n|}\Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big],$$

then $|T(q) * (1-q)| > 0$, that is $T(q) * (1-q) \neq 0$.

Since the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore, $T(q) \neq 0$ for

$$|q + R - 1| > \frac{1}{|a_n|}\Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big].$$

In other words, all the zeros of $T(q)$ lie in

$$|q + R - 1| \leq \frac{1}{|a_n|}\Big[(R\alpha_n + |\alpha_0| - \alpha_\ell) + (R\beta_n + |\beta_0| - \beta_\ell) + (R\gamma_n + |\gamma_0| - \gamma_\ell) + (R\delta_n + |\delta_0| - \delta_\ell) + M_\ell\Big],$$

and this completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Again consider the polynomial

$$T(q) * (1-q) = \phi(q) - q^n Ra_n + q^n Ra_n - q^{n+1}a_n,$$

where

$$\phi(q) = a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(Ra_n - a_{n-1}).$$

For $|q| = 1$, we have

$$\begin{aligned}|\phi(q)| &\leq |a_0| + \sum_{\nu=1}^{\ell}|a_\nu - a_{\nu-1}| + \sum_{\nu=\ell+1}^{n-1}|a_\nu - a_{\nu-1}| + |Ra_n - a_{n-1}| \\ &\leq |a_0| + N_\ell + \sum_{\nu=\ell+1}^{n-1}\Big[(|a_\nu| - |a_{\nu-1}|)\cos\theta + (|a_\nu| + |a_{\nu-1}|)\sin\theta\Big] \\ &\qquad + (R|a_n| - |a_{n-1}|)\cos\theta + (R|a_n| + |a_{n-1}|)\sin\theta \qquad \text{(by Lemma 4.1)} \\ &= (|a_0| - |a_\ell|) + N_\ell + R|a_n|(\cos\theta + \sin\theta) + |a_\ell|(1 - \cos\theta - \sin\theta) + 2\sin\theta\sum_{\nu=\ell}^{n-1}|a_\nu| \\ &\leq (|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta\sum_{\nu=\ell}^{n-1}|a_\nu| + N_\ell,\end{aligned}$$

where $N_\ell = \sum_{\nu=1}^{\ell} |a_\nu - a_{\nu-1}|$.

Notice that, we have

$$\max_{|q|=1} \left| q^n * \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| q^n \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} |\phi(q)|,$$

it is clear that $q^n * \phi(1/q)$ has the same bound on $|q| = 1$ as ϕ, that is

$$\left| q^n * \phi\left(\frac{1}{q}\right) \right| = \left| q^n \phi\left(\frac{1}{q}\right) \right| \le (|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=l}^{n-1} |a_\nu| + N_\ell$$

for $|q| = 1$.

Then, by the Maximum Modulus Theorem (Theorem E), we have for $|q| \le 1$,

$$\left| q^n \phi\left(\frac{1}{q}\right) \right| \le (|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=\ell}^{n-1} |a_\nu| + N_\ell.$$

Equivalently, for $|q| \ge 1$,

$$|\phi(q)| \le \left[(|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=\ell}^{n-1} |a_\nu| + N_\ell \right] |q|^n. \tag{2}$$

For $|q| \ge 1$, we have

$$\begin{aligned} |T(q) * (1-q)| &= |\phi(q) - q^n R a_n + q^n a_n - q^{n+1} a_n| \\ &\ge |q|^n |a_n| |q + R - 1| - |\phi(q)| \\ &\ge \left[|a_n| |q + R - 1| - \left\{ (|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=l}^{n-1} |a_\nu| + N_\ell \right\} \right] |q|^n, \end{aligned}$$

by (2). Hence, if

$$|q + R - 1| > \frac{1}{|a_n|} \left[(|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=l}^{n-1} |a_\nu| + N_\ell \right],$$

then $|T(q) * (1-q)| > 0$, and therefore $T(q) * (1-q) \ne 0$.

Note that $R \ge 1$, and $\theta \in [0, \pi/2]$, therefore

$$\begin{aligned} \frac{1}{|a_n|} \left[(|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=\ell}^{n-1} |a_\nu| + N_\ell \right] &\ge R(\cos\theta + \sin\theta) \\ &\ge \cos\theta + \sin\theta \\ &\ge 1. \end{aligned}$$

Thus,

$$|q + R - 1| > \frac{1}{|a_n|} \left[(|a_0| - |a_\ell|) + R|a_n|(\cos\theta + \sin\theta) + 2\sin\theta \sum_{\nu=l}^{n-1} |a_\nu| + N_\ell \right],$$

implies also that $|q+R-1|>1$. Since the only zeros of $T(q)*(1-q)$ are $q=1$ and the zeros of $T(q)$, therefore, $T(q)\neq 0$ for

$$|q+R-1|>\frac{1}{|a_n|}\left[(|a_0|-|a_\ell|)+R|a_n|(\cos\theta+\sin\theta)+2\sin\theta\sum_{\nu=\ell}^{n-1}|a_\nu|+N_\ell\right].$$

That is, all the zeros of $T(q)$ lie in

$$|q+R-1|\leq\frac{1}{|a_n|}\left[(|a_0|-|a_\ell|)+R|a_n|(\cos\theta+\sin\theta)+2\sin\theta\sum_{\nu=\ell}^{n-1}|a_\nu|+N_\ell\right],$$

and this completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. Consider the polynomial

$$\begin{aligned}T(q)*(1-q^2)&=a_0+qa_1+q^2(a_2-a_0)+q^3(a_3-a_1)+\cdots+q^{n-1}(a_{n-1}-a_{n-3})\\&\quad+q^n(a_n-a_{n-2})-q^{n+1}a_{n-1}-q^{n+2}a_n\\&=\psi(q)-q^{n+1}a_{n-1}-q^{n+2}a_n,\end{aligned}$$

where $\psi(q)=a_0+qa_1+q^2(a_2-a_0)+\cdots+q^{n-1}(a_{n-1}-a_{n-3})+q^n(a_n-a_{n-2})$.

For $|q|=1$, we have

$$\begin{aligned}|\psi(q)|&=\left|a_0+qa_1+\sum_{\nu=2}^{n}q^\nu\,(a_\nu-a_{\nu-2})\right|\\&\leq|a_0|+|a_1|+|a_2-a_0|+|a_3-a_1|+\cdots+|a_{n-1}-a_{n-3}|+|a_n-a_{n-2}|\\&\leq(\alpha_0+|\beta_0|+|\gamma_0|+|\delta_0|)+(\alpha_1+|\beta_1|+|\gamma_1|+|\delta_1|)\\&\quad+(\alpha_2-\alpha_0+|\beta_2|+|\beta_0|+|\gamma_2|+|\gamma_0|+|\delta_2|+|\delta_0|)\\&\quad+(\alpha_3-\alpha_1+|\beta_3|+|\beta_1|+|\gamma_3|+|\gamma_1|+|\delta_3|+|\delta_1|)\\&\quad\vdots\\&\quad+(\alpha_{n-1}-\alpha_{n-3}+|\beta_{n-1}|+|\beta_{n-3}|+|\gamma_{n-1}|+|\gamma_{n-3}|+|\delta_{n-1}|+|\delta_{n-3}|)\\&\quad+(\alpha_n-\alpha_{n-2}+|\beta_n|+|\beta_{n-2}|+|\gamma_n|+|\gamma_{n-2}|+|\delta_n|+|\delta_{n-2}|)\\&=\alpha_n+\alpha_{n-1}+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|+2\sum_{\nu=0}^{n-2}\Big(|\beta_\nu|+|\gamma_\nu|+|\delta_\nu|\Big).\end{aligned}$$

Proceeding similarly as in the proof of Theorem 3.1, it follows that for $|q|\geq 1$,

$$|\psi(q)|\leq\left[\alpha_n+\alpha_{n-1}+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|)+2\sum_{\nu=0}^{n-2}(|\beta_\nu|+|\gamma_\nu|+|\delta_\nu|)\right]|q|^n. \tag{3}$$

For $|q|\geq 1$, by (3), we have

$$\begin{aligned}|T(q)*(1-q^2)|&=|\psi(q)-q^{n+1}a_{n-1}-q^{n+2}a_n|\\&\geq|q|^n\Big\{|q||qa_n+a_{n-1}|-\Big[\alpha_n+\alpha_{n-1}+|\beta_n|+|\beta_{n-1}|\\&\qquad+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|+2\sum_{\nu=0}^{n-2}(|\beta_\nu|+|\gamma_\nu|+|\delta_\nu|)\Big]\Big\}.\end{aligned}$$

Hence, if

$$\left| q + \frac{a_{n-1}}{a_n} \right| > \frac{1}{|a_n|} \left[\alpha_n + \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| + 2 \sum_{\nu=0}^{n-2} (|\beta_\nu| + |\gamma_\nu| + |\delta_\nu|) \right],$$

then $|T(q) * (1 - q^2)| > 0$, that is, $T(q) * (1 - q^2) \neq 0$. Since the only zeros of $T(q) * (1 - q^2)$ are $q^2 = 1$ and the zeros of $T(q)$, and therefore, $T(q) \neq 0$ for

$$\left| q + \frac{a_{n-1}}{a_n} \right| > \frac{1}{|a_n|} \left[\alpha_n + \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| + 2 \sum_{\nu=0}^{n-2} (|\beta_\nu| + |\gamma_\nu| + |\delta_\nu|) \right].$$

In other words, all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} \right| \leq \frac{1}{|a_n|} \left[\alpha_n + \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| + 2 \sum_{\nu=0}^{n-2} (|\beta_\nu| + |\gamma_\nu| + |\delta_\nu|) \right],$$

which completes the proof of Theorem 3.3. □

5. Conclusion

The regular functions of a quaternionic variable have been introduced and intensively studied in the past decade. They have proven to be a fertile topic in analysis, and their rapid development has been largely driven by its numerous applications in many areas of scientific disciplines. We point out that after the study of the structure of zero sets and the Fundamental Theorem of Algebra for regular polynomials (which lead to the complete identification of the zeros of polynomials in terms of their factorization), it became interesting to establish the regions containing some or all the zeros of a regular polynomial of quaternionic variable. In the literature, we could not find much about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. Here, we obtain regions containing all the zeros of a regular polynomial of quaternionic variable when the real and imaginary parts of its coefficients are restricted by virtue of a maximum modulus theorem and the structure of the zero sets established in the newly developed theory of regular functions and polynomials of a quaternionic variable.

References

[1] P.B. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer Verlag, New 23 York, 1995.
[2] N. Carney, R. Gardner, R. Keaton, A. Powers The Eneström-Kakeya theorem for polynomials of a quaternionic variable, J. Approx. Theory **250** (2020), (Art. 105325), 1–10.
[3] L. Coroianu, S. G. Gal, On the inequalities of Turán, Bernstein and Erdös-Lax in quaternionic setting, RACSAM **115** (2021) (Art. 187), 1–20.
[4] S.G. Gal, I. Sabadini, On the Bernstein and Erdős-Lax inequality for quaternionic polynomials, C. R. Acad. Sci. Paris Ser. I, **353** (2015), 5–9.
[5] R.B. Gardner, N.K. Govil, G.V. Milovanović, Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials, Elsevier/Academic Press, London, 2022.
[6] G. Gentili, C. Stoppato, Zeros of regular function and polynomials of a quaternionic variable, Michigan Math. J. **56** (2008), 655–667.
[7] G. Gentili, D.C. Struppa, A new theory of regular function of a quaternionic variable, Adv. Math. **216** (2007), 279–301.
[8] G. Gentili, D.C. Struppa, On the multiplicity of zeros of polynomials with quaternionic coefficients, Milan J. Math. **76** (2008), 15–25.
[9] N. K. Govil, Q. I. Rahman, On the Eneström-Kakeya theorem, Tôhoku Math. Jour. **20** (1968), 126–136.
[10] A. Joyal, G. Labelle, Q.I. Rahman, On the location of zeros polynomials, Canad. Math. Bull. **10** (1967), 53–63.
[11] T.Y. Lam, A First Course in Noncommutative Rings, Springer, New York, 1991.
[12] M. Marden, Geometry of Polynomials, Math. Surveys, American Mathematical Society, Providence, 1966.
[13] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, Topics in Polynomials, Extremal Problems, Inequalities, Zeros, World Scientific, Singapore, 1994.
[14] J. Niven, Equations in quaternions, Am. Math. Mon. **48** (1941), 654–661.
[15] J. Niven, The roots of a quaternion, Am. Math. Mon. **49** (1942), 386–388.
[16] Q.I. Rahman, G. Schmeisser, Analytic Theory of Polynomials, Clarendon Press, Oxford, 2002.
[17] A. Sudbery, Quaternionic analysis, Math. Proc. Cambridge Philos. Soc. **85** (1979), 199–224.
[18] D. Tripathi, A note on Eneström-Kakeya theorem for a polynomial with quaternionic variable, Arab. J. Math. **9** (2020), 707–714.